Chapter 9

Algebraic spin liquids

The possibility for a quantum magnet to support algebraic correlations, a situation intermediate between long-range order and exponentially decaying correlations, is well documented for half-integer spins in 1D. This can be most easily demonstrated in the case of the spin-1/2 XY chain by mapping the problem onto free fermions using the Jordan-Wigner transformation. The extension to 2D requires another mapping of spin operators onto fermions known as Abrikosov fermions, and the resulting problem can be solved with mean-field theory, leading in some cases to a Dirac spectrum with algebraic correlations.

9.1 The spin-1/2 XY chain

In this section, we consider the simplest example of a quantum magnet for which correlations can be proven to be algebraic, the spin-1/2 XY chain described by the Hamiltonian

$$H = J\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) = \frac{J}{2}\sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+).$$

This Hamiltonian can be diagonalized exactly by mapping it onto free fermions.

9.1.1 Jordan-Wigner transformation

In 1D, spin-1/2 operators can be represented in terms of spinless fermions with a non-local transformation known as the Jordan-Wigner transformation and defined by:

$$\left\{ \begin{array}{l} S_i^+ = c_i^\dagger \exp(i\pi \sum_{j<i} c_j^\dagger c_j) = c_i^\dagger (-1)^{\sum_{j<i} n_j} \\ S_i^- = \exp(-i\pi \sum_{j<i} c_j^\dagger c_j)\ c_i = (-1)^{\sum_{j<i} n_j} c_i \\ S_i^z = c_i^\dagger c_i - \frac{1}{2} \end{array} \right. \tag{9.1}$$

with $n_j \equiv c_j^\dagger c_j$.

It is easy to check that these operators satisfy the SU(2) algebra. Indeed, since $c_i^\dagger$ and c_i commute with $(-1)^{\sum_{j<i} n_j}$, one can write

$$[S_i^+, S_i^-] = c_i^\dagger c_i - c_i c_i^\dagger = c_i^\dagger c_i - (1 - c_i^\dagger c_i) = 2c_i^\dagger c_i - 1 = 2S_i^z$$

and

$$[S_i^z, S_i^+] = [c_i^\dagger c_i, c_i^\dagger (-1)^{\sum_{j<i} n_j}] = [c_i^\dagger c_i, c_i^\dagger](-1)^{\sum_{j<i} n_j} = c_i^\dagger (-1)^{\sum_{j<i} n_j} = S_i^+.$$

Now, although the relation between spins and fermions is non local, the Hamiltonian keeps a local form in fermionic language. This is due to the remarkable property:

$$S_i^+ S_{i+1}^- = c_i^\dagger c_{i+1}.$$

Indeed, using the identity $(-1)^{n_i} = 1 - 2n_i$ for a fermion operator, one gets

$$S_i^+ S_{i+1}^- = c_i^\dagger (-1)^{n_i} c_{i+1} = c_i^\dagger (1 - 2n_i) c_{i+1} = c_i^\dagger c_{i+1}.$$

The Hamiltonian in fermionic language is thus just a simple tight-binding model:

$$H = \frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + \text{H.c.}).$$

It is readily diagonalized by a Fourier transform:

$$H = \sum_k J \cos k\ c_k^\dagger c_k.$$

The ground state is defined by

$$\langle c_k^\dagger c_k \rangle = \left\{ \begin{array}{l} 1 \text{ for } J\cos k < 0 \\ 0 \text{ for } J\cos k > 0 \end{array} \right.$$

If $J < 0$, the ground state is thus given by a half-filled Fermi sea:

$$|\psi_o\rangle = \prod_{-\pi/2 \le k \le \pi/2} c_k^\dagger\ |0\rangle.$$

For $J > 0$, the ground state would be obtained by filling the other states. But the physical properties are the same as for $J < 0$ since the two cases are related by a canonical transformation, namely a rotation by π around z of every other spin. So we will continue assuming $J < 0$.

