## II. QUANTUM FIELD THEORY, GREEN'S FUNCTION, AND PERTURBATION THEORY: SOME ADDITIONAL NOTES

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### A. Link cluster theorem

Expanding the exponential on the rhs we obtain the perturbation series in power of $V$

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \cdots \int_0^\beta d\tau_n \left\langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_n) \right] \right\rangle_0 \tag{1}$$

where $\langle \ldots \rangle_0$ is the average corresponding to the reference Hamiltonian $H_0$.

Let us first focus on the integrand of the $n$th order term

$$\left\langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_n) \right] \right\rangle_0 = \sum \langle T_\tau \left[ V(\tau_1) \right] \rangle_{0c} \langle T_\tau \left[ V(\tau_2) \ldots V(\tau_m) \right] \rangle_{0c} \cdots \langle T_\tau \left[ V(\tau_{m'}) \ldots V(\tau_m) \right] \rangle_{0c} \tag{2}$$

where the summation goes over all possible decompositions into connected terms, respecting the order of the imaginary time labelling, $\tau_i$, e.g. $\langle T_\tau [V(\tau_1)] \rangle_{0c} \langle T_\tau [V(\tau_2) \ldots V(\tau_m)] \rangle_{0c}$ and $\langle T_\tau [V(\tau_1) \ldots V(\tau_{m-1})] \rangle_{0c} \langle T_\tau [V(\tau_m)] \rangle_{0c}$ have to be considered as different contributions. Once integrated over all imaginary times, all connected terms which only differ by their imaginary time labels equally contribute, and we can group together connected diagrams which involves the same number of interactions

$$\begin{aligned}
\int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \left\langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_n) \right] \right\rangle_0 &= \sum_{m_1=0}^{n} \left[ \int_0^\beta d\tau_1 \langle T_\tau \left[ V(\tau_1) \right] \rangle_{0c} \right]^{m_1} \\
&\times \sum_{m_2=0}^{n/2} \left[ \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle T_\tau \left[ V(\tau_1) V(\tau_2) \right] \rangle_{0c} \right]^{m_2} \cdots \times F(m_1, m_2, \ldots)
\end{aligned}$$

where $F$ is just a combinatorical factor we will determine in the following.

On the rhs we have $m_i$ connected diagrams of order $i$ in $V$, which gives $m_i i$ interaction terms. Since we are only considering the $n$th order term on the lhs, we must impose $\sum_{i=1}^n m_i i = n$. Thee are $n!$ ways of diffents order of the imaginary time labels of the $n$ interactions, however, the $i!$ ways of ordering inside any connected diagram of order $i$ do not count as different contributions. Similarly, if we have $m$ connected diagrams of same order, the $m!$ ways of arranging them with respect to each other do not lead to independent contributions either.

Putting together, gives a factor of $\delta_{\sum_i i m_i, n} \frac{n!}{\prod_i m_i! i!}$ so that we have (We can understand this combinatorical also, by saying that we want only to fix the integration label by increasing $\tau_i$ from $1, \ldots m_n$ to $1, \ldots m_{n'}$ with $n < n'$...)

$$\begin{aligned}
\int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n \left\langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_n) \right] \right\rangle_0 &= n! \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \left[ \frac{1}{1!} \int_0^\beta d\tau_1 \langle T_\tau \left[ V(\tau_1) \right] \rangle_{0c} \right]^{m_1} \\
&\times \sum_{m_2=0}^{\infty} \frac{1}{m_2!} \left[ \frac{1}{2!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle T_\tau \left[ V(\tau_1) V(\tau_2) \right] \rangle_{0c} \right]^{m_2} \cdots \\
&\times \sum_{m_i=0}^{\infty} \frac{1}{m_i!} \left[ \frac{1}{i!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \cdots \int_0^\beta \tau_i \langle T_\tau \left[ V(\tau_1) V(\tau_2) \cdots V(\tau_i) \right] \rangle_{0c} \right]^{m_i} \cdots \\
&\times \delta_{\sum_i m_i i, n}
\end{aligned}$$

