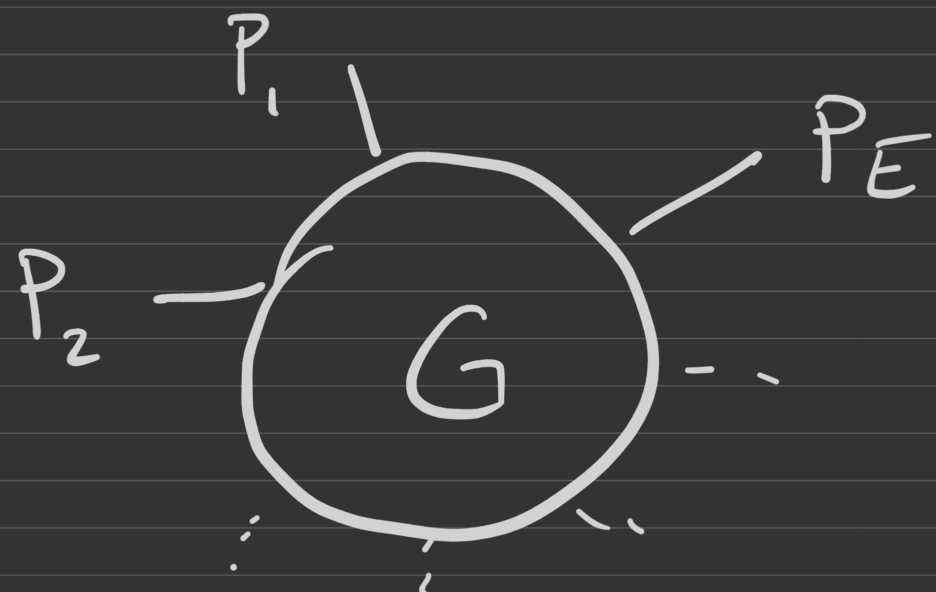
Power Counting loop integrals

- superficial degree of divergence of graph G : $\delta(G)$



$E = \#$ external legs

$V = \#$ vertices

$I = \#$ internal legs

$L = \#$ loops $\qquad L = I - V + 1$

$$G = \int \prod_{\ell=1}^{L} d^D k_\ell \prod_{a=1}^{I} \Delta(p_a) \prod_{\alpha=1}^{V} f_\alpha(p_{a_\alpha})$$

$$\sim K^{LD} \frac{dK}{K} K^{-2I} K^{\sum_\alpha d_\alpha} = K^{\delta(G)} \frac{dK}{K} \qquad \sum d_\alpha \equiv d_v$$

$$\boxed{\delta(G) = LD - 2I + d_v}$$

- $L = 1$ $\quad \delta(G) < 0 \iff G$ is UV finite

- $L > 1$ $\quad$ there can be "subdivergences" $\longrightarrow$ more tricky

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- $n_\alpha \equiv$ # legs in vertex α $\quad \Longrightarrow \quad \sum_\alpha' n_\alpha = E + 2I$

- $\underbrace{L = I - V + 1}$

$(\partial\phi)^2$

$[\phi] = \frac{D-2}{2}$

$$\Longrightarrow \delta(G) = D - \left(\frac{D-2}{2}\right) E - \sum_\alpha \left(-\frac{D-2}{2} n_\alpha - d_\alpha + D \right)$$

$$\equiv D - \Delta_\phi E - \sum_\alpha \Delta_\alpha \qquad = D - \sum_{e=1}^{E} \Delta_e - \sum_\alpha \Delta_\alpha$$

α-vertex $\longrightarrow \left[g_\alpha \, \partial^{d_\alpha} \phi^{n_\alpha} \right] = \delta$

$$[g_\alpha] = D - d_\alpha - \frac{D-2}{2} n_\alpha$$

- if $\Delta_\alpha \geq 0 \quad \forall \alpha \implies \delta(G) < 0$ for E large enough

On notation

$$I = \int d^4x\, d^4y\ \phi(x)\phi(y)\, G(x-y)$$

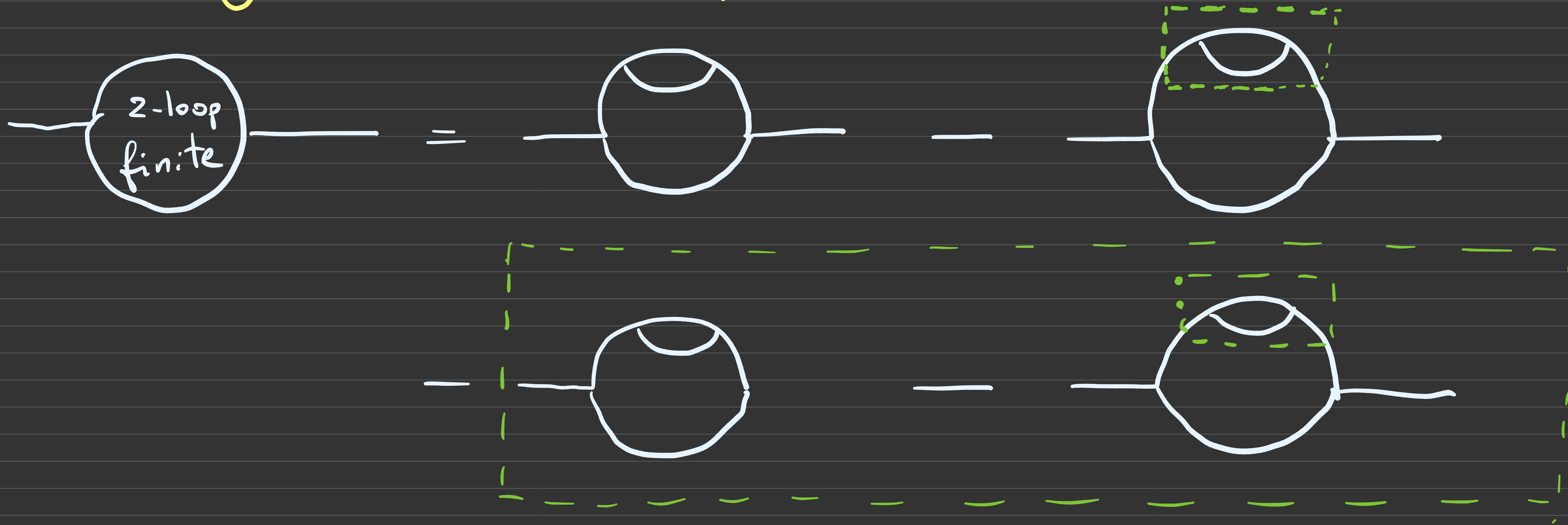
$$= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4}\ \hat{\phi}(p_1)\hat{\phi}(p_2)\, G(p_1, p_2)$$