9.1.2 Spin-spin correlations

The calculation of the spin-spin correlation function is very simple for the z component because the correspondence between spin and fermionic operators is purely local. So let's concentrate on the correlation function

$$\langle S_i^z S_j^z \rangle = \langle (n_i - \frac{1}{2})(n_j - \frac{1}{2}) \rangle = \langle n_i n_j \rangle - \frac{1}{2}\langle n_i \rangle - \frac{1}{2}\langle n_j \rangle + \frac{1}{4},$$

where the expectation values are calculated in the ground state.

In Fourier space, the operator n_i is given by

$$n_i = \frac{1}{L} \sum_{k_1,k_2} e^{i(k_1-k_2)i} c_{k_1}^\dagger c_{k_2},$$

where L is the number of sites, and its ground state expectation value is given by

$$\langle n_i \rangle = \frac{1}{L} \sum_{k_1,k_2} e^{i(k_1-k_2)i} \langle c_{k_1}^\dagger c_{k_2} \rangle.$$

Now, $\langle c_{k_1}^\dagger c_{k_2} \rangle = \langle \psi_0 | c_{k_1}^\dagger c_{k_2} | \psi_0 \rangle = 0$ except if $k_1 = k_2$ and k_2 is occupied. So,

$$\langle n_i \rangle = \frac{1}{L} \sum_{k_2 \text{ occ.}} 1 = \frac{N}{L} = \frac{1}{2}$$

because N, the number of fermions in the ground state, has to be equal to $L/2$ since the system is half-filled. The result that $\langle n_i \rangle = 1/2$ is of course expected since the system is translationally invariant, so that $\langle n_i \rangle$ is independent of i, and since $N = \langle \sum_i n_i \rangle = L/2$.

Let's turn to the two-point correlation function. In Fourier space, the expectation value is given by

$$\langle n_i n_j \rangle = \frac{1}{L^2} \sum_{k_1,k_2,k_3,k_4} e^{i(k_1-k_2)i} e^{i(k_3-k_4)j} \langle c_{k_1}^\dagger c_{k_2} c_{k_3}^\dagger c_{k_4} \rangle.$$

In the ground state, $\langle c_{k_1}^\dagger c_{k_2} c_{k_3}^\dagger c_{k_4} \rangle = 0$ except if ($k_1 = k_2$ and $k_3 = k_4$), or ($k_1 = k_4$ and $k_2 = k_3$).

First case: $k_1 = k_2$ **and** $k_3 = k_4$

The contribution of this case is given by

$$\frac{1}{L^2} \sum_{k_2,k_4 \text{ occ.}} 1 = \frac{N^2}{L^2} = \frac{1}{4}.$$

Second case: $k_1 = k_4$ and $k_2 = k_3$

The contribution of this case is given by

$$\frac{1}{L^2}\sum_{k_4\ \mathrm{occ.}}\sum_{k_3\ \mathrm{empty}} e^{i(k_4-k_3)(i-j)} = \frac{1}{L}\sum_{k_4\ \mathrm{occ.}} e^{ik_4 r} \times \frac{1}{L}\sum_{k_3\ \mathrm{empty}} e^{-ik_3 r}.$$

where we have put $i - j = r$. The first factor is given by

$$\frac{1}{L}\sum_{k\ \mathrm{occ.}} e^{ikr} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{ikr} dr = \frac{1}{\pi r}\sin\frac{\pi}{2}r,$$

and the second one by

$$\frac{1}{L}\sum_{k\ \mathrm{empty}} e^{-ikr} = \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} e^{-ikr} dr + \int_{\pi/2}^{\pi} e^{-ikr} dr\right) = \frac{1}{\pi r}[\sin\pi r - \sin\frac{\pi}{2}r].$$

Putting all contributions together, the correlation function is given by

$$\langle S_i^z S_{i+r}^z\rangle = \frac{1}{\pi^2}\frac{1}{r^2}\sin\frac{\pi}{2}[\sin\pi r - \sin\frac{\pi}{2}r].$$

