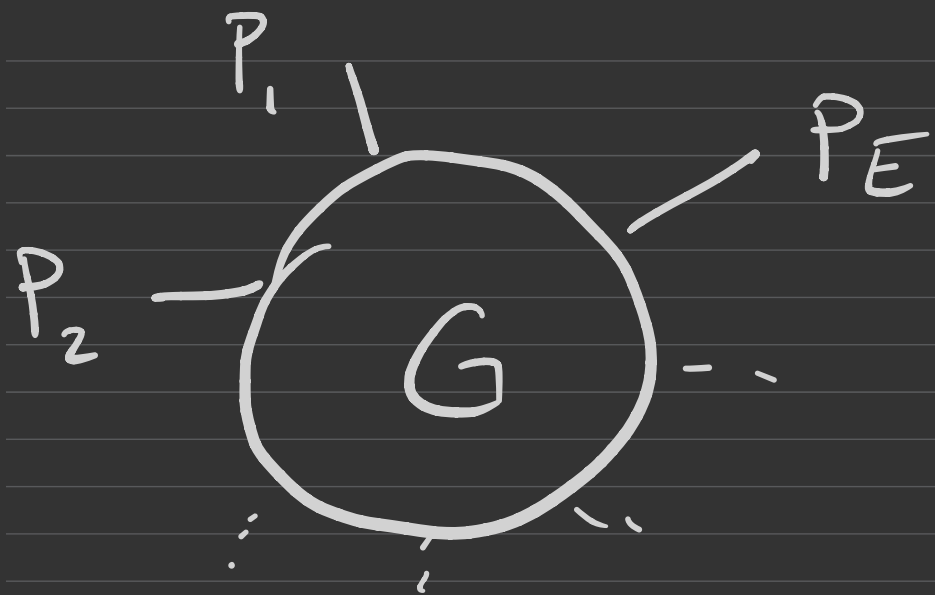




$E = \#$ external legs
 $V = \#$ vertices
 $I = \#$ internal legs
 $L = \#$ loops

$$\sum d_\alpha \equiv d_v$$

$$\begin{aligned}
 G_L &= \int \prod_{\ell=1}^L d^D k_\ell \prod_{\alpha=1}^I \Delta(p_\alpha) \prod_{\alpha=1}^v f_\alpha(p_{\alpha_\alpha}) \\
 &\sim \int K^{LD} \frac{dk}{K} K^{-2I} K^{d_v} = \int K^{\delta(G)} \frac{dk}{K}
 \end{aligned}$$

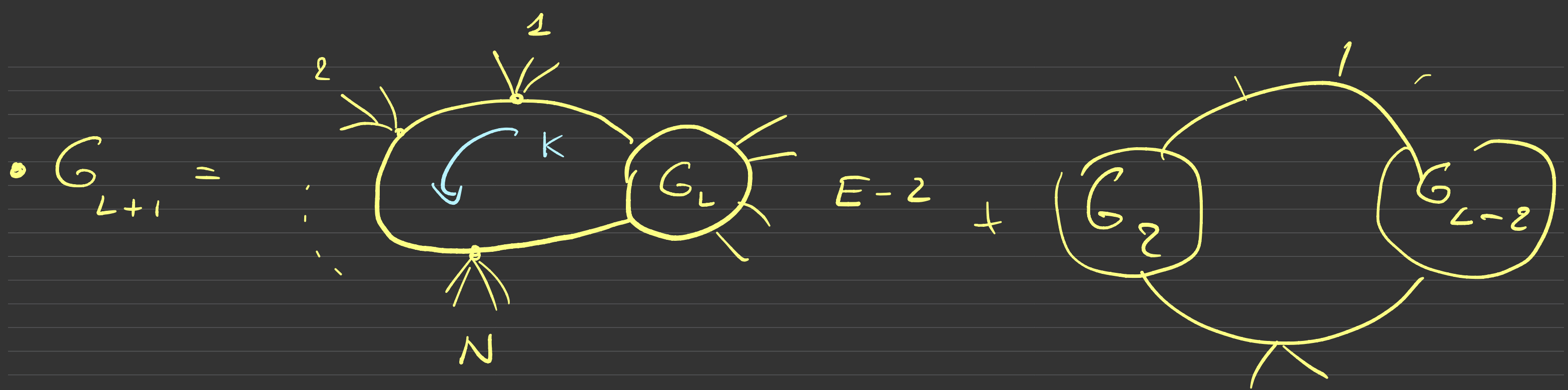
$$\delta(G_L) = LD - 2I + d_v$$

Renormalized $\Rightarrow$

$$G_L \sim p^{\delta(G_L)} F_0(\log p) + p^{\delta(G_L)-2} F_1(\log p) + \dots$$

$$\begin{aligned}
 &\downarrow \\
 &(\log p)^L + (\log p)^{L-1} + \dots
 \end{aligned}$$

by Weinberg Th



$$\approx \left(\frac{1}{K^2} \right)^{N+1} K^{\sum_{i=1}^N d_i} \left(K^{\delta(G_L)} + K^{\delta(G_L)-2} + \dots \right) d^D K$$

EX $= K^{\delta(G_{L+1})} \left(1 + K^{-2} + \dots \right) \frac{dK}{K}$

• $\partial_P^n G_{L+1} \sim K^{\delta(G_{L+1})-n} \frac{dK}{K}$

▶ Reading in Collins

- 5.1 — 5.4

- 5.5.1 (5.5.2 if you want details)