$$G(p_1, p_2) = (2\pi)^4\, \delta(p_1+p_2)\, G(p_1)$$

$$= \int \frac{d^4p}{(2\pi)^4}\ \hat{\phi}(p)\hat{\phi}(-p)\, G(p)$$

$$G(p_1, p_2) \equiv (2\pi)^4 \frac{\delta}{\delta\hat{\phi}(p_1)} (2\pi)^4 \frac{\delta}{\delta\hat{\phi}(p_2)}\, I$$

Algorithmic formulation



⚠ $\Gamma(\phi) \Longrightarrow \underbrace{1PI}_{\text{1-particle-irreducible}}$ n-point functions

Zinn-Justin

$$\frac{\delta^2 \Gamma}{\delta\phi_a \,\delta\phi_b} = \Gamma_{ab} = \left(W^{(2)}\right)^{-1}_{ab}$$

ex 1 scalar $\quad \Gamma = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \hat{\phi}(-p)\, \Gamma(p)\, \hat{\phi}(p) + O(\hat{\phi}^3) \cdots$

$$(2\pi)^8 \frac{\delta^2\Gamma}{\delta\hat{\phi}(p_1)\,\delta\hat{\phi}(p_2)} = (2\pi)^4\, \delta(p_1+p_2)\, \Gamma(p_1)$$

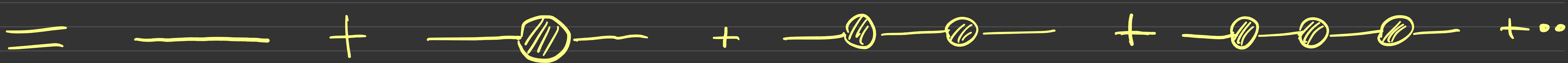
- In pert theory $\quad \Gamma(p) = p^2 + m^2 + \Sigma(p^2)$

$\hookrightarrow$ loops

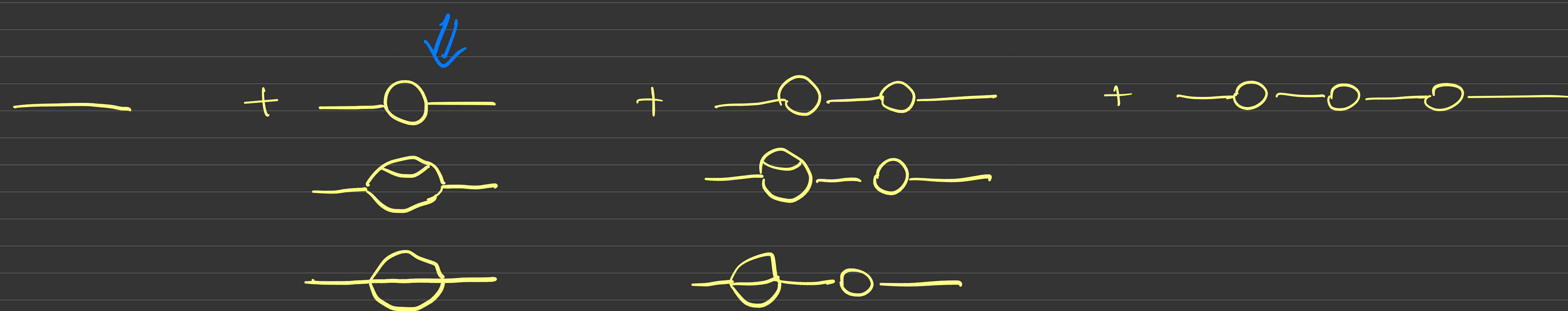
$$G(p) \equiv W^{(2)}(p) = \Gamma(p)^{-1} = \frac{1}{p^2+m^2+\Sigma(p^2)} =$$

$$= \frac{1}{p^2+m^2}\left(\frac{1}{1+\frac{\Sigma}{p^2+m^2}}\right) \qquad G_0(p) = \frac{1}{p^2+m^2}$$

$$= G_0\left(\frac{1}{1+\Sigma G_0}\right) = G_0 - G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \cdots$$



Ex



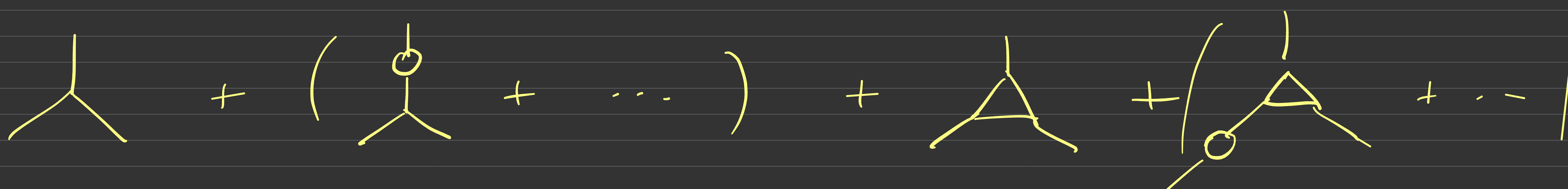
$\Rightarrow \quad -G_0 \Sigma G_0 =$ 

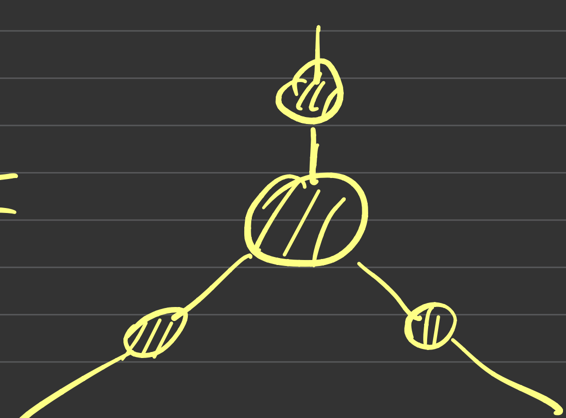
$\Sigma = -$  $=$ 1 particle irreducible 2-pt function

