Chapter 4

Ground state correlations of Ising models

If the ground state of the Ising model on a given lattice is degenerate, the ground state correlations must be calculated by averaging over all ground states. As a consequence, the spin-spin correlation is given by the average over the configurations C belonging to the ground state manifold:

$$\langle \sigma_i \sigma_j \rangle = \sum_{C \epsilon GS} (\sigma_i \sigma_j)_C$$

This is the average of numbers that can be positive or negative, and whether the system possesses long-range or short-range correlations, i.e. the behavior of $\langle \sigma_{\vec{R}_i} \sigma_{\vec{R}_j} \rangle$ in the limit $\|\vec{R}_i - \vec{R}_j\| \to +\infty$, is a non-trivial issue. In fact, four different behaviours have been identified:

- Long-range order. This is the case of the 3-state Potts model on the dice lattice (the dual lattice of the kagome lattice).
- Algebraic order: the correlations decay as a power law. This is realized in the AF Ising model on the triangular lattice or on the fully frustrated Villain model on the square lattice.
- Dipolar correlations: the correlations decay as the interaction between dipoles. This is the case of the AF Ising model on the checkerboard and pyrochlore model.
- Exponential decay of correlations. This is the case of the AF Ising model on the kagome lattice or on the fully furstrated model on the honeycomb lattice.

In this chapter, we will come up with arguments in favour of the last three behaviours. They are based on two approaches:

Mapping on dimer models If the Ising model can be mapped onto a dimer model that can be solved by Kasteleyn's method, then it is possible to predict the nature of the long-distance correlations just by studying the eigenvalues of the Kasteleyn matrix:

- If there is no gap between positive and negative eigenvalues, as for dimer coverings on the honeycomb lattice or on the square lattice, in which case the Kasteleyn matrix has a Dirac spectrum, the correlations decay as a power law.
- If there is a gap between positive and negative eigenvalues, as for dimer coverings on the triangular lattice, the correlations decay exponentially.

Mapping on height models If a mapping on a height model is possible, then algebraic correlations are predicted if the height model is in the rough phase, and the precise form of the correlations depend on the connection between the spins and the height variables:

- Purely algebraic if the spin is directly connected to the height variable, as in the Ising model on the triangular lattice.
- Dipolar if the spin is related to the gradient of the height field, as in the Ising model on the checkerboard lattice.

4.1 Ground state correlations functions of dimer models

The Pfaffian of a skew-symmetric matrix A_{ij} can be expressed as

$$\mathrm{Pf}(A) = \int \prod_i \mathrm{d}a_i \exp\left(\frac{1}{2}\sum_{i,j} a_i A_{ij} a_j\right) \tag{4.1}$$

where a_i are Grassman variables:

- $a_i a_j = -a_j a_i$
- $a_i^2 = 0$

- $\int \mathrm{d}a_i = 0$
- $\int \mathrm{d}a_i \; a_i = 1$, and when calculating multiple integrals, the a_i's have to arranged to be in contact with their corresponding $\mathrm{d}a_i$.

In the following, we will consider the case where the matrix A_{ij} is the matrix $a(i,j)$ of Chapter 2, and we will use the notation $Z = \mathrm{Pf}(A)$.

Proposition The dimer-dimer correlation functions

$$\langle n_{12} n_{34} \dots n_{2n-1} n_{2n} \rangle \,, \tag{4.2}$$

where $(1,2),(3,4),\dots,(2n-1,2n)$ are pairs of nearest-neighbor sites, can be expressed as

$$\langle a_1 a_2 a_3 a_4 \dots \rangle = \frac{1}{Z} \int \prod_i \mathrm{d}a_i \, (a_1 a_2 \dots a_{2n}) \exp\left(\frac{1}{2} \sum_{i,j} a_i A_{ij} a_j \right) \tag{4.3}$$

Proof The mean value $\langle n_{12} n_{34} \dots n_{2n-1} n_{2n} \rangle$ is equal to the number of configurations where $n_{12} = n_{34} = \dots = n_{2n-1} n_{2n} = 1$, divided by Z. But the number of configurations where $n_{12} = n_{34} = \dots = n_{2n-1} n_{2n} = 1$ is the Pfaffian of A deprived from the lines and columns $1, 2, \dots, 2n$. Now,

$$\exp\left(\sum_{i<j} a_i A_{ij} a_j \right) = \prod_{i<j} \exp\left(a_i A_{ij} a_j \right) = \prod_{i<j} \left(1 + a_i A_{ij} a_j \right) \tag{4.4}$$

The first equality comes from the fact that terms containing pairs of Grassman variables commute, and the second one from the property $a_i^2 = 0$.