- 5.8

• Schematically

$$G_{L+1} = \int d\mu_1 \dots d\mu_{L+1} F(P, \kappa_1, \dots, \kappa_{L+1})$$

• can consider $L+1$ distinct integration regions $R_1, \dots, R_{L+1}$

$$R_i : \min(|\kappa_1|, \dots, |\kappa_{L+1}|) = |\kappa_i|$$

$$\bullet G_{L+1} = \sum_{i=1}^{L+1} \int d\mu_i \underbrace{\int_{R_i} d\mu_1 \dots d\mu_i \dots d\mu_{i+1} \dots d\mu_{L+1} F(P, \kappa_1, \dots, \kappa_{L+1})}_{G_L^i \times \Pi \Delta(P, \kappa_i)}$$

$$G_L^i \times \Pi \Delta(P, \kappa_i)$$

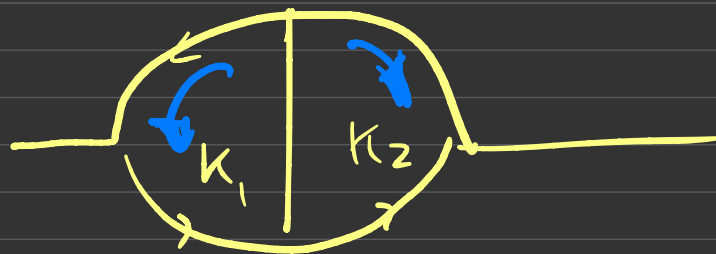
Ex $R_1: |k_1| \leq |k_{i>1}|$

$$\int d\mu_2 \dots d\mu_{L+1} F(p, k_1, \overset{\overline{p}}{\sim} k_2 \dots k_{L+1}) = G_L(\overset{\overline{p}}{\sim} p, k) \cdot \Pi \Delta(p, k)$$

• $G_L(\overline{p}) \sim \overline{p}^{\delta(G_L)} + \overline{p}^{\delta(G_L)-2} + \dots$

• use discussion of last time

Ex 2-loop self energy in ϕ^3



$\Rightarrow$ • $\delta(G_{L+1}) < 0 \Rightarrow \text{finite}$

• $\delta(G_{L+1}) > 0 \Rightarrow \text{divergence} \equiv \text{degree } \delta(G_{L+1}) \text{ polynomial}$

renormalized by adding a
local counterterm

▴ Structure of needed counter-terms specified by
the $\delta(G)$ of the various n -point functions

$$\bullet \delta(G) = D - \overbrace{\left(\frac{D-2}{2}\right) E}^{\Delta_\phi} + \overbrace{\sum_\alpha \left(\frac{D-2}{2} n_\alpha + d_\alpha - D\right)}^{-\Delta_\alpha}$$

$$\boxed{\Delta_\alpha = D - d_\alpha - \frac{D-2}{2} n_\alpha}$$

$$\Downarrow \int_\alpha \partial^{d_\alpha} \phi^{n_\alpha}$$

⚠ Crucial distinction: $\boxed{\Delta_\alpha \geq 0}$ versus $\boxed{\Delta_\alpha < 0}$

① $\Delta_\alpha \geq 0$: $\bullet \delta(G) \geq 0 \Rightarrow E \leq \frac{2D}{D-2} \begin{cases} \sqrt{D=4} & \leq 4 \\ \sqrt{D=6} & \leq 3 \\ \sqrt{D=3} & \leq 6 \end{cases}$

C.T $\sim \int \delta(G) \phi^E$ with $\delta(G) + \frac{D-2}{2} E \leq D$

Ex $\mathcal{L} = (\partial\phi)^2 + m^2\phi^2 + g\phi^3$ in $D=6$

$$E \leq \frac{12}{4} = 3$$

possible c.T. $\phi^3, \phi^2, \phi, (\partial\phi)^2$

- $\Delta_2 \geq 0 \Rightarrow$ finite class of counterterms
 $\equiv$ Renormalizable

2. $\Delta_\alpha < 0$ for some vertex

$$\delta(G) = D - \frac{D-2}{2} E - \sum_\alpha \Delta_\alpha$$

• $\forall E$ $\delta(G)$ can be made > 0 by sufficient insertions of vertex with $\Delta_\alpha < 0$

C.T. $\mathcal{O}^{\delta(G)} \phi^E$ $\delta(G) + \frac{D-2}{2} E = \overset{= D - \sum_\alpha \Delta_\alpha}{\downarrow} \text{unbounded}$

$\Rightarrow$ class of necessary counter-terms is infinite

Non-Renormalizable

▲ Modern perspective

Non-Renormalizable QFT $\equiv$ Effective QFT

• notice $\mathcal{L} \sim p_* \partial^d \phi^E$

$$[p_*] = -d - \frac{D-2}{2}E + D < 0$$

$$p_* = \frac{1}{M^{d + \frac{D-2}{2}E - D}} \equiv M^{-D_L}$$

At every E what is the size of effects of p_* ?

$$\overline{p_*} = \left(\frac{E}{M}\right)^{d + \frac{D-2}{2}E - D}$$

Loop counting

Ex

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 + \frac{g}{3!} \phi^3$$

$$\phi \rightarrow \frac{\phi}{g} \Rightarrow$$

$$\mathcal{L} \rightarrow \frac{1}{g^2} \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{3!} \phi^3 \right]$$

$$g^2 \sim \hbar$$

$$\left. \begin{array}{l} \text{---} \propto g^2 \\ \text{---} \propto 1/g^2 \end{array} \right\}$$



$$\sim (g^2)^{I-V} = (g^2)^{L-1}$$

$$\sqrt{\hbar} \sim g$$

Ex

$$\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_4}{4!}\phi^4$$

$$\phi \rightarrow \frac{1}{\sqrt{\lambda}}\phi$$



$$\frac{1}{\lambda_4} \left[\frac{1}{2}(\partial\phi)^2 + \frac{\phi^4}{4!} \right]$$

$$\lambda_4 \sim \hbar$$



$$\lambda_4 \stackrel{\text{I-V}}{=} \lambda_4$$

$$= \lambda_4^{L-1}$$

$$\lambda_4 \equiv g_4^2$$

$$= (g_4^2)^{L-1}$$

Ex

$$\frac{1}{2}(\partial\phi)^2 + \frac{\lambda_6}{6!}\phi^6$$

$$\phi \rightarrow \frac{1}{\lambda^{1/4}}\phi$$



$$\frac{1}{\sqrt{\lambda_6}} \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{6!}\phi^6 \right)$$