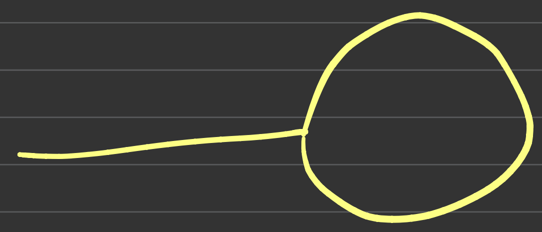
$$-\Gamma^3(p_1, p_2, p_3) = -(2\pi)^{12} \frac{\delta^3 \Gamma}{\delta\hat{\phi}(p_1)\,\delta\hat{\phi}(p_2)\,\delta\hat{\phi}(p_3)} \qquad -W^3 = G(p_1)G(p_2)G(p_3)\Gamma^{(3)}$$

$$= +\, G^{-1}(p_1)\, G^{-1}(p_2)\, G^{-1}(p_3)\, W^{(3)}(p_1, p_2, p_3)$$

$=$  $\equiv$ all loops that correct 3-pt function stripped of external legs



$=$ 



$$= \frac{g}{2}\int \frac{d^d \ell}{\ell^2+m^2} \xrightarrow{m=0} \int \frac{d^d \ell}{\ell^2} = 0$$

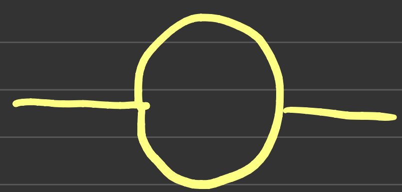
$$[\phi^3] = \phi^3 - c\phi$$

$$\delta(G) \equiv D - \Delta_\phi E - \sum_\alpha \Delta_\alpha$$

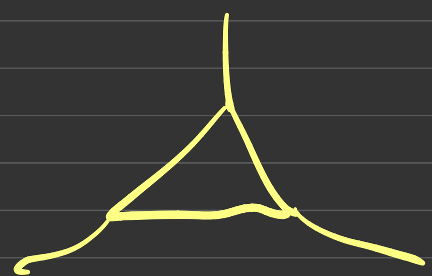
$(\partial\phi)^2$

$g\phi^3$

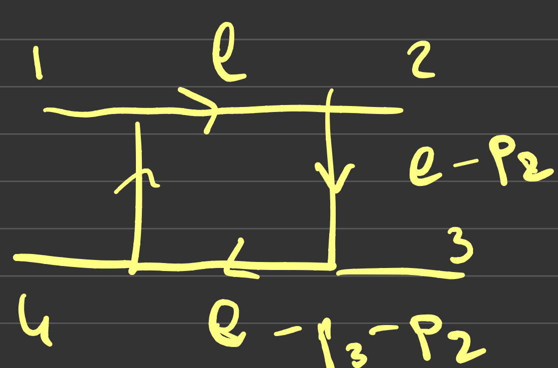
$\hookrightarrow \Delta_g = 0$



$$= 6 - 2\cdot 2 - 0 = 2$$



$$= 6 - 2\cdot 3 - 0 = 0$$



$$= 6 - 2\cdot 4 - 0 = -2$$ convergent

$\hookrightarrow$

$$\int \frac{d^6\ell}{(2\pi)^6} \frac{1}{(\ell+p_1)^2} \cdot \frac{1}{\ell^2} \cdot \frac{1}{(\ell-p_2)^2} \frac{1}{(\ell-p_2-p_3)^2}$$

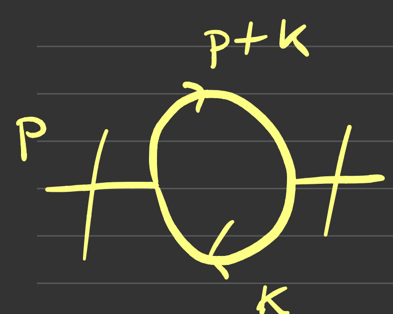
$$\sim \int d^6\ell \, \frac{1}{\ell^8} \longrightarrow \text{convergent}$$

Towards the general theory of renormalization

ϕ^3 in 6D $\qquad \mathcal{L}_R = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi^3$

in $d = 6-\varepsilon$

$[\phi] = \frac{1}{2}(d-2) \Rightarrow [g] = d - 3/2(d-2) = 3 - \frac{d}{2} = \frac{\varepsilon}{2} \Rightarrow g \to g\mu^{\varepsilon/2}$



$$= \frac{g^2\mu^\varepsilon}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k+p)^2} =$$

$$= \frac{g^2\mu^\epsilon}{2}\left(\frac{1}{4\pi}\right)^{d/2}\frac{\Gamma(2-d/2)\Gamma(\frac{d}{2}-1)^2}{\Gamma(2)\,\Gamma(d-2)}(p^2)^{\frac{d}{2}-2}$$

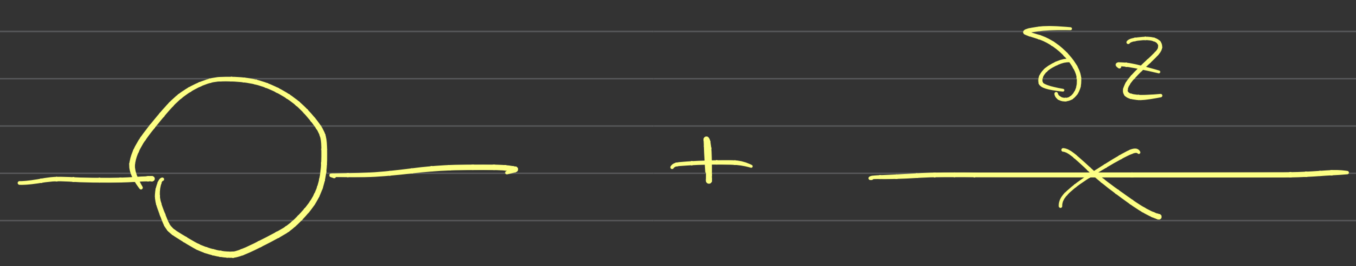
$$p^2(p^2)^{\frac{d}{2}-3} = p^2 p^{-\frac{\varepsilon}{2}}$$