All factors containing $a_1, a_2, \dots$ or a_{2n} will only contribute to a 1 to

$$\int \prod_i \mathrm{d}a_i \, (a_1 a_2 \dots a_{2n}) \exp\left(\sum_{i<j} a_i A_{ij} a_j \right) \tag{4.5}$$

which is thus equal to

$$\int \prod_{i \neq 1,2,\dots} \mathrm{d}a_i \exp\left(\sum_{\substack{i<j \\ i,j \neq 1,2,\dots}} a_i A_{ij} a_j \right) \tag{4.6}$$

This is just the Pfaffian of the matrix A deprived from lines and columns $1, 2, \dots, 2n$. QED

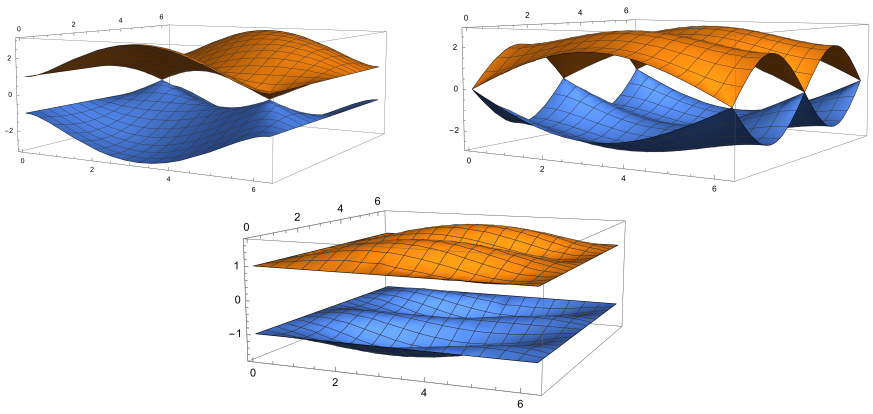

Figure 4.1: Spectrum of the Kasteleyn matrix for three lattices: honeycomb (top left), square (top right), triangular (bottom).

Now, a correlation function of the form $\langle a_1 a_2 a_3 a_4 \dots \rangle$ can be decomposed using Wick's theorem, which stipulates that it is equal to the sum of all the terms that consist of products of two-variable averages (all called a Green's function) obtained by pairing Grassman variables in all possible ways, and affected by the sing of the permutation that leads to this pairing. For instance,

$$\langle n_{12} n_{pq} \rangle = \langle a_1 a_2 a_p a_q \rangle = G_{12} G_{pq} - G_{1p} G_{2q} + G_{1q} G_{2p} \tag{4.7}$$

where the Green's function G_{ij} is defined by

$$G_{ij} = \langle a_i a_j \rangle . \tag{4.8}$$

This is nothing but a Gaussian integral over Grassman variables, and it can be shown that

$$\langle a_i a_j \rangle = \left(A^{-1}\right)_{ij} . \tag{4.9}$$

So the nature of the long distance decay (exponential or algebraic) or any correlation function will be controlled by that of $\left(A^{-1}\right)_{ij}$. Now, for a dimer covering problem, A is block diagonal after Fourier transform, with blocks of size $n \times n$, where n is the number of sites per unit cell in the Kasteleyn lattice (the lattice with arrows). Let's call such an $n \times n$ block $A(\vec{k})$, and $A^{-1}(\vec{k})$ the inverse matrix. If $i \in \alpha$ sublattice and $j \in \beta$ sublattice, then

$$\left(A^{-1}\right)_{ij} = \int_{\beta Z} \mathrm{d}\vec{k} \left(A^{-1}\right)_{\alpha\beta} e^{i\vec{k}\cdot(\vec{R}_i - \vec{R}_j)} \tag{4.10}$$

($\vec{R}_i$, $\vec{R}_j$: position of unit cells where i, j are located). This implies that, if $\left|\left(A^{-1}\right)_{\alpha\beta}(\vec{k})\right| \leq C > 0 \ \forall \vec{k}$, the correlations decay exponentially, otherwise, they decay algebraically. One can easily show that this condition is equivalent to $|\lambda_i(\vec{k})| \geq D > 0, i = 1, ..., n \ \forall \vec{k}$, where the $\lambda_i(\vec{k})$, $i = 1, ..., n$, are the eigenvalues of the Kasteleyn matrix $A(\vec{k})$.

In other words, the correlations are expected to decay algebraically if the spectrum of the Kasteleyn matrix is gapless, and exponentially if it has a gap between negative and positive eigenvalues. The spectra of the Kasteleyn matrices for dimer coverings on the honeycomb, square and triangular lattice are shown in Fig. 4.1. The spectrum is gapless for the honeycomb and square lattices, implying algebraic correlations, and gapped for the triangular lattice, implying exponential correlations.