$$\sim \lambda_6 \stackrel{\text{I-V}}{=} \lambda_6$$

$$= \lambda_6^{\frac{L-1}{2}}$$

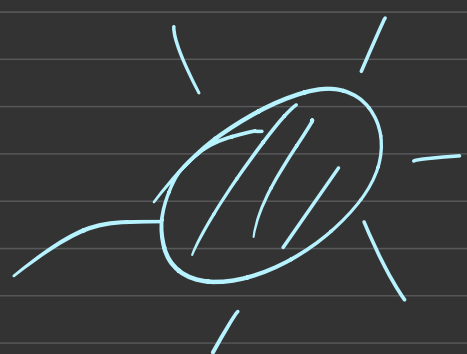
$$= \frac{1}{\sqrt{\lambda_6}} \lambda_6^{L/2}$$

$$\lambda_6 \equiv g_6^4$$

E_x

$$\frac{1}{2} (\partial \phi)^2 + \frac{1_5}{5!} \phi^5$$

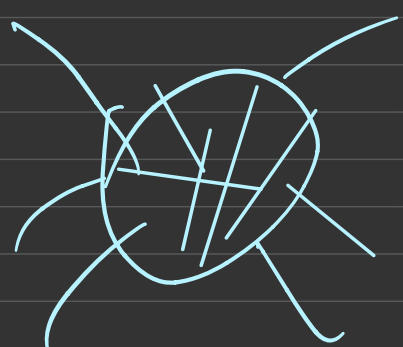
$$\lambda_5 \sim \rho_5^3$$


$$\sim \left(\rho_5^2 \right)^{L-1} = \lambda_5^{\frac{2}{3}(L-1)}$$

E_x

$$\frac{1}{2} (\partial \phi)^2 + \frac{1_n}{n!} \phi^n$$

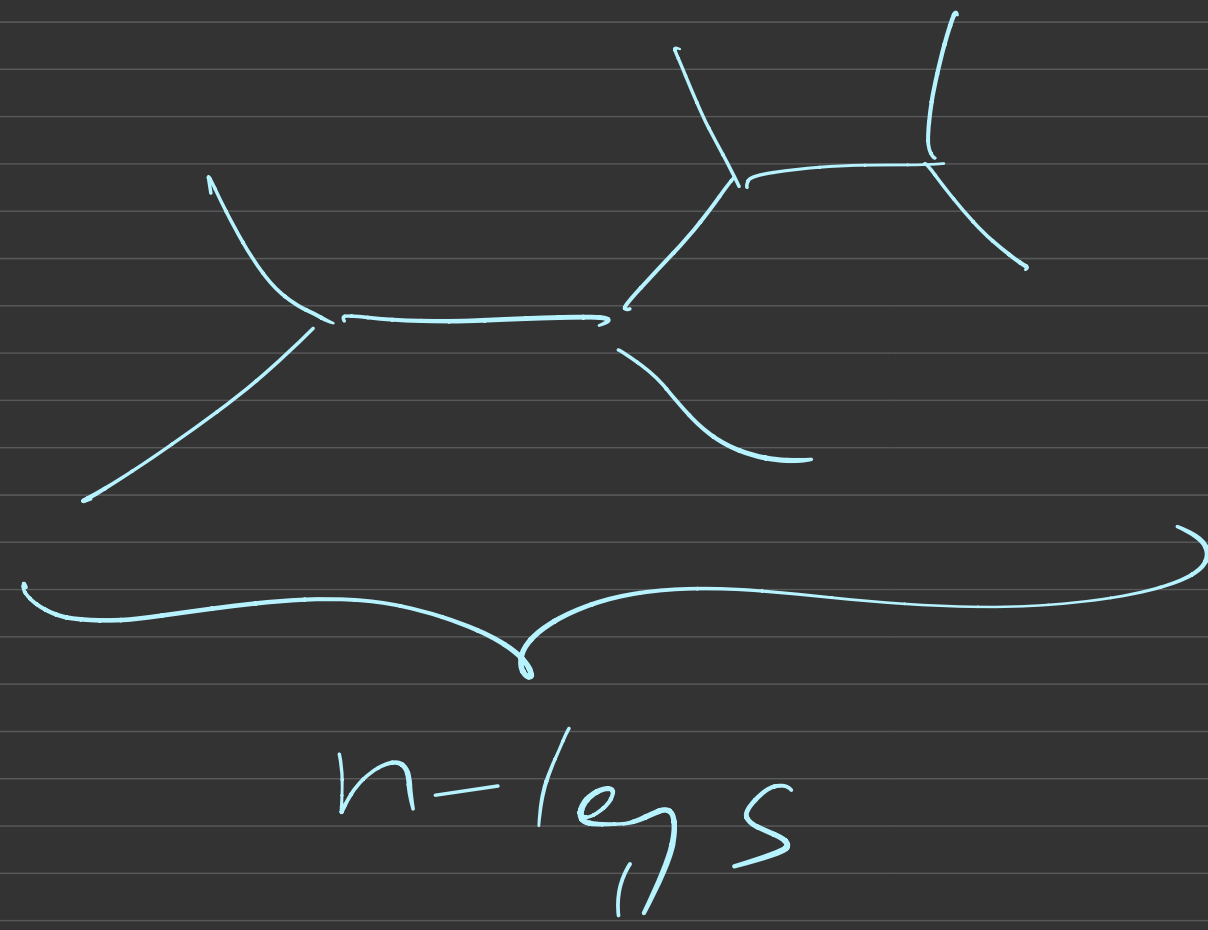
$$\lambda_n \sim \underline{\underline{\rho_n^{n-2}}}$$