$$p^2 p^{-\varepsilon/2} \simeq p^2(1 - \varepsilon/2 \ln p^2 + O(\varepsilon^2))$$

$$= \frac{g^2 p^2}{6(4\pi)^3}\left[\frac{1}{d-6} + \frac{1}{6}(3\gamma-8) + \frac{1}{2}\ln\frac{p^2}{4\pi\mu^2}\right]$$

$$-\delta Z^{(1)}\, p^2$$

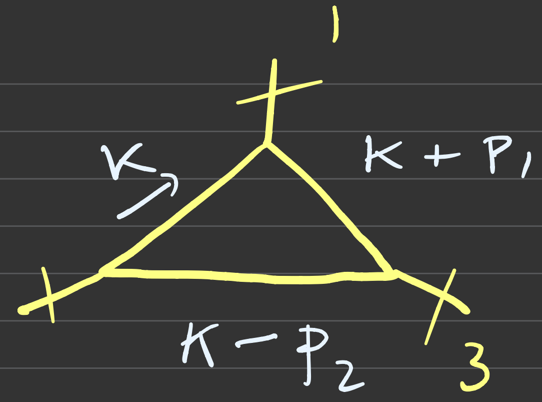
$$\delta\mathcal{L}^{(1,2)}_{cT} = \delta Z^{(1)}\,\frac{(\partial\phi)^2}{2} \qquad \delta Z^{(1)} = \underbrace{\frac{g^2}{6(4\pi)^3}\cdot\frac{1}{d-6}}$$



$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+p)^2} = \left(\frac{1}{4\pi}\right)^{d/2} \frac{\Gamma\left(2-d/2\right)\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(2)\,\Gamma(d-2)} \left(p^2\right)^{\frac{d}{2}-2}$$

- UV $\longrightarrow \Gamma\left(2-d/2\right) \longleftarrow \int_{k>p} \frac{k^{d-1}\,dk}{k^4} \sim \int k^{d-4}\frac{dk}{k} \qquad d-4 \geq 0$

- IR $\longrightarrow \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \longleftarrow \int_{k<p} \frac{k^{d-1}\,dk}{k^2\quad p^2} \sim \int k^{d-2}\frac{dk}{k} \qquad d \leq 2$

- 

$$-g^3\mu^{\frac{3}{2}\epsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}\,\frac{1}{(k+p_1)^2}\,\frac{1}{(k+p_2)^2}$$

$p_1+p_2+p_3=0$

$$= -\frac{g^3\mu^{3/2\,\epsilon}}{(4\pi)^{d/2}}\,\Gamma\left(3-\frac{d}{2}\right)\cdot\int\frac{\delta(1-x-y-z)\,dx\,dy\,dz}{\left(p_1^2xz+p_2^2yz+p_3^2xy\right)^{3-d/2}}$$

$$= -\frac{g^3\mu^{\epsilon/2}}{64\pi^3}\left[\frac{1}{6-d}+\text{finite}\right]$$

CT =  $= +\frac{g^3\mu^{\epsilon/2}}{64\pi^3}\,\frac{1}{6-d} \Longrightarrow \delta\mathcal{L}^{(1,3)}_{CT} = -\frac{1}{3!}\left(\frac{g^3\mu^{\epsilon/2}}{64\pi^3}\,\frac{1}{6-d}\right)\phi^3$

At 1-loop

$$\mathcal{L}_{tot} = \frac{1}{2}(\partial\phi)^2 + \frac{g\mu^{\epsilon}}{3!}\phi^3 + \underline{\underline{\delta\mathcal{L}_{CT}^{(1,2)}}} + \delta\mathcal{L}_{CT}^{(1,3)}$$

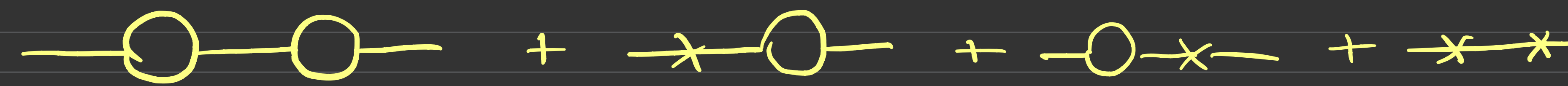
$$= \frac{1}{2}\left(1 + \frac{g^2}{6(4\pi)^3}\frac{1}{d-6}\right)(\partial\phi)^2 + \frac{g\mu^{\epsilon/2}}{3!}\left(1 + \frac{g^2}{(4\pi)^3}\frac{1}{d-6}\right)\phi^3$$

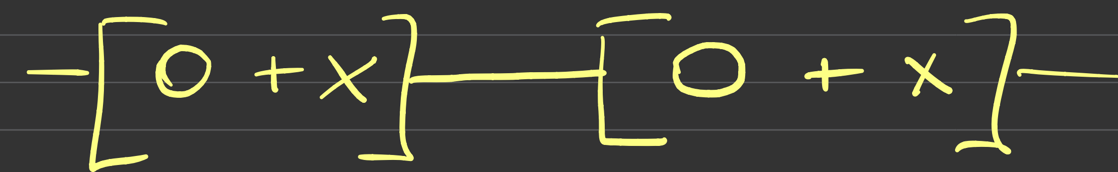
$$\equiv \frac{1}{2}Z_1(\partial\phi)^2 + \frac{g_0 Z^{3/2}}{3!}\phi^3$$