Implications for Ising models If an Ising model can be mapped onto a dimer covering problem, then the product of two spin variables can be expressed in terms of dimer variables. Indeed, for a pair (i, j) of nearest neighbour sites,

$$\sigma_i \sigma_j = 2n_{ij} - 1. \tag{4.11}$$

For a pair of sites that are not nearest neighbors, one should first find a path connecting them through nearest neighbors, e.g. a set of sites $1, ..., n$ such that $(i, 1), (1, 2), ..., (n-1, n), (n, j)$ are nearest nearest neighbours. Then

$$\sigma_i \sigma_j = \sigma_i \sigma_1 \ \sigma_1 \sigma_2 \ \ldots \sigma_{n-1} \sigma_n \ \sigma_n \sigma_j = (2n_{i1} - 1)(2n_{12} - 1) \ldots (2n_{nj} - 1) \tag{4.12}$$

where the first equality comes from the property $\sigma_k^2 = 1$, and the second from the expression of $\sigma_k \sigma_l$ in terms of the dimer variables n_{kl} for nearest neighbors.

As a consequence, the nature of the long-distance decay of the spin-spin correlation is similar to that of the dimer-dimer correlation and can be read off from the spectrum of the Kasteleyn matrix. This simple argument thus predicts that the Ising model on the triangular lattice and on the fully frustrated square lattice have algebraic correlations, while the Ising model on the fully frustrated honeycomb lattice has exponentially decaying correlations.

4.2 Mapping on height models

The presence of algebraic correlations can also be seen as the consequence of yet another mapping, this time onto height models. Such models represent the fluctuations of the surface of a solid by assigning to each point a height.

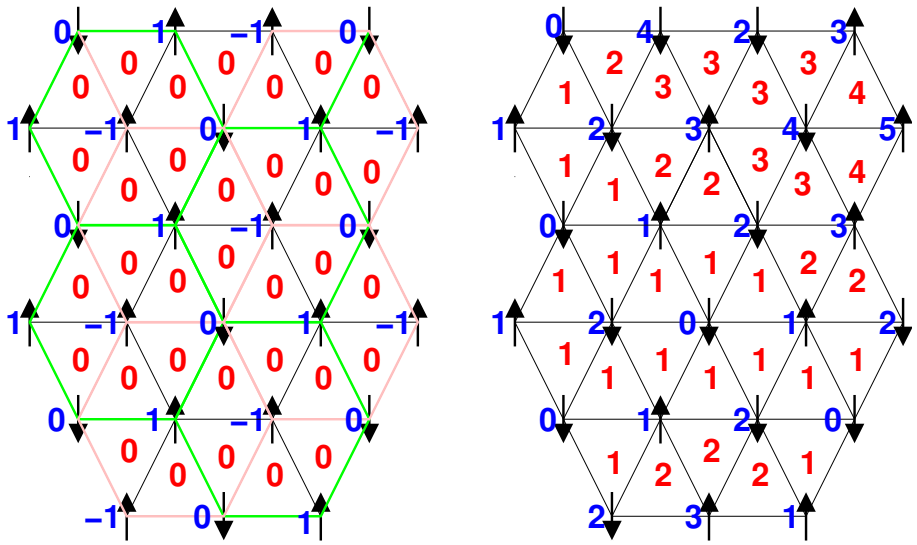

Figure 4.2: Example of height configurations for the Ising model on the triangular lattice. The blue numbers correspond to the local height variables $z(\vec{r})$, the red ones to the coarse-grained variables $h(\vec{r})$. Left: Flat surface corresponding to a maximally flippable configuration. All the spins sitting at the center of a green or pink hexagon can be flipped without leaving the ground state manifold. Right: A random configuration generating a non-uniform height configuration.

They describe the roughening transition of a solid between a flat surface, where the height difference is bounded from above, and a rough surface, where height differences correlations diverge logarithmically. There is no general recipe to map an Ising model onto a height model. This is actually not always possible. It is only possible if there is a local rule that leads to a consistent mapping. In this section, we will discuss two such mappings: for the triangular lattice, and for the checkerboard lattice

4.2.1 Antiferromagnetic Ising model on the triangular lattice

The mapping onto a height model proceeds in two steps. First, one defines *local* height variables on the original lattice. The recipe depends on the lattice. For the triangular lattice, one associates to each dimer covering on the dual lattice a local height configuration $z(\vec{r})$ according to the following prescriptions:

1. Choose the height of one site
2. Going clockwise around an up triangle, the height difference between neighbouring sites is given by:
   - 2 if one crosses a dimer
   - -1 if one does not cross a dimer

With this prescription, the height differences around up triangles add up to zero, and the signs are opposite when going clockwise around down triangles. This implies that the height differences around all triangles ($\bigtriangledown$ or $\bigtriangleup$) are equal to zero, hence that the assignment is consistent. What makes this mapping possible is the local rule that, for each plaquette $\sum_i \sigma_i = \pm 1$, never 3 or -3.

Then, from the local variables $z(\vec{r})$, one defines coarse-grained height variables $h(\vec{r})$ on the dual lattice by

$$h(\vec{r}) = \frac{1}{3}[z(\vec{r}_1) + z(\vec{r}_2) + z(\vec{r}_3)]$$

For a general configuration, the height variables can take any integer value. There is however one special configuration that leads to a flat surface, i.e. a surface where $h(\vec{r})$ is constant. This configuration is illustrated in Fig.4.2. In this configuration, two of the three honeycomb sublattices that can be obtained by depleting the triangular lattice by one third of the sites (as done to find a lower bound of the residual entropy in the introduction) have Néel order. This implies that all spins inside the hexagons of the two sublattices, 2/3 of the spins in total, can be flipped. Since on each triangle one spin cannot be flipped to stay in the ground state, one cannot do better. This configuration is said to be *maximally flippable*.