$$\sim \left(\rho_n^2 \right)^{L-1} = \left(1_n \right)^{\frac{2(L-1)}{n-2}}$$

$$\int_n^{n-2} \phi_n$$

Ex

$$\int \phi^3$$



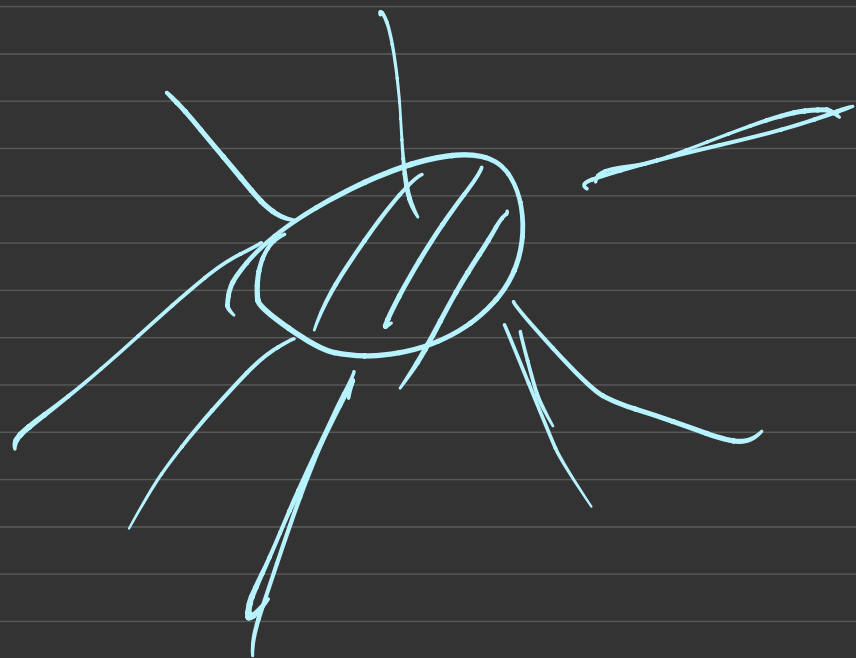
$$\sim \int^{n-2}$$

Ex scalar dynamics,

$$\left. \begin{array}{l} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right\} e^{n-2}$$

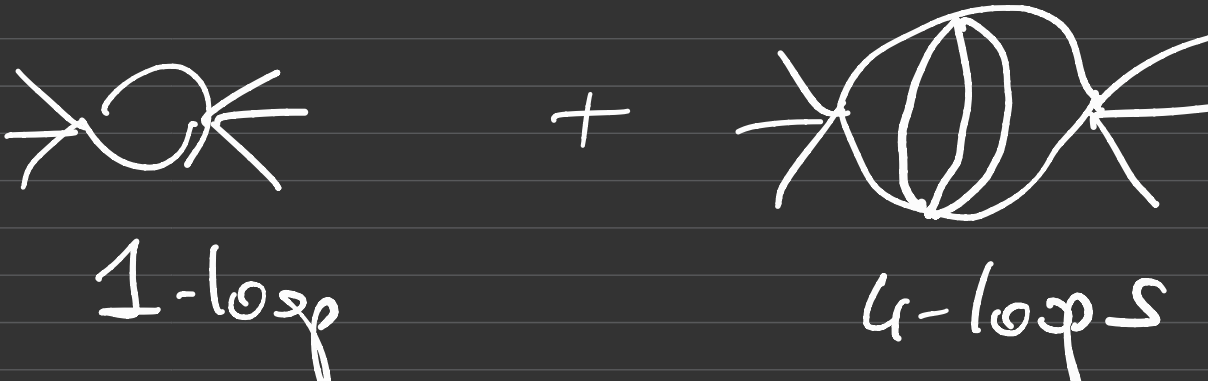
• Ex in ϕ^3 $\mu = \frac{\Gamma(p^2 - u^2)}{\sqrt{z}}$ G

$$\mu_{E-1\text{egs}} \sim \frac{1}{g^E} \cdot g^{2E} (g^2)^{I-V} \sim g^{E+2L-2} = (g^{E-2}) g^{2L}$$



$$\langle \phi \phi \rangle \sim \frac{g^2}{p^2} \xrightarrow{\text{Lst}} g \sim \sqrt{z}$$

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \lambda_5 \phi^5$$

$$\mathcal{K}_{3 \rightarrow 3} = \text{1-loop} + \text{4-loop}$$


▴ The Renormalization Group

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

Ex ϕ^4

$$\mathcal{L}_R = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$\mathcal{L}_{CT} = \frac{\delta z}{2}(\partial\phi)^2 + \frac{1}{2} \left[\overset{\Delta m^2}{(m^2 + \delta m^2)}(1 + \delta z) - m^2 \right] \phi^2 + \frac{(\overset{\Delta \lambda}{\lambda + \delta \lambda})(1 + \delta z)^2 - \lambda}{4!} \phi^4$$

$$\begin{aligned} \Rightarrow \mathcal{L}_R + \mathcal{L}_{CT} &= \frac{z}{2}(\partial\phi)^2 + \frac{z}{2}m_0^2\phi^2 + \frac{z^2}{4!}\lambda_0\phi^4 \\ &\equiv \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{4!}\lambda_0\phi_0^4 \end{aligned}$$

$$Z \equiv 1 + \delta Z$$

$$\phi_0 \equiv \sqrt{Z} \phi$$

$$u_0^2 \equiv u^2 + \delta u^2$$

$$\lambda_0 \equiv \lambda + \delta \lambda$$

Dim Reg.

$$\lambda_0 = \lambda \mu^\varepsilon \left[1 + \frac{Q_1(\lambda)}{\varepsilon} + \frac{Q_2(\lambda)}{\varepsilon^2} + \dots \right]$$

P.V.

