Chapter 8

Quantum dimer model

In this chapter, we investigate the properties of a minimal model devised to describe resonances between dimer singlet coverings. This model goes under the name of quantum dimer model (QDM). Its phase diagram depends on the lattice. On the triangular lattice, it possesses a resonating valence bond phase with a gap to all excitations, a ground state degeneracy that depends on the topology of the cluster, and non local elementary excitations. This last property is most simply derived from a closely related gauge theory that can be mapped onto a frustrated transverse field Ising model and solved using linear spin-wave theory.

8.1 The Rokhsar-Kivelson model

If the low energy singlet sector of a frustrated quantum model can be reasonably well described by products of singlets on nearest-neighbour bonds, the effective Hamiltonian must contain kinetic processes that exchange the position of some dimers and potential energy terms that favour or penalise some local arrangements of dimers. The simplest model on a square lattice can be written as:

$$H_{\mathrm{QDM}} = -t \sum_{\square} \left(\left|\parallel\right\rangle \left\langle = \right| + \text{h.c.} \right) + v \sum_{\square} \left(\left|\parallel\right\rangle \left\langle \parallel \right| + \left| = \right\rangle \left\langle = \right| \right) \qquad (8.1)$$

8.1.1 The Rokhsar-Kivelson point

Rokhsar and Kivelson asked the question whether, for some values of $\frac{v}{t}$, such a model can sustain a resonating valence bond (RVB) phase. As a first step toward an answer to this question, they proved the following theorem:

Theorem

At the RK-point defined by $v = t > 0$, the sum of all configurations with equal weight

$$|\psi\rangle_{\mathrm{RK}} = \sum_c |c\rangle$$

is a ground state.

Proof

Let's write $H_{\mathrm{QDM}} = \sum_i H^i_{\mathrm{QDM}}$, where the sum over i runs over all plaquettes of the lattice, and let us evaluate the effect of H^i_{QDM} on a given configuration c. There are two cases:

1. c does not contain parallel dimers on plaquette i (plaquette i is not flippable in c). Then $H^i_{\mathrm{QDM}}|c\rangle = 0$.

2. c does contain parallel dimers. Then there is a companion configuration c' equal to c on all plaquettes except i, and with the other orientation of dimers. Wtih this definition, it follows that

$$\begin{aligned}
H^i_{QDM}|c\rangle &= -t|c'\rangle + v|c\rangle \\
H^i_{QDM}|c'\rangle &= -t|c\rangle + v|c'\rangle \\
\Rightarrow H^i_{QDM}\left(|c\rangle + |c'\rangle\right) &= (v-t)\left(|c'\rangle + |c\rangle\right).
\end{aligned}$$

   Therefore, if $v = t$, $H^i_{\mathrm{QDM}}(|c\rangle + |c'\rangle) = 0$. This implies that

$$H^i_{\mathrm{QDM}} \sum_c |c\rangle = 0, \quad \forall i$$

   hence that

$$H_{\mathrm{QDM}}|\psi\rangle_{\mathrm{RK}} = 0.$$

   So $|\psi\rangle_{\mathrm{RK}}$ is an eigenstate of H_{QDM} with eigenvalue 0.

Let us now prove that this is the ground state energy. For that purpose, let's enlarge the Hilbert space to allow for empty sites (sites not connected to a dimer). Let's define the operator $a_{\|}$ as the operator that destroys $\|$ on a given plaquette provided there are such dimers in a configuration $|c\rangle$, and gives 0 otherwise. The adjoint operator is denoted by $a^\dagger_{\|}$. Then the Hamiltonian

$$H = -t\sum_{\square}\left(a^\dagger_{\|}a_{=} + a^\dagger_{=}a_{\|}\right) + v\sum_{\square}\left(a^\dagger_{\|}a_{\|} + a^\dagger_{=}a_{=}\right) \tag{8.2}$$

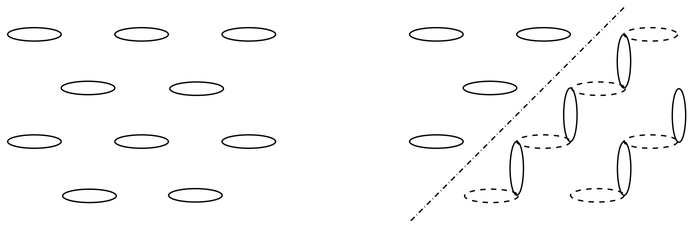

Figure 8.1: Examples of non flippable states on the square lattice. Left: Horizontal alternating columnar state. Right: A state with a diagonal domain wall between a horizontal alternating columnar domain and a vertical alternating columnar domain obtained by shifting dimers along diagonal rows. An arbitrary number of such domain walls can be created simultaneously.