Since r is integer, $\sin\pi r = 0$, and this simplifies to

$$\langle S_i^z S_{i+r}^z\rangle = -\frac{1}{\pi^2}\frac{1}{r^2}\sin^2\frac{\pi}{2}r,$$

leading to

$$\langle S_i^z S_{i+r}^z\rangle = \begin{cases} -\frac{1}{\pi^2}\frac{1}{r^2} \text{ if } r \text{ odd} \\ \\ 0 \text{ if } r \text{ even} \end{cases}$$

So the longitudinal correlation function decays as $1/r^2$. The transverse correlations require a more sophisticated approach because of the string that appears between the fermionic operators due to the non-local correspondence between S_i^+ and S_i^- and the fermionic operators. This can for instance be achieved with bosonization, with the result that the transverse correlation function $\langle S_i^+ S_{i+r}^-\rangle$ decays as $1/r^{1/2}$. For the antiferromagnetic Heisenberg model, both correlations decay as $1/r$.

9.2 Algebraic spin liquids in 2D

As shown by the example of the XY chain, the fermionic representation of spins is ver well adapted to the description of algebraic correlations. However, the Jordan-Wigner transformation does not lead to free fermions in higher dimension. Indeed, it is not possible any more for all pairs of nearest-neighbour sites to correspond to consecutive integers in a numbering of the sites from 1 to N, and several of the spin-spin interactions between nearest neighbour would appear as a fermionic hopping with a phase factor that depends on the number of fermions along some path. So to describe spins as fermions in dimension larger than 1, it is more convenient to use another representation.

9.2.1 Abrikosov fermions

This representation, which goes under the name of Abrikosov fermions, relies on two types of fermions with an internal spin degree of freedom ↑ and ↓ according to:

$$\begin{cases} S_i^+ = c_{i\uparrow}^\dagger c_{i\downarrow} \\ S_i^- = c_{i\downarrow}^\dagger c_{i\uparrow} \\ S_i^z = \frac{1}{2}\left(n_{i\uparrow} - n_{i\downarrow}\right) \end{cases}$$

with the constraint $n_{i\uparrow} + n_{i\downarrow} = 1$. Without the constraint, these fermionic operators would simply represent a spin-1/2 particle.

This representation of the spin algebra leads to the following fermionic form of the $S = \frac{1}{2}$ Heisenberg Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} J_{ij}\left[\frac{1}{2}\left(c_{i\uparrow}^\dagger c_{i\downarrow} c_{j\downarrow}^\dagger c_{j\uparrow} + \text{h.c.}\right) + \frac{1}{4}\left(c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}\right)\left(c_{j\uparrow}^\dagger c_{j\uparrow} - c_{j\downarrow}^\dagger c_{j\downarrow}\right)\right]$$

All terms are quartic in fermionic opertors. It is thus impossible to diagonalize the Hamiltonian with the standard approach for quadratic Hamiltonians (Fourier transform and, if necessary, Bogoliubov transformation). To get a tractable model, one has to perfrom a mean-field decoupling of the quartic terms using quadratic operators. In the present context, a very useful quadratic operator is given by:

$$\chi_{ij} = c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}$$

Indeed, in terms of this operator, the scalar product of two spin operators can be written as:

$$\vec{S}_i \cdot \vec{S}_j = \frac{1}{4} - \frac{1}{2}\chi_{ij}^\dagger \chi_{ij}$$