$$\phi\sqrt{Z} = \phi_0$$

$$\equiv \frac{1}{2}(\partial\phi_0)^2 + \frac{g_0}{3!}\phi_0^3$$

2-pt function at 2 loops

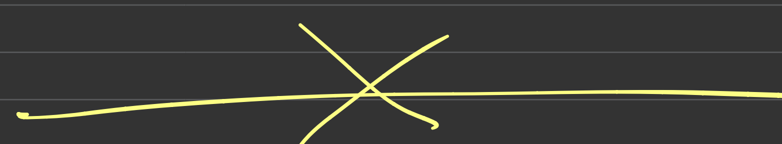
- non 1PI 

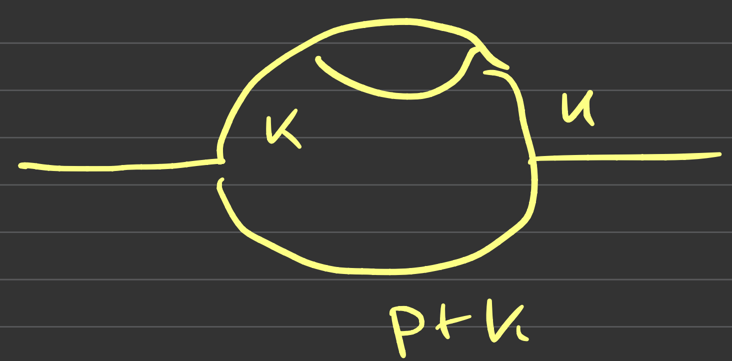
  $=$ 

$\Rightarrow$ 1-loop 1PI finite $\longrightarrow$ 2-loop non-1PI finite

- 2-loop 1PI diagrams
  -  "nested"
  -  "overlapping"

Nested $\Longrightarrow$

 $-\delta Z\, p^2$



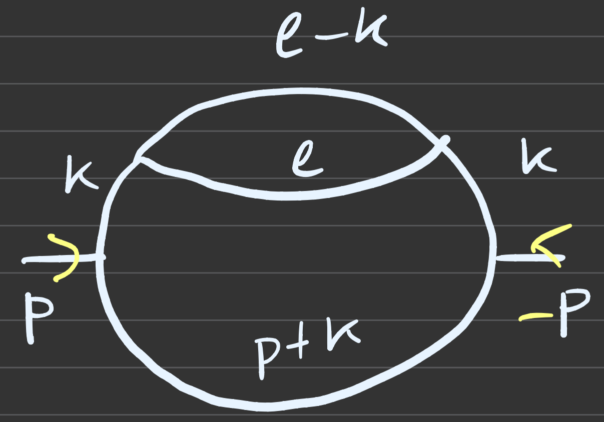
$$\sim \int d\sigma_k \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \quad k^2 \ln \frac{\Lambda^2}{k^2}$$

$$= \int d\sigma_k \frac{1}{k^2 (p+k)^2} \quad \ln \frac{\Lambda^2}{k^2}$$

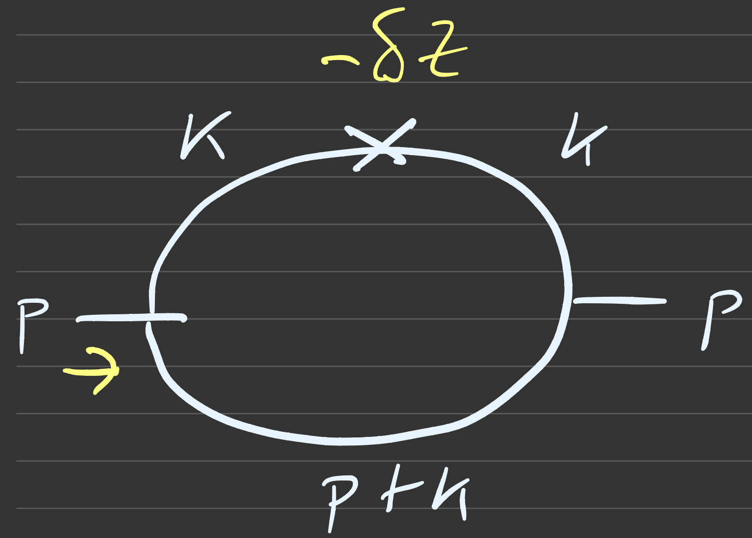
$$\sim \left(\ln \frac{\Lambda^2}{p^2}\right)^2 = \left(\ln \Lambda^2 - \ln p^2\right)^2$$

$\Longrightarrow$ non-local div

$$d\sigma = \frac{d^d k}{(2\pi)^d}$$



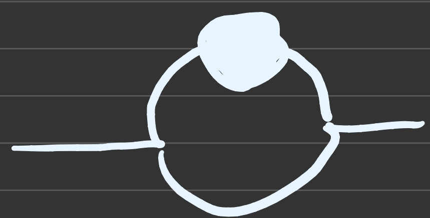
$$\frac{g^4 \mu^{2\epsilon}}{2} \int d\sigma_k \, d\sigma_e \left(\frac{1}{K^2}\right)^2 \frac{1}{(p+k)^2} \, \frac{1}{e^2} \, \frac{1}{(e-k)^2}$$



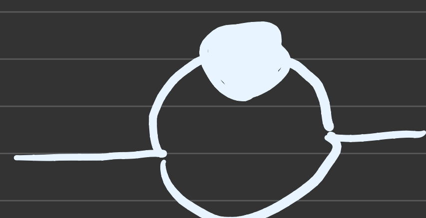
$$g^2 \mu^{\epsilon} \int d\sigma_k \left(\frac{1}{K^2}\right)^2 \frac{1}{(p+k)^2} \left(-\frac{g^2}{6(4\pi)^3} \frac{K^2}{d-6}\right)$$

$$= g^4 \mu^{\epsilon} \int d\sigma_k \left(\frac{1}{K^2}\right)^2 \frac{1}{(p+k)^2} \left[\underbrace{\left(\frac{\mu^{\epsilon}}{2} \int d\sigma_e \frac{1}{e^2 (e-k)^2}\right) - \frac{K^2}{6(4\pi)^3 (d-6)}}_{-\Sigma'(K) \,\equiv\, \text{finite}} \right]$$

• explicit computation: $\Sigma(k) = k^2 \left(A \ln k^2 + B\right)$

 $= g^4 \mu^\varepsilon \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right) \frac{1}{(p+k)^2} \left(A \ln k^2 + B\right)$

diverges for $d \longrightarrow 6$

• ∂_p^3  $\sim \int d^d k \, \frac{1}{k^2} \frac{1}{(p+k)^5} \left(A \ln k^2 + B\right)$

$\overset{d \to 6}{\sim} \int \frac{dk}{k^2} \ln k^2 \longrightarrow$ finite

$\Longrightarrow$ divergent part is a degree 2 polynomial in p $\equiv \boxed{\text{local}}$

 $+$  $=$

$$\frac{p^2}{2} \cdot \frac{g^4}{(4\pi)^6} \cdot \left\{ -\frac{1}{18}\frac{1}{\varepsilon^2} + \frac{11}{216}\frac{1}{\varepsilon} + \frac{\left(791 - 324\gamma + 36\gamma^2 - 324 \cdot \mathrm{Log} + 72\gamma\, \mathrm{Log} + 36\, \mathrm{Log}^2\right)}{2592} \right\}$$

$$\mathrm{Log} \equiv \ln \frac{p^2}{4\pi\mu^2}$$

$m = 0$ + Lorentz $\Longrightarrow$ divergent part $\propto p^2$

☆ Comments

A) 2-loop counterterm contains $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^2}$ terms: lowest instance of general result.