is equal to H_{QDM} when restricted to the subspace where all sites belongs to a dimer. By the variational principle, this implies that the ground state energy of H_{QDM} must be greater than or equal to the ground state of H: $E_{\mathrm{GS}}(H_{\mathrm{QDM}}) \geq E_{\mathrm{GS}}(H)$. Now, for $v = t > 0$, one can write

$$H = v \sum_{\square} \left(a^\dagger_{\|} - a^\dagger_{=}\right)\left(a_{\|} - a_{=}\right) = v \sum_i A_i^\dagger A_i$$

with $A_i^\dagger = a^\dagger_{\|} - a^\dagger_{=}$, so that

$$\langle\psi|H|\psi\rangle = v \sum_i \left\langle \psi \middle| A_i^\dagger A_i \middle| \psi \right\rangle = v \sum_i \| |A_i\psi\rangle \|^2 \geq 0.$$

This implies that

$$E_{\mathrm{GS}}(H) \geq 0$$

and thus that

$$E_{\mathrm{GS}}(H_{\mathrm{QDM}}) \geq 0.$$

So indeed $|\psi\rangle_{\mathrm{RK}}$ is a ground state of H_{QDM} when $v = t > 0$.

8.1.2 The resonating valence-bond phase on the triangular lattice

Does this imply that the QDM on the square lattice at the RK point is an RVB liquid? Not quite for several reasons. First of all, there are other ground states. Indeed all configurations with no flippable plaquette are zero energy eigenstates. Examples of such states are shown in Fig.8.1. This is not a final blow though. For $v \neq t$, the non-flippable states will still have zero energy while the energy of the RK state will have a correction of the form $a(v-t)$, $a > 0$. So, for $v > t$, the non-flippable configurations can be expected to be the only ground states, but for $v < t$, the state emanating from $|\psi\rangle_{\mathrm{RK}}$ can be expected to be lower. However, this state will be a unique ground state for $v < t$ if and only if, apart from the non-flippable states, the spectrum is gapped at the RK point. Now, according to the general correspondence between the quantum gap and the correlation length, this will be the case if the correlations in the RK state decay exponentially. But, since the dimer counting operator is diagonal in the configuration basis, the dimer-dimer correlations in the RK ground state are exactly the same as the average of the dimer-dimer correlations over all dimer configurations already calculatedin Chapter 4. As we saw, the decay of these correlation is itself related to the behaviour of the Kasteleyn matrix: If all eigenvalues stay at a finite distance of 0, dimer-dimer correlations decay exponentially. Otherwise, they decay algebraically.

On the square lattice, we saw that the Kasteleyn matrix has no gap. This implies that the correlations decay algebraically. In that case, the perturbation can be expected to lead to some form of long-range dimer order, and the ground state state is not an RVB state.

By contrast, for the triangular lattice, the Kasteleyn matrix is gapped and the correlations decay exponentially. In that case, perturbation theory predicts that the ground state for $v < t$ is adiabatically connected to the RK state, at least down to a critical ratio, building an RVB phase, as first predicted by Moessner and Sondhi. The phase diagram of this model is shown in Fig.8.2

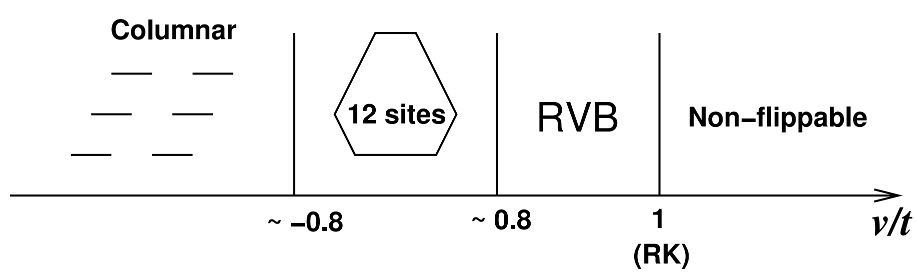

Figure 8.2: Schematic phase diagram of the Quantum Dimer Model on the triangular lattice