Being maximally flippable, this configuration is connected to a large number of configurations that differ from it only by flipping a few spins. These configurations are entropically favored. This leads to the idea of describing the ground state manifold in terms of height variables as fluctuations around the flat surface. The simplest assumption is to assume a gaussian weight that penalizes height configurations according to their gradient. Going to a continuous height variable, one thus assumes that the weight of a configuration is given by $\exp(-F[h(\vec{x})])$, where $F[h(\vec{x})]$ is a functional of $h(\vec{x})$ of the form:

$$F[h(\vec{x})] = \int \mathrm{d}\vec{x} \frac{K}{2} \left|\vec{\nabla} h(\vec{x})\right|^2 = \sum_{\vec{q}} \frac{K}{2} q^2 \left|h(\vec{q})\right|^2 . \tag{4.13}$$

For a mode $h(\vec{q})$, the mean value of $h(\vec{q})^2$ is given by:

$$\left\langle |h(\vec{q})|^2 \right\rangle = \frac{\int \mathrm{d}h h^2 \exp\left(-\frac{K}{2}q^2h^2\right)}{\int \mathrm{d}h \exp\left(-\frac{K}{2}q^2h^2\right)} \tag{4.14}$$

The denominator is a gaussian integral, and the numerator is proportional to its derivative with respect to K, leading to:

$$\left\langle |h(\vec{q})|^2 \right\rangle = \frac{1}{Kq^2}$$

Going back to real space,

$$\left\langle \left|h(\vec{r}+\vec{R}) - h(\vec{R})\right|^2 \right\rangle = \sum_{\vec{q}_1,\vec{q}_2} h(\vec{q}_1)h(\vec{q}_2)e^{i\vec{q}_1.\vec{R}}e^{i\vec{q}_2.\vec{R}}(e^{i\vec{q}_1.\vec{r}}-1)(e^{i\vec{q}_2.\vec{r}}-1)$$

which implies that

$$\frac{1}{N}\sum_{\vec{R}} \left\langle \left|h(\vec{r}+\vec{R}) - h(\vec{R})\right|^2 \right\rangle = 2\sum_{\vec{q}} |h(\vec{q})|^2(1-e^{i\vec{q}.\vec{r}}).$$

Now, $\left\langle \left|h(\vec{r}+\vec{R}) - h(\vec{R})\right|^2 \right\rangle$ must be independent of $\vec{R}$. This implies that

$$\frac{1}{2}\left\langle \left|h(\vec{r}) - h(\vec{0})\right|^2 \right\rangle = \int \mathrm{d}q \left(1-e^{i\vec{q}\cdot\vec{r}}\right)\frac{1}{Kq^2} \propto \frac{1}{2\pi K}\ln\frac{\pi r}{\alpha} \quad \text{when} \quad r \to +\infty$$

where a is the lattice parameter. Up to constant, the left hand side is equal to the height correlation $\langle h(\vec{r})h(\vec{0})\rangle$. At large distance, it diverges logarithmically. This behaviour is typical of the rough phase. This phase is not always stable however. For $K > \pi/2$, it can be shown using renormalization group arguments that a periodic potential is a relevant perturbation, and that it will lock the surface into a flat configuration where $\left\langle \left|h(\vec{r}) - h(\vec{0})\right|^2 \right\rangle$ is bounded. So this description in terms of the rough phase of a height model is only possible if $K < \pi/2$.

Let us now see how spin variables are related to height variables. For that purpose, let us start from a spin configuration on a triangle and flip spins around the triangle one after the other without breaking the rule of two spins up and one spin down or two spins downs and one spin up (see Fig.4.3). The height h of the first and last configurations differ by 6, but they correspond to the same spin configuration. Therefore the spins have to be periodic functions of the height h of period 6, a logical conclusion since there are only 6 possible spin configurations on a triangle whereas the height

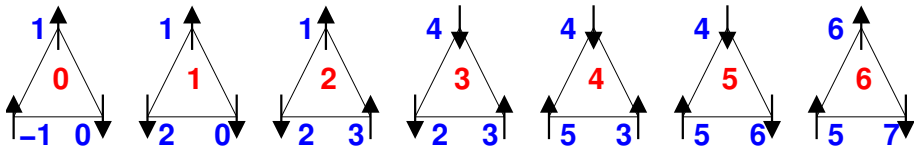

Figure 4.3: Illustration of the periodicity of the spin variables with the height. The configurations are obtained by flipping spins one after the other while respecting the ground state rule. The last configuration is the same as the first one, but the height has increased by 6.

variables can take arbitrary integer values. This implies that a spin variable can be expanded in Fourier series as:

$$\sigma(\vec{r}_i) = \sum_{G\neq 0} O_G e^{iGh(\vec{r})}$$

with $G = \frac{2\pi}{6} \times n$, n integer. Now, since this is a gaussian theory, the correlation function of a Fourier component of the height field is given by

$$\left\langle e^{iGh(\vec{r})} e^{-iGh(\vec{0})} \right\rangle = e^{-\frac{1}{2}G^2 \left\langle |h(\vec{r})-h(\vec{0})|^2 \right\rangle}$$

So the slowest decay will be given by the Fourier component with the smallest value of G, namely $\frac{2\pi}{6}$.