where we have extended all upper limits of the summation to infty, since they are trunced by the final constraint of the discrete delta function. Since $(-1)^n = (-1)^{\sum_i i m_i}$ we can write

$$\frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \cdots \int_0^\beta d\tau_n \left\langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_n) \right] \right\rangle_0 = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \delta_{\sum_i i m_i, n} \prod_i \frac{1}{m_i!} \Xi_i^{m_i} \tag{3}$$

where $\Xi_m$ the connected diagram of order $m$, together with its combinatorial factor and a potential minus sign

$$\Xi_m = \frac{(-1)^m}{m!} \int d\tau_1 \int d\tau_2 \cdots \int d\tau_m \, \langle T_\tau \left[ V(\tau_1) V(\tau_2) \ldots V(\tau_m) \right] \rangle_{0,c} \tag{4}$$

Adding all contributions from all orders in $n$, the constraint of the discrete delta of each order in $n$, e.g. in the rhs of Eq. (3), becomes irrelevant, and we can write the perturpation expansion of Eq. (1) in the following way

$$\frac{Z}{Z_0} = \sum_{m_1=0}^{\infty} \frac{\Xi_1^{m_1}}{m_1!} \sum_{m_2=0}^{\infty} \frac{\Xi_2^{m_2}}{m_2!} \cdots \sum_{m_n=0}^{\infty} \frac{\Xi_n^{m_n}}{m_n!} \cdots = \exp\left[ \sum_{m=1}^{\infty} \Xi_m \right] \tag{5}$$

and we directly obtain Eq. (1). We see that the disconnected graphs actually do not contribute to the logarithm of the partition function. In contrast to disconnected diagrams, connected diagrams are all proportional to the total number of particles and the linked cluster theorem ensures therefore the extensitivity of the free energy. We have managed to arrange perturbation theory in order to obtain a meaningful expansion of the free energy in the sense that its extensitivity is garantied in each order in perturbation theory, considering only connected graphs of order $n$.

### B. Feynman diagrams for the free energy

Feynman diagrams for the free energy then reduce to draw all connected close diagrams using $m$ interaction vertices and $2m$ propagators $G_0$. To obtain the $m$th order term for $\Xi_m$, we must further multiply by $(-1)^m$ and devide by $m!$. Of the $m!$ ways of time labelling the different $V$s, only $(m-1)!$ will give different diagrams, as the $m$ cyclic changes, e.g. $\tau_1 \to \tau_2, \ldots, \tau_{m-1}$, will not lead an indepedent new Feynman diagram. Still, any of different diagrams actually merely correpond to different orderings and thus give the same contribution to the integral in $\Xi_m$. Therefore, we can restrict the drawings of Feyman diagrams discarding different time orderings of the interaction, and multiply the resulting integral by the factor $(m-1)!/m! = 1/m$ which takes into account all possible different time ordered contributions. For fermions, we have to multiply our result by a possible sign, $(-1)^F$, where $F$ is the number of loops of the corresponding Feynman diagram which we will discuss below.

**Fermion loops.** When using Wick's theorem, we have to consider a sign which arises since the ordering of the operators $V$ is fixed by the Feynman diagram. The signe is fixed by the number of commutations involved to put the $\Psi^\dagger$ corresponding to one $G_0$ term in Wick's decomposition in front of the corresponding $\Psi$ (Remember the structure: $G_0(\tau_1; \mathbf{r}_2) = -T_\tau \langle \Psi(\tau_1) \Psi^\dagger(\tau_2) \rangle_0$). Now, the natural order of operators arising in the interaction is

$$\Psi^\dagger(\mathbf{r}, \tau) \Psi^\dagger(\mathbf{r}', \tau) \Psi(\mathbf{r}', \tau + \epsilon) \Psi(\mathbf{r}, \tau + \epsilon) = \Psi^\dagger(\mathbf{r}, \tau) \Psi(\mathbf{r}, \tau + \epsilon) \Psi^\dagger(\mathbf{r}', \tau) \Psi(\mathbf{r}', \tau + \epsilon) \tag{6}$$

where $\epsilon > 0$ is an infinitesimal small number to avoid ambigitious of the time ordering. The egality holds for bosons and fermions, as we have twice commuted the annihilation operator originally on the very right.