Proof

$$\begin{aligned}
\chi_{ij}^\dagger \chi_{ij} =& \left(c_{j\uparrow}^\dagger c_{i\uparrow} + c_{j\downarrow}^\dagger c_{i\downarrow}\right)\left(c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}\right) \\
=& c_{j\uparrow}^\dagger c_{i\uparrow} c_{i\uparrow}^\dagger c_{j\uparrow} + c_{j\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{j\downarrow} + c_{j\downarrow}^\dagger c_{i\downarrow} c_{i\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{i\downarrow} c_{i\downarrow}^\dagger c_{j\downarrow} \\
=& c_{j\uparrow}^\dagger c_{j\uparrow} - n_{i\uparrow} n_{j\uparrow} - S_j^+ S_i^- - S_j^- S_i^+ + c_{j\downarrow}^\dagger c_{j\downarrow} - n_{i\downarrow} n_{j\downarrow}.
\end{aligned}$$

Using the identity

$$S_i^z S_j^z + \frac{1}{4} n_i n_j = \frac{1}{2}\left(n_{i\uparrow} n_{j\uparrow} + n_{i\downarrow} n_{j\downarrow}\right)$$

and the fact that $n_i \equiv n_{i\uparrow} + n_{i\downarrow} = 1$, one gets

$$\chi_{ij}^\dagger \chi_{ij} = n_j - 2 S_i^z S_j^z - \frac{1}{2} n_i n_j - S_j^+ S_i^- - S_j^- S_i^+ = -2 \vec{S}_i \cdot \vec{S}_j + \frac{1}{2}$$

So, up to a constant, and for a model with nearest-neighbor interactions of magnitude J, the Hamiltonian is given by

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \chi_{ij}^\dagger \chi_{ij}.$$

If we define the order parameter of the mean-field decoupling by

$$\chi_{ij}^0 = \langle \chi_{ij} \rangle$$

the mean-field decoupling corresponds to the approximation:

$$\chi_{ij}^\dagger \; \chi_{ij} \simeq \chi_{ij}^0 \; \chi_{ij}^\dagger + \chi_{ij}^{0*} \; \chi_{ij} - |\chi_{ij}^0|^2.$$

As usual, the term that has been left out is the product of the fluctuations: $(\chi_{ij} - \chi_{ij}^0)(\chi_{ij}^\dagger - \chi_{ij}^{0*})$.

In the next two sections, we discuss two mean-field solutions for the square lattice antiferromagnet with nearest-neighbour interactions.

9.2.2 Dimerized solution

Let us suppose that $\chi_{ij}^0 = \chi^0$ real on a dimer covering of the square lattice, and 0 otherwise. For a bond that belongs to the dimer covering, the mean-field Hamiltonian is given, up to a constant, by:

$$H_{\langle ij \rangle} = -\frac{J\chi^0}{2} \sum_\sigma \left(c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma}\right).$$

With the help of the operators

$$\begin{cases} a_\sigma^\dagger = \frac{1}{\sqrt{2}}\left(c_{i\sigma}^\dagger + c_{j\sigma}^\dagger\right) & a_\sigma = \frac{1}{\sqrt{2}}\left(c_{i\sigma} + c_{j\sigma}\right) \\ b_\sigma^\dagger = \frac{1}{\sqrt{2}}\left(c_{i\sigma}^\dagger - c_{j\sigma}^\dagger\right) & b_\sigma = \frac{1}{\sqrt{2}}\left(c_{i\sigma} - c_{j\sigma}\right) \end{cases}$$

it takes the diagonal form

$$H = -\frac{J\chi^0}{2}\sum_\sigma \left(a_\sigma^\dagger a_\sigma - b_\sigma^\dagger b_\sigma\right)$$

In terms of these operators, the constraint is given by: $\sum_\sigma \left(a_\sigma^\dagger a_\sigma + b_\sigma^\dagger b_\sigma\right) = 2$.