- at n-loop order $\longrightarrow \frac{1}{\epsilon}, \frac{1}{\epsilon^2}, \ldots, \frac{1}{\varepsilon^n}$ terms

- $\frac{1}{\varepsilon} \longleftrightarrow \ln \Lambda$ in P.V. or hard-cut off

- $(\ln \Lambda)^n \Longrightarrow$ hierarchically ordered loop momenta

$$k_1 \gg k_2 \gg \ldots \gg k_n$$

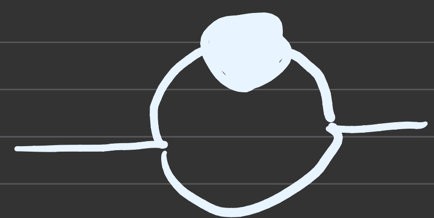
B)

$$\cancel{\text{divs}} \left\{ \begin{array}{lll} \frac{1}{\varepsilon}, \ \ln \lambda & \Longleftrightarrow & \ln p \\ \frac{1}{\varepsilon^2}, \ (\ln \lambda)^2 & \Longleftrightarrow & (\ln p)^2 \end{array} \right]$$

finite terms

$A + B \equiv$ Weinberg Theorem

in hard-momentum cut-off

 $$= g^4 \mu^{\varepsilon} \int^{\Lambda} \frac{d^6 k}{(2\pi)^d} \left(\frac{1}{k^2}\right) \frac{1}{(p+k)^2} \left(A \ln k^2 + B\right)$$

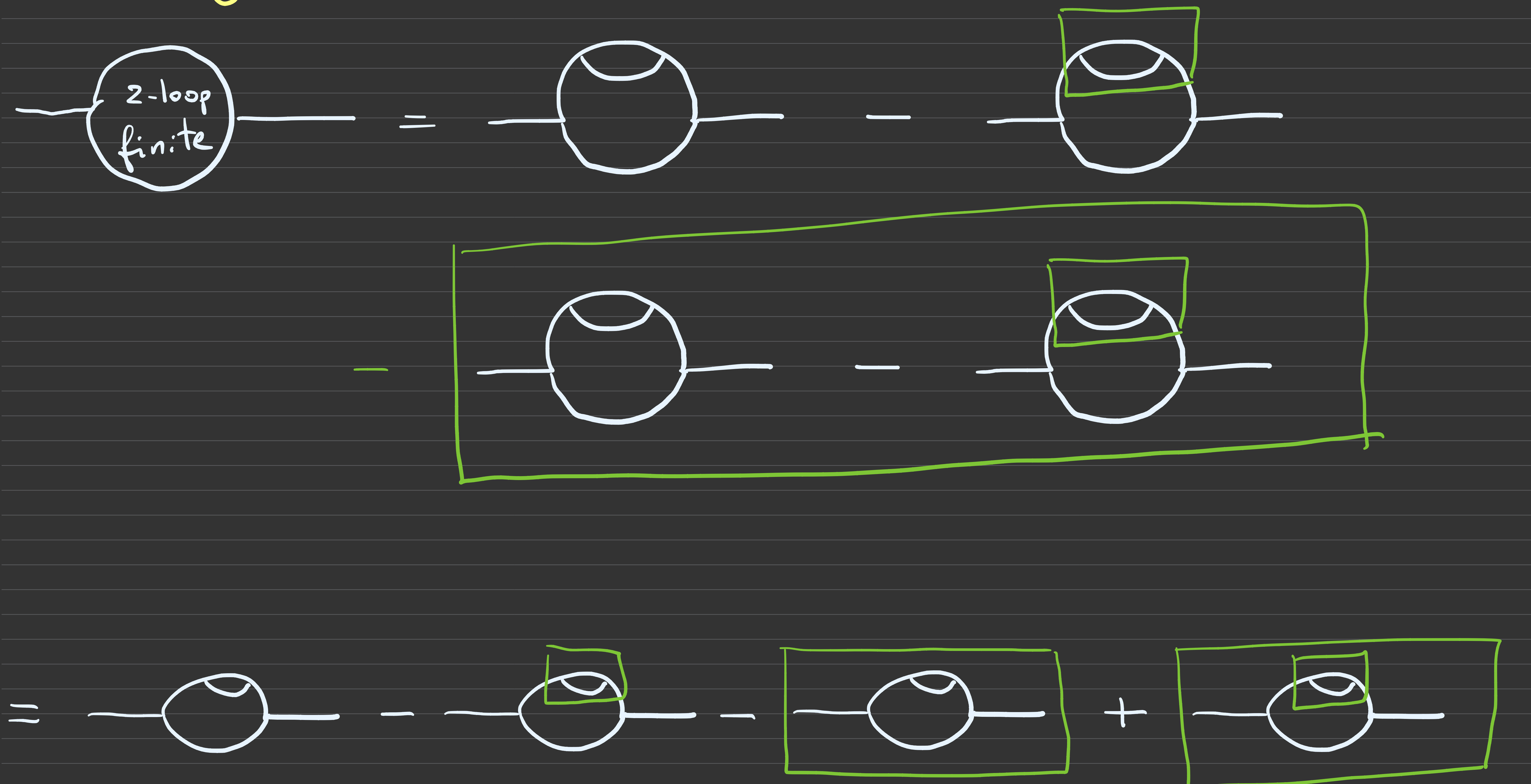
∂_p^2  $$\sim \int^{\Lambda} d^6 k \, \frac{1}{k^2 (k+p)^4} \ln k^2/\mu^2$$

$$\xrightarrow{\text{P.V.}} \int_{\mu}^{\Lambda} \frac{d^6 k}{k^6} \ln k^2/\mu^2 \sim \int_{\mu}^{\Lambda} \frac{dk}{k} \ln \frac{k}{\mu}$$