$$\lambda_0 = \lambda \left[1 + \tilde{Q}_1(\lambda) \ln \frac{\Lambda}{\mu} + \tilde{Q}_2(\lambda) \ln^2 \frac{\Lambda}{\mu} + \dots \right]$$

$$Q_e \sim e\text{-loop} + (e+1)\text{-loop} + \dots$$

$$= Q_{e,e} \lambda^e + Q_{e,e+1} \lambda^{e+1} + \dots$$

• Similarly $\omega_0^2 = \omega^2 \left[1 + \frac{b_1(\lambda)}{\varepsilon} + \frac{b_2(\lambda)}{\varepsilon^2} + \dots \right]$

$$Z = \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \frac{c_2(\lambda)}{\varepsilon^2} + \dots \right]$$

with $b_e \sim \lambda^e + \dots$

$$c_e \sim \lambda^e + \dots$$

- Basic structure fixed by dimensional analysis

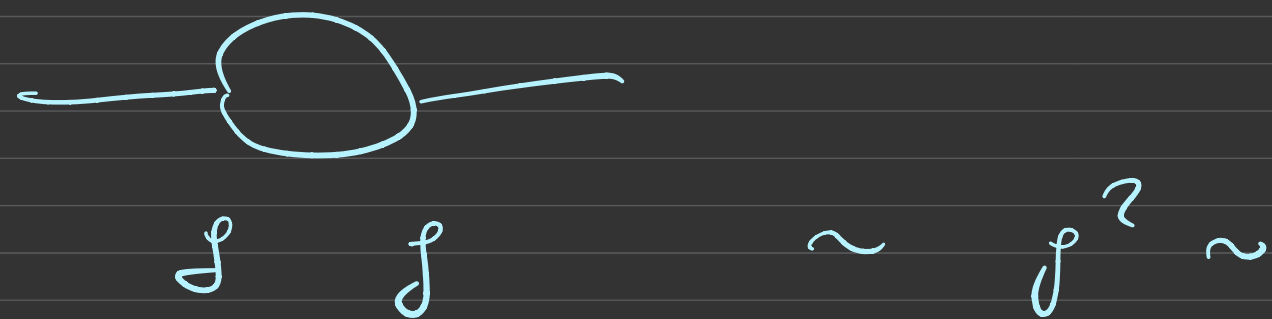
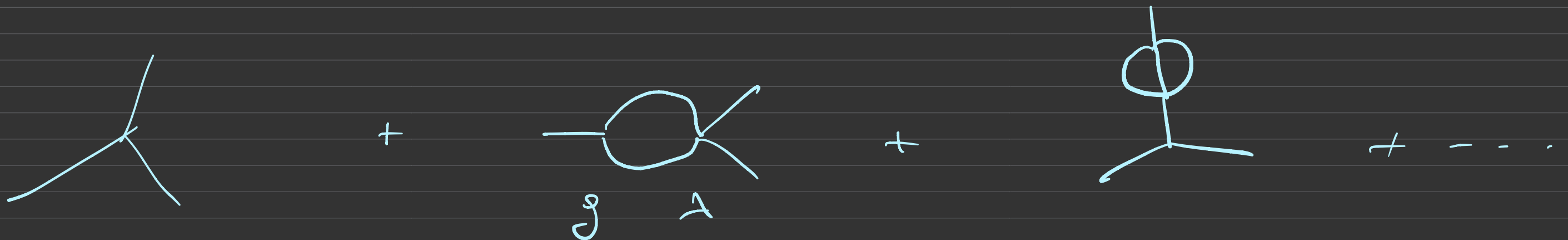
Ex $\mathcal{L} = \frac{1}{2} (\partial\phi)^2 + \eta\phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi^3 + \frac{1}{4!} \lambda \phi^4$

$$g_0 = g \mu^{2-D/2} \left[1 + \frac{d_1(\lambda)}{\varepsilon} + \frac{d_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$m_0^2 = m^2 \left[1 + \frac{b_1}{\varepsilon} + \dots \right] + g^2 \left[\frac{h_1(\lambda)}{\varepsilon} + \frac{h_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$\eta_0 = \eta \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \dots \right] + g m^2 \left[\frac{f_1(\lambda)}{\varepsilon} + \dots \right] + g^3 \left[\frac{m_1(\lambda)}{\varepsilon} + \dots \right]$$

$$g_0 = g \mu^{2-D/2} \left[1 + \frac{d_1(\lambda)}{\varepsilon} + \frac{d_2(\lambda)}{\varepsilon^2} + \dots \right]$$



⊙ Amplitudes & Correlators = 0

$$\mathcal{O} = \lambda^9 \left[\mathcal{O}_0 + \mathcal{O}_1(\lambda) \ln \frac{E}{\mu} + \mathcal{O}_2(\lambda) \ln^2 \frac{E}{\mu} + \dots \right]$$

$$\mathcal{O}_e = \mathcal{O}_{e,e} \lambda^e + \mathcal{O}(\lambda^{e+1})$$

Ex

$$\mu(2 \rightarrow 2) = -\lambda \left[1 + \frac{\lambda}{32\pi^2} \left(3\delta - 6 + \ln \frac{stu}{(4\pi\mu^2)^3} - i\pi \right) + \dots \right]$$

$$\sim -\lambda - \frac{6\lambda^2}{32\pi^2} \left(c + \ln \frac{E}{\mu} \right) + \mathcal{O}(\lambda^3)$$

$$A) \mathcal{M} \equiv \mu(\lambda, \mu) = -\hat{\lambda} - \frac{6\hat{\lambda}^2}{32\pi^2} \ln E + O(\hat{\lambda}^3)$$

$$\cdot \hat{\lambda} \equiv \lambda - \frac{6\lambda^2}{32\pi^2} \ln \mu$$

$$(\lambda', \mu') \sim (\lambda, \mu) \quad \text{if}$$

$$\lambda' - \frac{6\lambda'^2}{32\pi^2} \ln \mu' = \lambda - \frac{6\lambda^2}{32\pi^2} \ln \mu$$

$$\Rightarrow \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \ln \frac{\mu}{\mu'} + O(\lambda^3)$$

B) $\frac{\lambda}{32\pi^2} \ll 1$ not enough to ensure perturbativity
if $\ln E/\mu$ sufficiently large

• $\frac{\lambda}{32\pi^2} \ln \frac{E}{\mu} \gtrsim 1 \Rightarrow$ pert. theory breaks down

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

Renormalization group

- Can this be made more precise?
- Can we still compute when $\frac{\lambda}{16\pi^2} \ln \frac{E}{\mu} \gtrsim O(1)$?