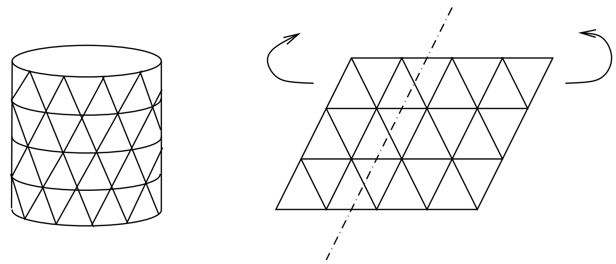

Figure 8.3: Quantum Dimer Model on the triangular lattice. Left: Cylindrical geometry. Right: Definition of the cut (unfolded representation of the cylinder).

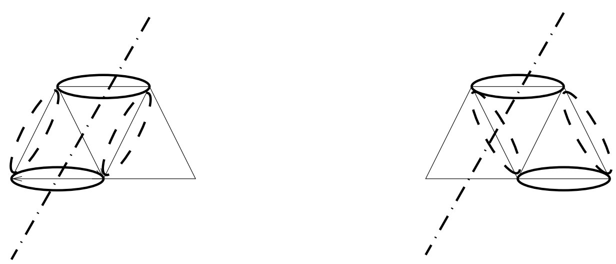

Figure 8.4: Illustration of the conservation of the parity of the number of dimers cut by a line under the Rokhsar-Kivelson plaquette kinetic term. The number of dimers remains equal to 2 on the left, and to 1 on the right.

Topological order in the RVB phase

Such a phase possesses some kind of topological order. Indeed, let's look at the triangular lattice QDM on a cylinder, and let's consider a cut parallel to the axis of the cylinder (see Fig. 8.3). Then, the parity of the number of dimers cut by the line is conserved (see Fig. 8.4). But both parities can be realized. Indeed, if the number of sites around the circumference is even, one can build a columnar state with a certain parity, and the state obtained by shifting one row has the other parity. And if the number of sites around the circumference is odd, one can still build a columnar state plus a column of tilted dimers, and shifting the top row including the top tilted dimer changes the parity. So there are two topological sectors. Since locally the configurations look the same in both sectors, one expects the ground state in the two sectors to be degenerate in the thermodynamic limit. This has been proven numerically.

Quite remarkably, the two ground states cannot be distinguished by any local operator. They thus constitute well-protected qubits. The only operators that can distinguish them are non local. For instance, the operator

$$\prod_{\text{cut}}(-1)^{n_{\text{dimer}}} \tag{8.3}$$

will be equal to +1 in one sector and −1 in the other sector.

Another remarkable property of this phase is that the topological degeneracy depends on the genus of the surface. On a torus for instance, one can keep track of the sign along two curs, and the degeneracy is equal to 4.

Elementary excitations in the RVB phase

As we shall see from the mapping onto a gauge theory, the elementary excitations are non local. From an eigenstate $\sum_c \alpha_c |c\rangle$, one can create elementary excitations as

$$|i\rangle = \sum_c \alpha_c (-1)^{\text{cut}(i)} |c\rangle\,, \quad \text{cut}(i) = \sum_{i\to\text{edge}} n_{\text{dimer}}.$$

These excitations are called“vortex” Ising excitations or “visons”.

8.2 Analytical approach to Qauntum Dimer Models

Beyond the RK point, it is difficult to derive any result analytically directly in this language. The only way is to write down a gauge model in terms of bond variables which is equivalent to the QDM in a certain limit, and which can be mapped onto an Ising model in a field that can be solved using a semicalssical approach.

8.2.1 Gauge theory

Gauge theories are formulated in terms of bond variables. In the present case, let us introduce a variable on each link of the triangular lattice:

$$\tau_l^x = \begin{cases} -1 & \text{if there is a dimer on link } l \\ 1 & \text{if there is no dimer on link } l \end{cases} \tag{8.4}$$

It is called τ_l^x with a superscript because it will soon be upgraded to a Pauli matrix. If we want to describe exacty the Hilbert space of the QDM, we have to impose the constraint that:

$$\sum_{l[i]} \tau_l^x = 4 \tag{8.5}$$

where $l[i]$ denotes the six links starting from a site i.