Let's suppose that $\chi^0 > 0$. Then, in the ground state

$$\left\langle a_\uparrow^\dagger a_\uparrow\right\rangle = \left\langle a_\downarrow^\dagger a_\downarrow\right\rangle = 1 \quad \text{and} \quad \left\langle b_\uparrow^\dagger b_\uparrow\right\rangle = \left\langle b_\downarrow^\dagger b_\downarrow\right\rangle = 0,$$

and the order parameter is given by

$$\begin{aligned}\chi^0 = \langle \chi_{ij}\rangle = \left\langle c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}\right\rangle &= \left\langle \frac{1}{2}\left[\left(a_\uparrow^\dagger + b_\uparrow^\dagger\right)(a_\uparrow - b\uparrow) + \left(a_\downarrow^\dagger + b_\downarrow^\dagger\right)(a_\downarrow - b\downarrow)\right]\right\rangle \\ &= \frac{1}{2}\langle (a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow)\rangle = 1.\end{aligned}$$

This is consistent with the hypothesis $\chi^0 > 0$.

Besides, for pairs of sites that do not correspond to a dimer of the dimer covering, the expectation value vanishes because each term of the Hamiltonian changes the fermionic occupancy of the two dimer bonds to which the sites belong.

So, this solution is self-consistent. It is thus a valid mean-field solution. It is highly degenerate because all dimer coverings lead to the same energy since the total energy is just the sum of the bond energies. This solution is not the lowest energy solution for the simple square lattice antiferromagnet, but a solution of this type can be stabilized for frustrated models, as suggested for the intermediate phase of the $J_1 - J_2$ model. This solution is also competitive when the spin operators are replaced by $SU(N)$ operators with the fundamental representation on one sublattice and the anti-fundamental representation on the other sublattice in the large N limit. Note also that, when going beyond mean-field, one generates an effective Hamiltonian in the subspace of the dimer coverings that takes the form of a Quantum Dimer Model.

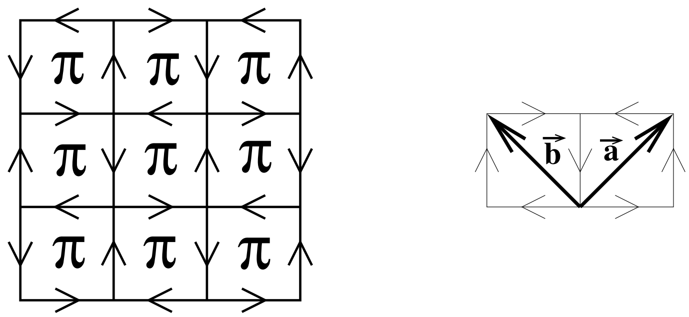

Figure 9.1: π-flux state. Each hopping term has a phase $\pi/4$ in the direction of the arrow of the left panel. The vectors used in the solution of the resulting tight-binding problem are defined in the right panel.

9.2.3 Flux phase

These fermionic mean-field theories are particularly attractive to describe exotic, gapless ground states. One of the first cases ever discussed is the π-flux state, a mean-field solution investigated by Affleck and Marston. It corresponds to the choice

$$\left|\chi_{ij}^0\right| = \chi_0 \text{ on all bonds,} \tag{9.2}$$

$$\theta_{ij} = \frac{\pi}{4} \text{ according to the arrows on Fig. 9.1} \tag{9.3}$$

If one interprets the phase as the circulation of a vector potential, the flux through each plaquette is equal to π, hence the name π-flux state.

The kinetic term of the mean field Hamiltonian corresponds to a tight-binding problem with two sites per unit cell. If we denote by $\vec{a}$ and $\vec{b}$ the unit vectors according to the right panel of Fig. 9.1, and by k_1 and k_2 the components of the wave-vector in reciprocal space

$$\vec{k} = k_1\vec{a}^\star + k_2\vec{b}^\star, \tag{9.4}$$

the eigenvalue problem takes the simple form

$$\begin{vmatrix} -E & \frac{J\chi_0 z}{2} \\ \frac{J\chi_0 z^*}{2} & -E \end{vmatrix} = 0$$