The long distance behaviour of the correlation function of the spin is thus given by

$$\begin{aligned} \left\langle \sigma(\vec{r})\sigma(\vec{0}) \right\rangle &\propto \left\langle e^{i\frac{2\pi}{6}h(\vec{r})} e^{-i\frac{2\pi}{6}h(\vec{0})} \right\rangle && (4.15)\\ &= e^{-\frac{1}{2}(\frac{2\pi}{6})^2 \langle |h(\vec{r})-h(\vec{0})|^2 \rangle} && (4.16)\\ &= e^{-(\frac{2\pi}{6})^2 \frac{1}{2\pi K} \ln \frac{\pi r}{a}} && (4.17)\\ &= \left(\frac{\pi r}{a}\right)^{-(\frac{2\pi}{6})^2 \frac{1}{2\pi K}} && (4.18) \end{aligned}$$

or

$$\left\langle \sigma(\vec{r})\sigma(\vec{0}) \right\rangle \propto \left(\frac{\pi r}{a}\right)^{-\frac{2\pi}{36K}}$$

Now, it is known from the exact solution that

$$\left\langle \sigma(\vec{r})\sigma(\vec{0}) \right\rangle \propto r^{-\frac{1}{2}}$$

This suggests that the Ising model on the triangular lattice can be mapped on a height model in the rough phase with $K = \frac{\pi}{9}$. This value of K is smaller than the critical valued of K, $\frac{\pi}{2}$, beyond which the rough phase is unstable to a periodic potential, so this mapping is fully consistent.

4.2.2 Antiferromagnetic Ising model on checkerboard lattice

The checkerboard lattice is not a planar graph (bonds are crossing), and Kasteleyn's approach cannot be implemented. The height mapping is in that case very useful because it still offers a simple way to get analytical insight into the nature of the correlations, and, as we shall see, it brings an unexpected new features of the correlation.

The Ising model on the checkerboard lattice can be rewritten as a sum over crossed plaquettes (plaquettes with diagonal bonds) as

$$E = \sum_{\boxtimes} e_{\boxtimes}$$

with

$$e_{\boxtimes} = \frac{1}{2}\left[\left(\sum_{i\in\boxtimes}\sigma_i\right)^2 - \sum_{i\in\boxtimes}\sigma_i^2\right] = \frac{1}{2}\left(\sum_{i\in\boxtimes}\sigma_i\right)^2 - 2$$

The energy is minimal as soon as $\sum_{i\in\boxtimes}\sigma_i = 0$ on each crossed plaquette. Thanks to this local rule, a height mapping can be constructed as follows.

First, we note that there are two sites per unit cell. This defines two sublattices. If one chooses the x and y directions such that the diagonal bonds the crossed plaquettes are along x or y, then one can choose the two sublattices 1 and 2 such that $\sigma_1(\vec{r})$ is between two neighboring empty plaquettes along x and $\sigma_2(\vec{r})$ between two neighboring empty plaquettes along y. One then defines *local* height variables $z(\vec{r})$ at the center of the empty plaquettes to which the spins are related by:

$$\begin{cases}\sigma^1(\vec{r}) = (-1)^{x+y}\Delta_x z\\ \sigma^2(\vec{r}) = (-1)^{x+y}\Delta_y z\end{cases} \tag{4.19}$$

where x and y are the integer coordinates of the centers of the empty plaquettes, $\Delta_x z \equiv z(x+1,y) - z(x,y)$ and $\Delta_y z \equiv z(x,y+1) - z(x,y)$. This can be visualized as follows (see Fig. 4.4): the spins are given by the height difference between neighboring plaquettes affected by a sign that depends on whether the bond linking the height is consistent (+) or inconsistant (−) with the arrows in the crossed plaquettes, which alternate between clockwise and counterclockwise.