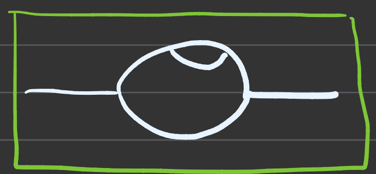
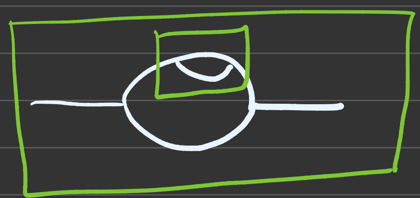
$$\frac{dk}{k} \equiv d \ln k/\mu$$

$$\sim \int_0^{\ln \Lambda/\mu} \left(d \ln \frac{k}{\mu}\right)\left(\ln \frac{k}{\mu}\right) = \frac{1}{2}\left(\ln \Lambda/\mu\right)^2$$

▲ Algorithmic formulation

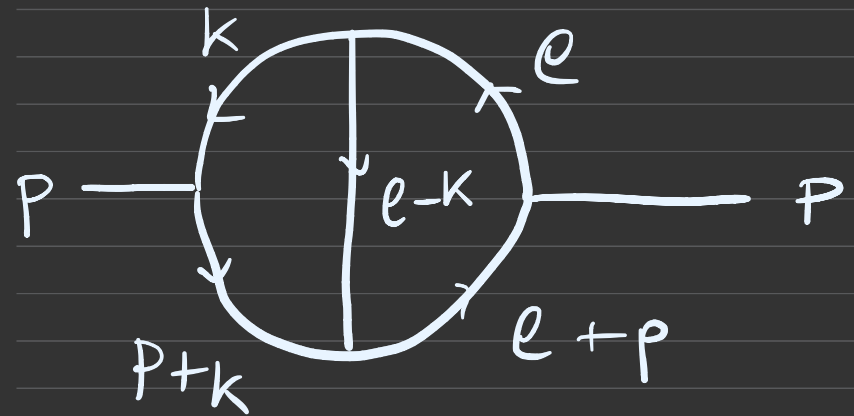


first instance of the Forest Formula

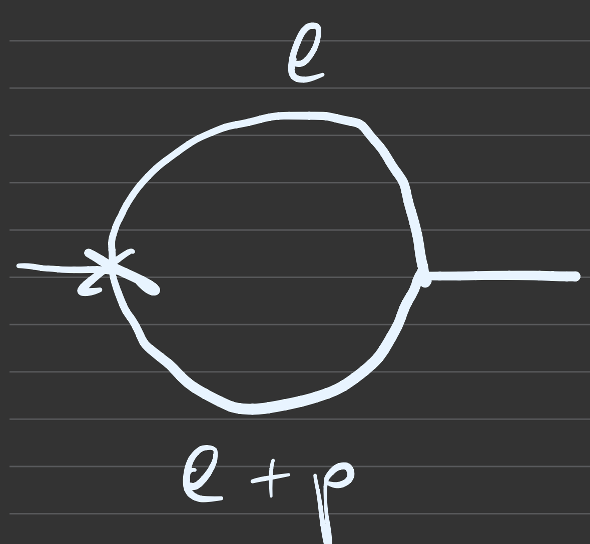
$-$  $+$  $$= + \underbrace{\frac{1}{4}\frac{g^4}{(4\pi)^6}\left(\frac{1}{18}\frac{1}{(d-6)^2} + \frac{11}{216}\frac{1}{d-6}\right)}_{-\delta Z^{(2)}} P^2$$

$$\Delta \mathcal{L}^{(2,2)}_{CT} = \frac{1}{2}\,\delta Z^{(2)} (\partial\phi)^2$$

Overlapping $\Longrightarrow$



$$= \frac{g^4 \mu^{2\varepsilon}}{2} \int d\mu_k \, d\mu_\ell \, \frac{1}{k^2} \, \frac{1}{(p+k)^2} \, \frac{1}{(\ell-k)^2} \, \frac{1}{\ell^2} \, \frac{1}{(\ell+p)^2}$$



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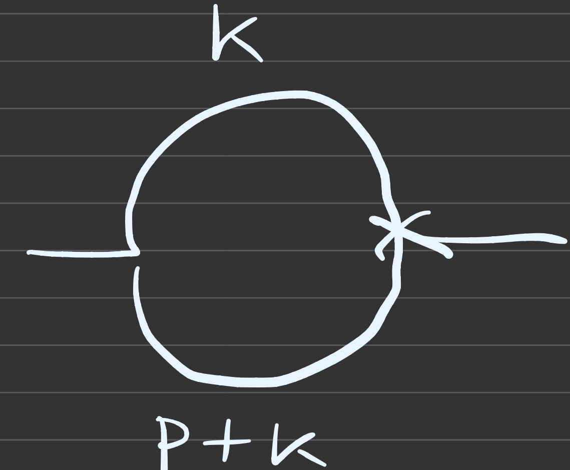
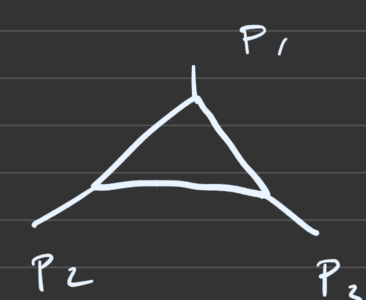


$$= \frac{\overset{\sim g^4 \mu^{\varepsilon}}{\overset{\Downarrow}{g \, \delta g \, \mu^{\varepsilon/2}}}}{2} \left[\int d\sigma_\ell \, \frac{1}{\ell^2} \, \frac{1}{(\ell+p)^2} + \int d\sigma_k \, \frac{1}{k^2} \, \frac{1}{(k+p)^2} \right]$$