Naive approach

The kinetic term can be written

$$-t \sum_{\boxbslash} \tau_1^z \tau_2^z \tau_3^z \tau_4^z,$$

where $\tau_1^z, \tau_2^z, \tau_3^z, \tau_4^z$ are Pauli matrices on the four external bonds of $\boxbslash$, because $\tau_l^z \left|\tau_l^x = \pm 1\right\rangle = \left|\tau_l^x \mp 1\right\rangle$. Finally, the repulsion term take the form:

$$\frac{V}{4} \sum_{\boxbslash} \left((1-\tau_1^x)(1-\tau_3^x) + (1-\tau_2^x)(1-\tau_4^x)\right)$$

Unfortunately, there is no simple approach to study this model, and this is not a useful representation of the model.

Ising gauge theory

The only models of that sort that lead to a simple development are Ising gauge theories, with a Hamiltonian of the form:

$$H = -J \sum_l \tau_l^x - \Gamma \sum_{\triangle} \tau_1^z \tau_2^z \tau_3^z \tag{8.6}$$

where the kinetic terms acts on the links of elementary plaquettes, here triangular plaquettes.

Let us show a few things about this model:

- It is equivalent to the $v = 0$ QDM in the limit $\Gamma/J \to 0$.
- It can be mapped onto an Ising model in a transverse field.
- The Ising model can be solved with linear spin-wave theory.
- Elementary excitations in the RVB phase are non local in the gauge language (visons).

Equivalence between gauge theory and QDM

In the gauge model, the number of bonds emanating from a given site is not fixed. However, it can be in two steps. First, we note that the parity of the number of bonds is a conserved quantity. Indeed the operator

$$\prod_{l[i]} \tau_l^x \tag{8.7}$$

which is equal to -1 if the number of dimers is odd and $+1$ if it is even, commutes with H since the kinetic terms flip 0 or 2 dimers from any site of the lattice. Such local conservation laws are typical of gauge theories. This gives the freedom to choose sectors according to the value ± 1 at each site of this operator.

Let us choose the sector

$$\prod_{l[i]} \tau_l^x = -1 \quad \text{for all } i. \tag{8.8}$$

In the limit $\Gamma/J \to 0$, J dominates, and for $J > 0$ the corresponding term in H selects configurations in which $\sum_l \tau_l^x$ is maximal. In the odd sector, the number of dimers emanating form a given site i is equal to 1, 3 or or 5. The J term selects the physical configurations where this number is equal to 1, which leads to $\sum_{l[i]} \tau_l^x = 4$, the largest possible value (the other possible values are -4 and 0). Then, Γ can be treated as a perturbation. When $\Gamma = 0$, the ground state is degenerate. It corresponds to all dimer coverings of the triangular lattice. Switching on Γ will lead to an effective Hamiltonian.

Now, to first order, the effective Hamiltonian is zero. Indeed, applying Γ takes us away from the degenerate subspace. To second order, it will be zero unless the plaquettes touch. In that case, the common bond drops from the effective Hamiltonian since $\tau_l^z \tau_l^z = \mathbb{I}$, and the effective term is simply a kinetic term of order 2 in Γ/t that flips dimers around a plaquette. The resulting effective model is just the QDM with $v = 0$ and $t = \Gamma^2/J$.

Pushing the expansion to higher order induces repulsion terms of order Γ^4/J^3 plus additional terms. The mapping is only exact in the limit $\Gamma/J \to 0$, but this strongly suggests to study the model as a function of Γ/J, with the hope to find the equivalent of an RVB phase upon increasing Γ.

8.2.2 Ising model in a transverse field

Gauge theories of this type can be mapped onto the Ising model through a duality transform. If we denote by a the sites of the dual lattice (the honeycomb lattice), the duality transformation takes the form:

$$\begin{cases} \sigma^x(a) = \prod_{l(a)} \tau_l^z \\ \sigma^z(a) = \prod_{l<a} \tau_l^x \end{cases} \tag{8.9}$$

where $l(a)$ denotes the three links around site a and $l < a$ all links cutting a straight path (say, horizontal to the left) starting at site a (we implicitly assume a finite lattice with open boundary conditions). One can check that these variables satisfy the commutation relations of Pauli matrices. Indeed, if $a \neq a'$, $\sigma^x(a)$ and $\sigma^z(a')$ have 0 or 2 links in common, and they commute. And if $a = a'$, there is only one common link and they anticommute.