with

$$z = e^{i\pi/4}(1 + e^{-ik_1+ik_2}) + e^{-i\pi/4}(e^{-ik_1} + e^{ik_2})$$

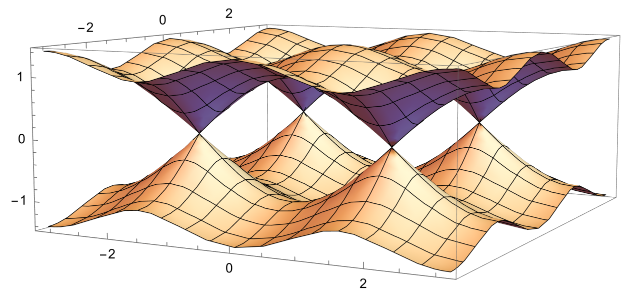

Figure 9.2: Dirac spectrum of the π-flux state.

The spectrum is given by $E^2 = \left(\frac{J\chi_0}{2}\right)^2 |z|^2$ with $|z|^2 = 4(\cos^2\frac{k_1+k_2}{2} + \cos^2\frac{k_1-k_2}{2})$. Defining $k_x = \frac{k_1+k_2}{2}$ and $k_y = \frac{k_1-k_2}{2}$, the spectrum takes the form

$$E = \pm J\chi_0\sqrt{\cos^2 k_x + \cos^2 k_y}$$

This spectrum has a very peculiar property. Since we are at half-filling, the Fermi energy $E_F = 0$. The Fermi surface thus reduces to 4 points, $k_x, k_y = \pm\pi/2$. This is a Dirac spectrum, qualitatively similar to that of graphene (see Fig. 9.2).

In such a ground state, correlations are algebraic. Furthermore, it has been argued that, because of the Dirac form of the spectrum, as opposed to an extended Fermi surface, such a mean-field solution is stable against fluctuations beyond mean field.

For the antiferromagnetic Heisenberg model on the square lattice, this solution is definitely not the ground state. Indeed, numerical simulations have shown beyond reasonable doubt that the ground state has Néel order. However a similar solution has been proposed for the kagome antiferromagnet with a flux π per hexagonal plaquette. Such an algebraic spin liquid ground state has been supported by some numerical simulations.

9.2.4 Related developments

In the mean field approach, the constraint $n_i = 1$ is treated on average. This constraint can be treated more precisely using Gutzwiller's projection. A technique known as variationnal Monte Carlo has been devised to calculate the expectation value of operators including a Gutzwiller projection:

$$\prod_i (1 - n_{i\uparrow}n_{i\downarrow}) \tag{9.5}$$

If one wants to compare the energy of different solutions to test the stability of a given solution, it is certainly better to perform such a projection. However, if one starts playing this game, one should use the decoupling parameters as free parameters in the context of a variational approach. This opens the way to more general decouplings including pairing operators. This line has been developed by Sorella and collaborators.

In that context, it is useful to note that, when formulated in terms of fermionic operators, the Hamiltonian has a local gauge invariance:

$$\begin{pmatrix} c_{i\uparrow}^\dagger \\ c_{i\downarrow} \end{pmatrix} \rightarrow e^{i\theta\vec{\sigma}\cdot\vec{n}} \begin{pmatrix} c_{i\uparrow}^\dagger \\ c_{i\downarrow} \end{pmatrix} \tag{9.6}$$

where $\vec{\sigma} =^t (\sigma_x, \sigma_y, \sigma_z)$. A simple case is:

$$\begin{cases} c_{i\uparrow}^\dagger \rightarrow c_{i\downarrow} \\ c_{i\downarrow} \rightarrow c_{i\uparrow}^\dagger \end{cases} \tag{9.7}$$

This implies that different mean-field theories can correspond to the same state. This is the case of the π-flux state, which is equivalent from the point of view of spins with a pairing instability with d-wave symmetry.

Finally, mean-field theories with fractional fluxes that break time-reversal symmetry have been proposed. These solutions are spin-liquid analogs to the Fractional Quantum Hall Effect. They possess non-trivial topological properties such as chiral edge-states and are often referred to as chiral spin liquids.