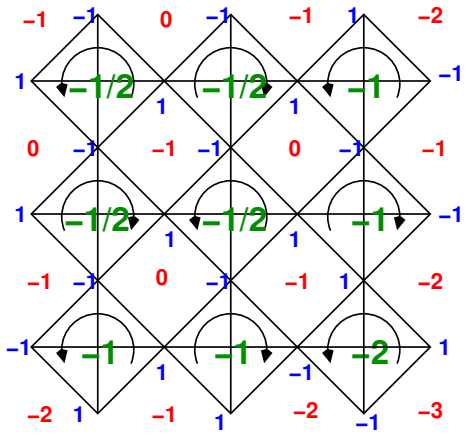

Figure 4.4: Example of height configurations for the Ising model on the checkerboard lattice. The blue numbers correspond to the spins, the red numbers to the local height variables $z(\vec{r})$, the green ones to the coarse-grained variables $h(\vec{r})$. The Néel order is realized on the upper left four crossed plaquettes, and the surface is locally flat, with a constant height variable.

As for the Ising model, there is a special configuration that will lead to a flat surface. This is the Néel state of the underlying square lattice (see Fig.4.4). In this state, one can flip the spins around every empty plaquette, making it the maximally flippable configuration. In this configuration, the local height varialbles alernate between 0 and 1. If one defines coarse-grained height variables by

$$h_{\boxtimes} = \frac{1}{4} \sum_{i(\boxtimes)} z(i),$$

where the sites $i(\boxtimes)$ are the centers of the empty plaquettes adjacent to a crossed plaquette, then the Néel state corresponds to a flat surface.

Turning to a field description, the spins are related to the height field by:

$$\sigma^i(\vec{r}) = (-1)^{x+y} \partial_{x_i} h + e^{i\pi h} + \dots \tag{4.20}$$

with $i =, 2$, $x_1 \equiv x$ and $x_2 \equiv y$. The first term comes from the fact that the spins are related to lattice differences of the local field, and the second term is the first non-zero Fourier component of with $G = 2\pi/2 = \pi$ since the spin variable is periodic in the height variable with period 2, as illustrated in Fig. 4.5.

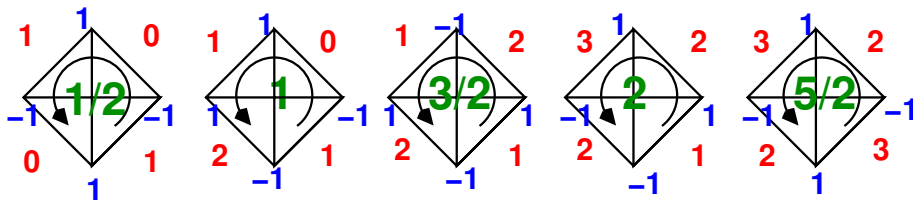

Figure 4.5: Illustration of the periodicity of the spin variables with the height for the checkerboard lattice.

The correlations coming from the first term take the form:

$$\left\langle \sigma^i(\vec{r})\sigma^j(\vec{0})\right\rangle = (-1)^{x+y}\partial_{x_i}\partial_{x_j}C(\vec{r}) \tag{4.21}$$

where $C(\vec{r}) = \frac{1}{2}\langle\left|h(\vec{r}) - h(\vec{0})\right|^2\rangle$ is the correlation function of the height field. Using the properties

$$\partial_x\partial_x \ln r = \frac{r^2 - 2x^2}{r^4}, \quad \partial_x\partial_y \ln r = -\frac{2xy}{r^4}, \tag{4.22}$$

this leads to

$$\left\langle \sigma^i(\vec{r})\sigma^j(\vec{0})\right\rangle = \frac{(-1)^{x+y}}{2\pi K}\partial_{x_i}\partial_{x_j}\ln\left(\frac{\pi r}{\alpha}\right) = \frac{(-1)^{x+y}}{2\pi K}\frac{r^2\delta_{ij} - 2x_ix_j}{r^4} \tag{4.23}$$

These correlations have a dipolar form.

The next term in the expansion of the spin operators gives a power law, as for the Ising model on the triangular lattice, with a decay $r^{-\pi/2K}$ since the first Fourier component is given by $G = \pi$. As for the triangular Ising model, the value of K must be extracted by comparing with other approaches. The exact solution of the six-vertex model, to which this model is equivalent, shows that $K = \pi/4$, leading to a decay $1/r^2$.

Structure factor Experimentally, one can use neutron diffraction to measure the *structure factor*, the Fourier transform of these correlations. In the present case, with two sites per unit cell, it is defined by

$$S(\vec{q}) = \sum_{i,j}\langle\sigma^i(\vec{q})\sigma^j(-\vec{q})\rangle$$

The dipolar contribution gives rise to a very peculiar pattern in Fourier space that is most simply derived by going back to the definition of the spin

operators in terms of height. Indeed,

$$\sigma^i(\vec{r}) = (-1)^{x+y}\partial_{x_i} h = e^{i\vec{Q}.\vec{r}}(-i)\sum_{\vec{q}} q_i e^{-i\vec{q}.\vec{r}} h(\vec{q}) \tag{4.24}$$

leading to

$$\sigma^i(\vec{q}) = (-i)(\vec{q}+\vec{Q})_i h(\vec{q}+\vec{Q}), \quad \sigma^j(-\vec{q}) = (-i)(-\vec{q}+\vec{Q})_j h(-\vec{q}+\vec{Q}) \tag{4.25}$$