Basic Idea

$$m^2, \lambda, \mu \rightarrow \mathcal{L}_0 = \frac{Z}{2} (\partial\phi)^2 + \frac{m_0^2}{2} Z \phi^2 + \frac{\lambda_0}{4!} Z^2 \phi^4$$

$$m'^2, \lambda', \mu' \rightarrow \mathcal{L}'_0 = \frac{Z'}{2} (\partial\phi')^2 + \frac{m_0'^2}{2} Z' \phi'^2 + \frac{\lambda_0'}{4!} Z'^2 \phi'^4$$

Require $\left. \begin{array}{l} \lambda_0 = \lambda_0' \\ m_0 = m_0' \end{array} \right\} \Rightarrow \sqrt{Z}\phi = \phi_0 \quad \text{and} \quad \sqrt{Z'}\phi' = \phi_0'$

identical correlators and
can be identified

$$\Rightarrow \mathcal{L}_0 = \mathcal{L}'_0$$

$$\Rightarrow \mathcal{L} + \mathcal{L}_{CT} = \mathcal{L}' + \mathcal{L}'_{CT} \equiv \mathcal{L}_0$$

$$\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$

$$\frac{1}{2} (\partial\phi')^2 + \frac{m'^2}{2} \phi'^2 + \frac{\lambda'}{4!} \phi'^4$$

$\Rightarrow$ the two construction simply correspond to different splittings of the same bare Lagrangian $\mathcal{L}_0$.

• $\forall \mu \in \mathbb{R}^+$ I can perform such equivalent construction

$$\lambda, u, \phi \longrightarrow \lambda(\mu), u(\mu), \phi(\mu)$$

• μ -choice optimal to study physics at $E \sim \mu$
where $\log \frac{E}{\mu} \lesssim \mathcal{O}(1)$

• Interpretation

$$\mathcal{L}_0 = \mathcal{L}_R(\mu) + \mathcal{L}_{CT}(\mu)$$

$\Downarrow$
compensates quantum fluctuations at $\hbar \sim \mu$

$$\triangle \mathcal{L}_0 = \frac{1}{2} (\partial \phi_0)^2 - \frac{m_0^2}{2} \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4$$

• μ -dependence of $\lambda(\mu)$, $m^2(\mu)$, $\phi(\mu)$ implicitly determined by

$$\mu \frac{d}{d\mu} \lambda_0 = 0$$

$$\mu \frac{d}{d\mu} m_0^2 = 0$$

$$\mu \frac{d}{d\mu} \phi_0 = 0$$

$\Downarrow$

Renormalization Group Equations

$$\bullet \mu \frac{d}{d\mu} \lambda_0 = \mu \frac{d}{d\mu} \left[\lambda \mu^\varepsilon \left(1 + \frac{Q_1}{\varepsilon} + \frac{Q_2}{\varepsilon^2} + \dots \right) \right]$$

$$\mu \frac{d}{d\mu} \lambda$$

$$= \left(\hat{\beta}(\lambda) + \varepsilon \lambda \right) \left(1 + \sum_n \frac{Q_n}{\varepsilon^n} \right) + \lambda \hat{\beta} \left(\sum_n \frac{Q'_n}{\varepsilon^n} \right) = 0$$

$$\Rightarrow \hat{\beta} = - \frac{\varepsilon \lambda + \sum_n \frac{\lambda Q_n}{\varepsilon^{n-1}}}{1 + \sum_n \frac{(1 + \lambda Q_n) Q_n}{\varepsilon^n}}$$

$$= - \left\{ \varepsilon \lambda - \lambda^2 Q'_1 + \frac{\lambda^2}{\varepsilon} \left[-Q'_2 + Q'_1 (Q_1 + \lambda Q'_1) \right] + \dots \right\}$$

⚠ $\mu \rightarrow \mu', \lambda(\mu) \rightarrow \lambda(\mu'), \dots \equiv$ invariance of UV finite observables

$$\Downarrow \\ \vec{\beta}(\lambda) \equiv \text{finite}$$

$\Rightarrow$ all $\frac{1}{\epsilon}, \frac{1}{\epsilon^2}, \dots$ terms in $\vec{\beta}$

Should vanish !!

Ex $-Q_2' + Q_1'(Q_1 + \lambda Q_1') = 0$

$Q_1 \rightarrow Q_2$ fixed by diff. eq.

$$Q_1 = c_1 \lambda + c_2 \lambda^2 + \dots$$

$$Q_1' = c_1 + 2c_2 \lambda + \dots$$

$$\begin{aligned} Q_2' &= (c_1 + 2c_2 \lambda) (c_1 \lambda + c_2 \lambda^2 + c_1 \lambda + 2c_2 \lambda^2) \\ &= c_1^2 \lambda + O(\lambda^2) \end{aligned}$$

————— $\xrightarrow{\text{Known}}$

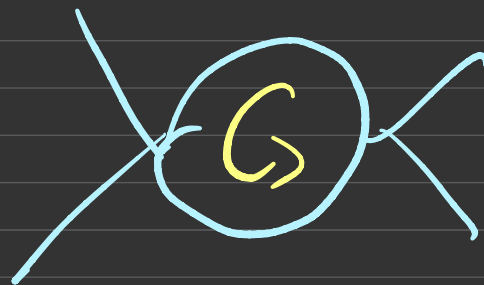
$$Q_2 = \frac{1}{2} c_1^2 \lambda^2 + O(\lambda^3)$$

$$\mu \frac{d}{d\mu} \left[\lambda \mu^\varepsilon \left(1 + \frac{Q_1}{\varepsilon} + \frac{Q_2}{\varepsilon^2} + \dots \right) \right]$$

