PHYS715 - Phase Retrieval exercices

Jonathan Dong

October 2024

1 Solution sets and metrics for phase retrieval

It is useful to recognize the space of solutions of phase retrieval problems. This will inform us on the symmetries of the problem and how to compute metrics to efficiently evaluate the result of a phase retrieval algorithm.

1. Let us assume that we have found a solution x to the equation $\mathbf{y} = |\mathbf{A}\mathbf{x}^*|^2$. Could you find an infinite number of solutions constructed from this solution x?

2. Let's now focus on the case of Fourier phase retrieval. In this setting, we have measurements $\mathbf{y} = |\mathbf{F}\mathbf{x}^*|^2$. Constraints such as finite support and positivity are required to make this problem solvable.
   When we do not make any assumption, could you find a generic strategy to construct infinitely many solutions even up to a global phase shift?

3. Would a metric $\|\mathbf{x} - \mathbf{x}^*\|^2$ be suitable to evaluate reconstruction performance?

4. Which metric would you suggest using to evaluate reconstruction performance? Several solutions are possible.

2 Gradient descent and link with error reduction algorithm

Here we are going to derive the usual formulas for gradient descent in phase retrieval. The formula will be computed for real-valued phase retrieval $\mathbf{y} = |\mathbf{A}\mathbf{x}^*|^2$ with A and x real-valued. This simplifies the formalism and the actual formulas to use are very close.

(The clean extension to complex-valued phase retrieval would require the concept of CR-calculus or Wirtinger derivatives. Derivatives of functions defined in a complex-valued space is a complicated field involving holomorphic functions, integrals along paths, etc. With CR-calculus or Wringer derivatives, we work in a real-valued space and just define derivatives on the real and imaginary parts independently.)

1. Warm up: linear regression
   Forward model $\mathbf{y} = \mathbf{A}\mathbf{x}^*$
   Loss function $l(\mathbf{x}) = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2$
   What is the component i of the gradient, i.e. $\frac{dl}{dx_i}$?
   Can you find a compact vectorial formula?

2. 1st loss: Intensity loss function
   Forward model $\mathbf{y} = (\mathbf{A}\mathbf{x}^*)^2$
   Loss function $l_1(\mathbf{x}) = \|\mathbf{y} - (\mathbf{A}\mathbf{x})^2\|^2$
   What is the component i of the gradient, i.e. $\frac{dl_1}{dx_i}$?
   Can you find a compact vectorial formula?

3. 2nd loss: amplitude loss function
   Forward model $\mathbf{y} = (\mathbf{A}\mathbf{x}^*)^2$
   Loss function $l_2(\mathbf{x}) = \|\sqrt{\mathbf{y}} - |\mathbf{A}\mathbf{x}|\|^2$
   What is the component i of the gradient, i.e. $\frac{dl_2}{dx_i}$?
   Can you find a compact vectorial formula?

4. Let us assume now that the forward model $\mathbf{A}$ is a Fourier transform and that $\mathbf{x}$ has a finite known support Γ.
   We consider the error reduction algorithm (Fienup '82) where we alternate between amplitude constraint and support constraint. Write the update equation of the error reduction algorithm. Can you make a link between gradient descent with the amplitude loss function and the error reduction algorithm?

3 Spectral method explained

Even though phase retrieval is a non-linear problem, spectral methods can be a convenient way to get a good initial estimate. They are designed for the random setting in which the $\mathbf{A}$ is iid random.

Let's consider measurements $\mathbf{y} = (\mathbf{A}\mathbf{x}^*)^2$ where $\mathbf{A}$ is an iid random matrix: each element of A is drawn independently from a complex-valued normal distribution ($a_{ij} \sim \mathcal{N}(0, 1/2) + j\mathcal{N}(0, 1/2)$). d is the dimension of x and n the dimension of y. We can also write it component by component: $y_i = |\mathbf{a}_i^H \mathbf{x}|^2$.

The weighted simplest weighted covariance matrix is defined as:

$$Z = \frac{1}{n} \sum_i y_i \mathbf{a}_i \mathbf{a}_i^H. \tag{1}$$

1. As Z is an empirical sum of a random variable, introduce an iid random vector $\mathbf{a}$ and write the law of large numbers to find the limit of Z as n goes to infinity.

2. Writing the scalar product $\mathbf{a}^H x$ explicitly, expand $y = |\mathbf{a}^H \mathbf{x}|^2$ into a double sum.

3. Deduce an expanded expression of Z.

4. Which terms have non-zero expectation? Find the limit of the weighted covariance matrix.

The study of the weighted covariance matrix in the general case and when both n and d go to infinity (with fixed oversampling ratio n/d) requires more notions from random matrix theory.