We will get two first order diagrams. In the first we contract $\Psi^\dagger(\mathbf{r}', \tau)$ with $\Psi(\mathbf{r}', \tau+\epsilon)$ leading to $-G_0(\mathbf{r}', \tau+\epsilon; \mathbf{r}', \tau)$ and, similar, we get $-G_0(\mathbf{r}, \tau+\epsilon; \mathbf{r}', \tau)$. Both propagator lines in the Feynman diagram start at the same position at $\tau$ and return to the same position at $\tau + \epsilon$, taking $\epsilon \to 0$), forming two loops and a contribution of $(-1)^{F=2} = +1$.

The second diagram is formed by contracting $\Psi^\dagger(\mathbf{r}', \tau)$ with $\Psi(\mathbf{r}, \tau + \epsilon$ leading to $+G_0(\mathbf{r}, \tau + \epsilon; \mathbf{r}', \tau)$, the plus sign since we do not need to commute anyting whereas $\Psi^\dagger(\mathbf{r}, \tau)$ needs three commutation to reach the place on the right of $\Psi(\mathbf{r}', \tau + \epsilon)$, giving $-G_0(\mathbf{r}', \tau + \epsilon; \mathbf{r}, \tau)$, and thus in total a minus sign. If we now follow the propagators, one from $\mathbf{r}, \tau$ going to $\mathbf{r}', \tau + \epsilon$ and continue with the second from $\mathbf{r}', \tau$ returning to $\mathbf{r}, \tau + \epsilon$, we can associate this diagram with one fermion loop, $F = 1$.

Let us go to 2nd order, with a structure of

$$\Psi^\dagger(\mathbf{r}_1, \tau_1) \Psi(\mathbf{r}_1, \tau_1 + \epsilon) \Psi^\dagger(\mathbf{r}_1', \tau_1) \Psi(\mathbf{r}_1', \tau_1 + \epsilon) \Psi^\dagger(\mathbf{r}_2, \tau_2) \Psi(\mathbf{r}_2, \tau_2 + \epsilon) \Psi^\dagger(\mathbf{r}_2', \tau_2) \Psi(\mathbf{r}_2', \tau_2 + \epsilon) \tag{7}$$

We can see that no sign will arise when any of the $\Psi^\dagger$ is contracted with any of the $\Psi$ on the left, since this involves an even number of permutations, however any contruction of $\Psi^\dagger$ with a $\Psi$ on the right will involve a minus sign. When starting with one particular $\Psi^\dagger(n_1)$, e.g. $\Psi^\dagger(\mathbf{r}_2', \tau_2)$, $n_1 \equiv (\mathbf{r}_2', \tau_2)$, and contract with any $\Psi(n_2)$ on the left, e.g. $\Psi(\mathbf{r}_1', \tau_1 + \epsilon)$ we can continue to follow the loop by following the contraction of the corresponding $\Psi^\dagger(n_2)$, e.g. $\Psi^\dagger(\mathbf{r}_1', \tau_1 + \epsilon)$. Closing the loop we need an odd number of contractions to the right, e.g. by contracting with $\Psi^\dagger(\mathbf{r}_1', \tau_1 + \epsilon)$ with $\Psi(\mathbf{r}_2', \tau_2 + \epsilon$, which gives us our rule of associating a minus sign with any fermion loop. This arguments extends to any order, since, actually I didn't use any particularities of 2nd order.