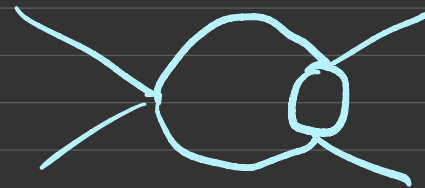
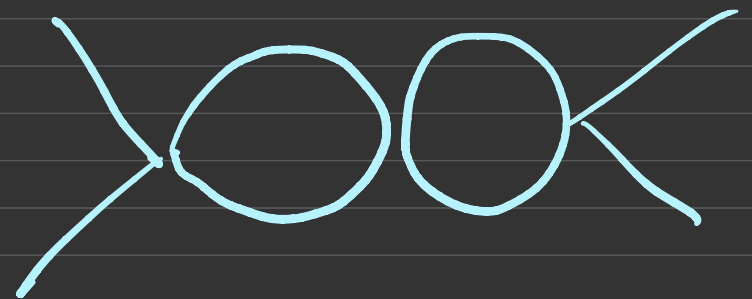
$$\lambda \mu^\varepsilon \left(1 + \frac{C_1 \lambda + \dots}{\varepsilon} + \frac{\frac{1}{2} C_1 \lambda^2 + \dots}{\varepsilon^2} \dots \right)$$

//

//



$\log \Lambda$



P.V. $\frac{1}{\varepsilon} \rightarrow \ln \Lambda$

$$\Rightarrow \hat{\beta}(\lambda) = -\varepsilon\lambda + \lambda^2 Q_1'$$

$$\equiv -\varepsilon\lambda + \beta(\lambda)$$

- For renormalized quantities $\varepsilon \rightarrow 0$ smooth

$$\Rightarrow \mu \frac{d}{d\mu} \lambda(\mu) \stackrel{d=4}{=} \lambda^2 Q_1'(\lambda) \equiv \beta(\lambda)$$

• Similar story for $u^2(\mu)$:

$$u_o^2 = u^2 \left[1 + \frac{b_1(\lambda)}{\varepsilon} + \frac{b_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$0 = \mu \frac{d}{d\mu} u_o^2 = \mu \frac{d}{d\mu} u^2 \left[1 + \sum_n \frac{b_n}{\varepsilon^n} \right] + u^2 \left[\sum_n \frac{b'_n}{\varepsilon^n} \right] (-\varepsilon\lambda + \lambda^2 \alpha'_1)$$

$$\Rightarrow \mu \frac{d}{d\mu} \ln u^2(\mu) = - \frac{\sum_n \frac{b'_n}{\varepsilon^n} [-\varepsilon\lambda + \lambda^2 \alpha'_1]}{\left[1 + \sum_n \frac{b_n}{\varepsilon^n} \right]}$$

$$\equiv -\gamma_u(\lambda)$$

$$\bullet \quad \phi_0 = \sqrt{Z} \phi(\mu)$$

$$Z = \left[1 + \frac{c_1(\lambda)}{\varepsilon} + \frac{c_2(\lambda)}{\varepsilon^2} + \dots \right]$$

$$0 = \mu \frac{d}{d\mu} \phi_0 \Rightarrow \mu \frac{d}{d\mu} \phi = -\gamma(\lambda) \phi$$

$$\gamma(\lambda) = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z = \frac{1}{2} \left(\frac{\partial}{\partial \lambda} \ln Z \right) (-\varepsilon \lambda + \lambda Q'_1)$$

• Formal solutions of RG eqs.

$$\begin{aligned} \bullet \quad \frac{d\lambda}{d\ln\mu} &= \beta(\lambda) \quad \Rightarrow \quad \ln \frac{\mu'}{\mu} = \int_{\lambda(\mu)}^{\lambda(\mu')} \frac{d\lambda}{\beta(\lambda)} \\ d\ln\mu &= \frac{d\lambda}{\beta(\lambda)} \end{aligned}$$

$$\begin{aligned} \bullet \quad \frac{d\ln u^2}{d\ln\mu} &= -\gamma_u(\lambda) \quad \Rightarrow \quad \ln \frac{u^2(\mu')}{u^2(\mu)} = - \int_{\ln\mu}^{\ln\mu'} \gamma_u(\lambda(\tilde{\mu})) d\ln\tilde{\mu} \\ &= - \int_{\lambda(\mu)}^{\lambda(\mu')} \frac{\gamma_u(\lambda)}{\beta(\lambda)} d\lambda \end{aligned}$$

- $\frac{d}{d \ln \mu} \phi(\mu) = -\gamma(\mu) \phi(\mu)$

$$\Rightarrow Z(\mu', \mu) \equiv e^{-\int_{\lambda(\mu)}^{\lambda(\mu')} \frac{\gamma(\lambda)}{\beta(\lambda)} d\lambda}$$

$$\begin{cases} \frac{d}{d \ln \mu'} Z(\mu', \mu) = -\gamma(\lambda(\mu')) Z(\mu', \mu) \\ Z(\mu, \mu) = 1 \end{cases}$$

$$\Rightarrow \phi(\mu') = Z(\mu', \mu) \phi(\mu) \Rightarrow \frac{d \phi(\mu')}{d \ln \mu'} = -\gamma(\mu') \phi(\mu')$$

Examples

① $\lambda \phi^4$

$$\cdot \lambda_0 = \lambda \mu^\varepsilon \left(1 + \frac{Q_1}{\varepsilon} + \dots \right)$$

$$\cdot Q_1 = \frac{3\lambda}{16\pi^2} \lambda + o(\lambda^2)$$

$$\cdot \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + o(\lambda^3) > 0$$

$$\mu \frac{d}{d\mu} \lambda(\mu) = \frac{3\lambda^2(\mu)}{16\pi^2} \implies$$

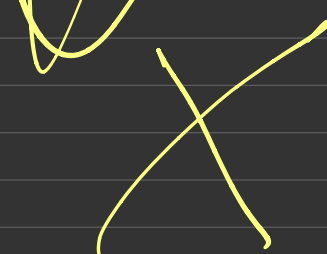
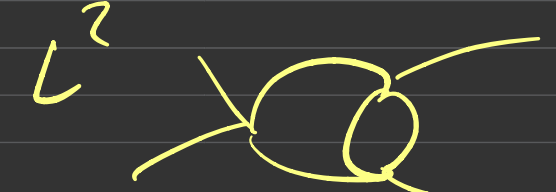
$$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\mu'}{\mu}}$$

$$\bullet \quad \sigma(E) \approx \frac{1}{E^2} \cdot \lambda(E)^2 \left(1 + c \lambda(E) + \dots \right)$$

$$\approx \frac{1}{E^2} \lambda(E)^2 \approx \frac{1}{E^2} \left[\frac{\lambda(\mu)}{1 - \frac{3\lambda(\mu)}{16\pi^2} \ln E/\mu} \right]^2$$

$$\approx \frac{1}{E^2} \lambda(\mu)^2 \left(\sum_{L=0}^{\infty} \left(\frac{3\lambda}{16\pi^2} \right)^L L^L \right)$$