Now, with these definitions, the product of $\sigma^z(a)$ on neighbouring sites is proportional to τ_l^x on the common bond l. To see this, let's look at the two possible cases. If a is left of a',

$$\sigma^z(a)\sigma^z(a') = \prod_{l'<a} \underbrace{(\tau_{l'}^x)^2}_{=\mathbb{I}} \; \tau_l^x = \tau_l^x.$$

And if a is on top of a', and if we denote by $i < l$ the set of sites of the triangular lattice left of a horizontal bond l, including the left site of bond l, we can write

$$\sigma^z(a)\sigma^z(a') = \prod_{i<l} \underbrace{\prod_{l'[i]} \tau_{l'}^x}_{=-1} \; \tau_l^x = (\pm 1)\tau_l^x.$$

If we consider two such pairs a and a' with adjacent horizontal common bonds, the sign will be -1 on one of them and $+1$ on the other one because the number of factors $\prod_{l'[i]} \tau_{l'}^x$ changes by 1 in the expression of $\sigma^z(a)\sigma^z(a')$ for the two bonds. Since the sign is $+1$ on all non-horizontal bonds, this implies that if we consider all pairs around a hexagon of the honeycomb lattice, the sign will be -1 on one of the bonds and $+1$ on all the other bonds.

So, the gauge model is equivalent to the frustrated Ising model in a transverse field on the honeycomb lattice defined by the Hamiltonian:

$$H = -J \sum_{\langle a,b \rangle} M_{ab}\sigma^z(a)\sigma^z(b) - \Gamma \sum_a \sigma^x(a) \tag{8.10}$$

where M_{ab} is -1 on one bond of each hexagon and $+1$ on all the others. In fact, it does not matter which bond is equal to -1. All models where the number of negative M_{ab} is odd on each hexagon are related by canonical transformations $\sigma^z(a) \to \pm\sigma^z(a)$ (a rotation by π around x).

The advantage of this mapping is that the Ising model does not have a constraint any more. It is a magnetic model that can be studied by standard approaches such as linear spin-wave theory. In the limit $\Gamma/J \to +\infty$ the spins are oriented along x. This can be proven to remain true classically down to $\Gamma/J = \sqrt{6}$. Below that point, the system undergoes a phase transition to a broken symmetry phase with a large unit cell. The size of the unit cell depends on the choice of M_{ab}'s, but in the gauge theory language, this choice is immaterial, and the unit cell has 12 sites. This agrees with the phase diagram of the QDM on the triangular lattice.

In the large Γ/J limit, the ground state is some kind of RVB state. Indeed it is given by

$$\bigotimes_a |\sigma^x(a) = 1\rangle = \bigotimes_a \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_a + |\downarrow\rangle_a\right) \tag{8.11}$$

in the local basis of eigenvectors of $\sigma^z(a)$. Expanding the sum, this implies that

$$\bigotimes_a |\sigma^x(a) = 1\rangle \propto \sum_{\{\sigma_a\}} \bigotimes_a |\sigma_a\rangle, \quad \sigma_a = \uparrow \text{ or } \downarrow. \tag{8.12}$$

In other words, up to a normalization factor, the ground state is the sum of all configurations with equal weights, in close analogy to the Rokhsar-Kivelson ground state.

In this RVB phase, the excitations are spin-waves. They correspond to applying $\sigma^z(a)$ to the ground state (which flips $\sigma^x(a)$), and to let this excitation propagate under the kinetic term

$$-J \sum_{\langle a,b\rangle} \prod_{ab} \sigma^z(a)\sigma^z(b) \tag{8.13}$$

Now, if we define $\sigma^-(a) = \sigma^z(a) - i\sigma^y(a)$, the only bilinear term of the Hamiltonian is $\sigma^z(a)\sigma^z(b)$, which is equal to $\frac{1}{4}\left(\sigma^+(a) + \sigma^-(a)\right)\left(\sigma^+(b) + \sigma^-(b)\right)$. So, if one performs a Holstein-Primakoff transformation, one get $a_a^+ a_b^+$ and $a_a a_b$ terms, leading to a Hamiltonian that can be diagonalized by a Bogoliubov transformation. The dispersion is gapped for large Γ, and becomes soft at $\Gamma/J = \sqrt{6}$. The dispersion is very similar to that obtained numerically for "visons" in the QDM.

Finally, in the gauge language,

$$\sigma^z(a) = \prod_{l<a} \tau_l^x = \prod_{l<a} (-1)^{n_{\text{dimer}}(l)}. \tag{8.14}$$

When applied to the RVB ground state, it corresponds to a non local excitation analogous to the “vison” excitation of the QDM.