where $\vec{Q}$ is defined by $e^{i\vec{Q}.\vec{r}} = (-1)^{x+y}$. It is the corner of the first Brillouin zone, and $2\vec{Q}$ is a vector of the reciprocal lattice, so $\sigma^j(-\vec{q})$ can be written

$$\sigma^j(-\vec{q}) = (-i)(-\vec{q}-\vec{Q})_j h(-\vec{q}-\vec{Q}) \tag{4.26}$$

leading to

$$\langle \sigma^i(\vec{q})\sigma^j(-\vec{q})\rangle = -(\vec{q}+\vec{Q})_i((\vec{q}+\vec{Q})_j\langle h(\vec{q}+\vec{Q})h(-\vec{q}-\vec{Q})\rangle$$

If we measure $\vec{q}$ from the Brillouin zone corner by defining the relative wavec vector $\vec{k}$ by $\vec{q} = \vec{Q} + \vec{k}$, we get:

$$\langle \sigma^i(\vec{k})\sigma^j(-\vec{k})\rangle = -\frac{1}{K}\frac{k_i k_j}{k^2}$$

These correlations vanish and change sign each time we cross the x or y axis, giving rise to a characteristic pattern known as *pinch points*.

The other contribution to the correlation function, which decays as $1/r^2$ in real space, gives rise to a logarithmic divergence at the zone center.

4.3 Mapping of 3D Ising models on gauge theories

In 3D, a local constraint leads to similar physics. However, the effective model is not a height model, but a gauge theory. To understand why, it is convenient to start by discussing mappings of dimer models.

4.3.1 Dimers on square and cubic lattices

Let us define vectors $\vec{n}(\vec{x})$ in dimension $D = 2$ or 3 of components $n_i(\vec{x})$, $i = 1, ..., D$ by

$$n_i(\vec{x}) = \begin{cases} 1 & \text{if bond between } \vec{x} \text{ and } \vec{x}+\hat{e}_i \text{ occupied by a dimer,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

This will define a dimer covering if the following closed pack, hard core constraint is satisfied:

$$\sum_i [n_i(\vec{x}) + n_i(\vec{x} - \hat{e}_i)] = 1 \tag{4.28}$$

where $\{\hat{e}_i\}$ are the vectors defining the lattice.

Let us also take advantage of the fact that the square and cubic lattices are bipartite and define the sign of site $\vec{x}$ by $\sigma_{\vec{x}} = +1$ on one sublattice and $\sigma_{\vec{x}} = -1$ on the other sublattice.

Next, let us define a field variable on the links by:

$$B_i(\vec{x}) = \sigma_{\vec{x}} \left[n_i(\vec{x}) - \frac{1}{z} \right] \tag{4.29}$$

where z is the coordinance of the lattice. The lattice divergence of this field satisfies:

$$\begin{aligned} &\sum_i [B_i(\vec{x}) - B_i(\vec{x} - \hat{e}_i)] \\ =&\sum_i \left(\sigma_{\vec{x}} \left[n_i(\vec{x}) - \frac{1}{z} \right] - \sigma_{\vec{x}-\hat{e}_i} \left[n_i(\vec{x} - \hat{e}_i) - \frac{1}{z} \right] \right) \\ =&\sigma_{\vec{x}} \sum_i \left[n_i(\vec{x}) + n_i(\vec{x} - \hat{e}_i) - \frac{2}{z} \right] \\ =&\sigma_{\vec{x}} \left(\sum_i [n_i(\vec{x}) + n_i(\vec{x} - \hat{e}_i)] - 1 \right) = 0 \end{aligned} \tag{4.30}$$

It is thus possible to define a field whose lattice divergence vanishes identically, hence the notation $B_i(\vec{x})$ by analogy with the magnetic field.

Solution in 2D: height field In 2D, and or a field defined in the continuum, the solution of the equation $\mathrm{div}\vec{B} = 0$ is given by $\vec{B} = \vec{\nabla} \times h$, where h is the scalar field, or, more explicitly:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times h = \begin{pmatrix} \frac{\partial h}{\partial y} \\ -\frac{\partial h}{\partial x} \end{pmatrix} \tag{4.31}$$

By analogy, the solution in the discrete case is given by

$$\vec{B}(\vec{x}) = \begin{pmatrix} B_1(\vec{x}) \\ B_2(\vec{x}) \end{pmatrix} = \vec{\nabla} \times h = \begin{pmatrix} h\left(\vec{x} + \frac{\hat{e}_1}{2} + \frac{\hat{e}_2}{2}\right) - h\left(\vec{x} + \frac{\hat{e}_1}{2} - \frac{\hat{e}_2}{2}\right) \\ -h\left(\vec{x} + \frac{\hat{e}_1}{2} + \frac{\hat{e}_2}{2}\right) + h\left(\vec{x} - \frac{\hat{e}_1}{2} + \frac{\hat{e}_2}{2}\right) \end{pmatrix} \tag{4.32}$$

This defines a height field on the dual lattice $h\left(\vec{x} \pm \frac{\hat{e}_1}{2} \pm \frac{\hat{e}_2}{2}\right)$.