$L \equiv \ln \frac{E}{\mu}$

L^0 
 $+$
 L^1 
 $+$
 L^2 
 $+$
 L^3 
 $+$...

$$\triangle \frac{d\lambda}{d\ln\mu} = c_1 \lambda^2 + c_2 \lambda^3 + \dots + c_e \lambda^{e+1} + \dots$$

$$\frac{d\ln\lambda}{d\ln\mu} = \lambda (c_1 + c_2 \lambda + \dots + c_e \lambda^{e-1} + \dots)$$

$$\downarrow$$

$$(\lambda L)^n$$

$$\lambda^n L^n$$

$$\downarrow$$

$$\lambda (\lambda L)^n$$

$$\lambda^{n+1} L^n$$

$$\downarrow$$

$$\lambda^{e-1} (\lambda L)^n$$

$$\lambda^{e+n-1} L^n$$

RG asymptotics of $\lambda \phi^4$ in 4D

$$\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\mu'}{\mu}}$$

• IR $\ln \frac{\mu'}{\mu} \rightarrow -\infty \Rightarrow \lambda(\mu') \approx \frac{16\pi^2}{3 \ln \frac{\mu}{\mu'}} \rightarrow 0$

• UV $\frac{3\lambda(\mu)}{16\pi^2} \ln \frac{\Lambda_*}{\mu} = 1 \Rightarrow \lambda(\Lambda_*) = \infty$ Landau Pole

$$\ln \frac{\Lambda_*}{\mu} = \frac{16\pi^2}{3\lambda(\mu)} \rightarrow \Lambda_* = \mu e^{\frac{16\pi^2}{3\lambda(\mu)}}$$

⚠ Landau Pole & bare couplings $\rightarrow \frac{1}{\epsilon}$ in DR

• Pauli-Villars : $\lambda_0 = \lambda(\mu) P\left(\lambda(\mu), \ln \frac{\Lambda}{\mu}\right)$

$$\sum_{n=0}^{\infty} p_n(\lambda) \left(\ln \frac{\Lambda}{\mu}\right)^n$$

$$p_0 = 1$$

$$p_n = c\lambda^n + \dots$$

• $\Lambda \simeq \mu$: $\lambda_0 = \lambda(\Lambda) \left[1 + a_1 \lambda(\Lambda) + a_2 \lambda^2(\Lambda) + \dots \right]$

hp weak
coupling

$$\lambda_0 \sim \lambda(\Lambda) \simeq \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\Lambda}{\mu}}$$

$$\lambda_0 = \lambda(\mu) + \frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\Lambda}{\mu} + \left(\frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\Lambda}{\mu} \right)^2 + \dots$$

we must work under assumption

$$\text{that } \frac{3}{16\pi^2} \lambda(\mu) \ln \frac{\Lambda}{\mu} \ll 1$$

$$\Lambda \ll \Lambda_* \equiv \text{Landau pole scale}$$

▴ Asymptotics of $g\phi^3$ in 6D

$$g_0 = g \mu^{\epsilon/2} \left(1 - \frac{g^2}{64\pi^3} \frac{1}{\epsilon} + \dots \right)$$

$$\mu \frac{d}{d\mu} g_0 = 0 \quad \Rightarrow \quad \hat{\beta}(g) = -\frac{\epsilon}{2} g - \frac{3g^3}{128\pi^3}$$

• 6D $\frac{dg}{d\ln\mu} = -\frac{3g^3}{128\pi^3} \quad \Rightarrow \quad \frac{dg^2}{d\ln\mu} = -\frac{3g^4}{64\pi^3}$

$$g^2(\mu') = \frac{g^2(\mu)}{1 + \frac{3}{64\pi^3} g^2(\mu) \ln \frac{\mu'}{\mu}}$$

- UV : $\ln \frac{\mu'}{\mu} \rightarrow \infty \implies g^2(\mu') \sim \frac{64\pi^3}{3 \ln \frac{\mu'}{\mu}} \rightarrow 0$

Asymptotic Freedom

- IR $g(\mu_*) = \infty$ for $\frac{3}{64\pi^3} g^2(\mu) \ln \frac{\mu_*}{\mu} = -1$

$$\mu_* = \mu e^{-\frac{64\pi^3}{3g^2(\mu)}}$$

dimensional
transmutation

△ Bare coupling (P.V.)

$$g_0^2 = \frac{g^2(\mu)}{1 + \frac{3}{64\pi^3} g^2(\mu) \ln \frac{\Lambda}{\mu}}$$

- $\left(g^2 \ln \frac{\Lambda}{\mu}\right)^n$ series of counter-terms makes sense
also when $g^2 \ln \frac{\Lambda}{\mu} \rightarrow \infty$

$$0 = \left[\sum_i \bar{\chi}_i + \mu \frac{d}{d\mu} \right] G$$

- $G = G(p_1, \dots, p_n, \lambda_a(\mu), \mu)$

$\lambda_a \equiv$ all parameters
 $\equiv$ all couplings &
 all masses

$$\left(\sum_i \bar{\chi}_i + \sum_a \beta_a \frac{\partial}{\partial \lambda_a} + \mu \frac{\partial}{\partial \mu} \right) G = 0$$