Solution in 3D: gauge field In 3D, the solution of the equation $\mathrm{div}\vec{B} = 0$ is given by

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{pmatrix} \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \\ \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \\ \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \end{pmatrix} \tag{4.33}$$

In the discrete case, the solution is defined on the dual lattice (the centers of the cube), and the components of the $\vec{B}$ field are related to those of the $\vec{A}$ field by

$$\begin{aligned} B_1(\vec{x}) =& A_2\left(\vec{x} + \frac{1}{2}\sum_i \hat{e}_i\right) - A_2\left(\vec{x} + \frac{1}{2}\sum_i \hat{e}_i - \hat{e}_3\right) \\ &- \left[A_3\left(\vec{x} + \frac{1}{2}\sum_i \hat{e}_i\right) - A_3\left(\vec{x} + \frac{1}{2}\sum_i \hat{e}_i - \hat{e}_2\right)\right] \end{aligned} \tag{4.34}$$

and similar relations for B_2 and B_3. Unlike the height field in 2D, the gauge field $\vec{A}$ is not uniquely defined

4.3.2 Antiferromagnetic Ising model on the pyrochlore lattice

Let us consider Ising spins on the pyrochlore with antiferromagnetic coupling. In the ground state

$$\sum_{\boxtimes} \sigma_i = 0 \tag{4.35}$$

The dual lattice of the pyrochlore lattice is the diamond lattice, a bipartite lattice. Let us divide it into two sublattices A and B and on each bond κ let us define a unit vector $\hat{e}_\kappa$ from A to B.

Next, let us define a field on the sites of the original lattice, hence on the bond of the diamond lattice, by:

$$\vec{B}_\kappa = \sigma_\kappa \hat{e}_\kappa \tag{4.36}$$

The integral of the divergence around a site of the diamond lattice is

$$\sum_{\kappa(\vec{x})} \vec{B}_\kappa \hat{e}_\kappa = \sum_{\kappa(\vec{x})} \sigma_\kappa = 0 \tag{4.37}$$

Hence, one can look in the continuum for a field with zero divergence.

Upon coarse graining, the configurations with small coarse grained field $\vec{B}$ are favoured. Indeed, to go from one ground state to the other one must flip the spins along aloop of alternating spins to fulfill the constraint

$\sum_{\boxtimes} \sigma_i = 0$. Along such loops, the sum of the field $\vec{B}_\kappa = \vec{0}$. So, configurations with a very small $\vec{B}$ will have small loops of $\vec{B}_\kappa$ and will be very flippable.

So, one can postulate a weight

$$S(\vec{B}(\vec{x})) = \exp\left[-\frac{K}{2}\int \mathrm{d}^3\vec{r}\vec{B}(\vec{r})^2\right] \tag{4.38}$$

Since $\mathrm{div}\vec{B} = 0$, one can choose a vector potential $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$. With the gauge $\mathrm{div}\vec{A} = 0$, $\|\vec{q}\times\vec{A}(\vec{q})\|^2 = q^2\sum_i A_i(\vec{q})^2$, leading to:

$$\left\langle |A_i(\vec{q})|^2\right\rangle = \frac{1}{Kq^2} \tag{4.39}$$

$$\rightarrow \left\langle A_i(\vec{r})A_j(\vec{0})\right\rangle = \frac{\delta_{ij}}{4\pi K r} \tag{4.40}$$

$$\rightarrow \left\langle B_i(\vec{r})B_j(\vec{0})\right\rangle = \frac{1}{4\pi K}\frac{3x_ix_j-\delta_{ij}r^2}{r^5} \tag{4.41}$$

$$\rightarrow \left\langle \sigma_\alpha(\vec{r})\sigma_\beta(\vec{0})\right\rangle = \frac{1}{4\pi K}\frac{3(\hat{e}_\alpha.\vec{r})(\hat{e}_\beta.\vec{r})-(\hat{e}_\alpha.\hat{e}_\beta)r^2}{r^5} \tag{4.42}$$

where α,β refer to the position of the spin in the unit cell. So the correlations are again of dipolar form. Note that only gauge invariant quantities are physical. So, unlike in the case of the height field in 2D, which is unique up to a constant, and which gives rise to additional contributions to spin operators, the spins are simply related to the $\vec{B}$ field by $\sigma_\kappa = \vec{B}_\kappa.\hat{e}_\kappa$.

In Fourier space, $\vec{B}(\vec{q}) = -i\vec{q}\times\vec{A}(\vec{q})$, leading to

$$\langle B_i(\vec{q})B_j(-\vec{q})\rangle = \frac{1}{K}(\delta_{ij}-\frac{q_iq_j}{q^2})$$

As for the checkerboard model in 2D, we expect to find pinch points in neutron scattering experiments.