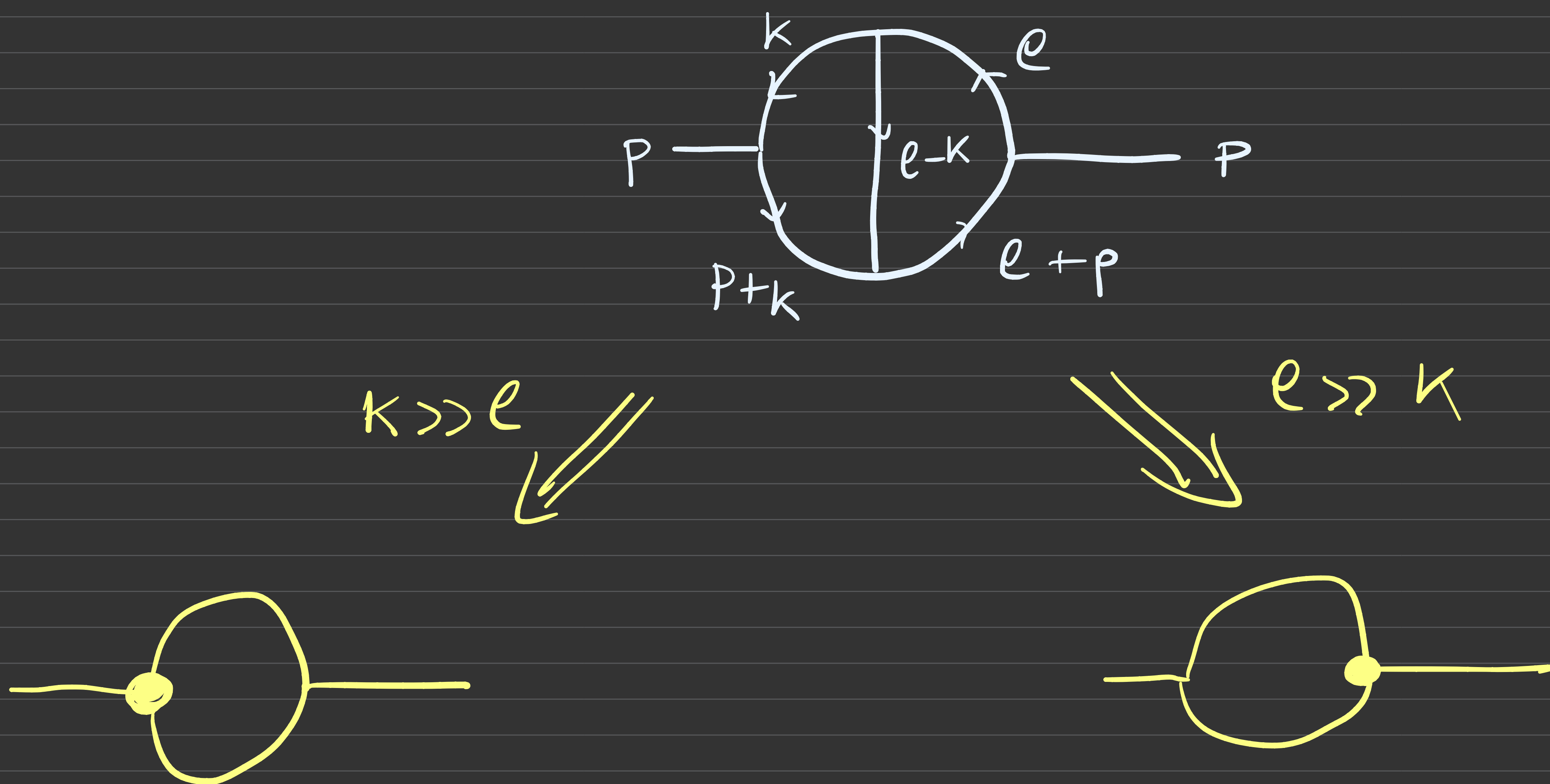
$$\delta g = \frac{g^3 \mu^{\varepsilon/2}}{(4\pi)^3} \, \frac{1}{d-6}$$



$+$

$$\propto g^3 \left(A \ln p + B \right)$$

Subdivergences

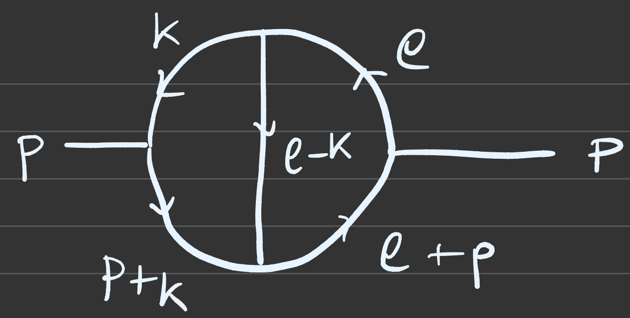


• Concretely :  $= \int\limits_{|k|>|l|} d\sigma_k \, d\sigma_l \, (\dots) + \int\limits_{|l|>|k|} d\sigma_k \, d\sigma_l \, (\dots)$

$\Downarrow$

$$\int_{|k| \gg |\ell|} + \int_{|k| \sim |\ell|} + \int_{|\ell| \gg |k|}$$

<u>Subs</u> $\qquad$ $\underline{\underline{\cancel{\text{sup}}}}$ $\qquad$ Subs



$$= \int d\mu_k \, d\mu_\ell \, \frac{1}{k^2} \frac{1}{(p+k)^2} \frac{1}{(\ell-k)^2} \frac{1}{\ell^2} \frac{1}{(\ell+p)^2}$$

$$d\mu_k = \frac{d^6 k}{(2\pi)^6}$$

- $\ell \gg k \gg p$

$$= \sim \int d\sigma_k \frac{1}{k^2 (p+k)^2} \int_{\ell \gg k,p} d\sigma_\ell \frac{1}{\ell^6} \left(1 + \frac{k\cdot p}{\ell^2} + \frac{k^2}{\ell^2} \cdots\right)$$

$$\int_k^\Lambda \frac{d\ell}{\ell}$$

$$\sim \int d\sigma_k \left(\frac{1}{k^2}\right)^2 \left(1 - \frac{2p\cdot k}{k^2} - \frac{p^2}{k^2} + \cdots\right) \times \left(\ln \Lambda/k + \cdots\right)$$

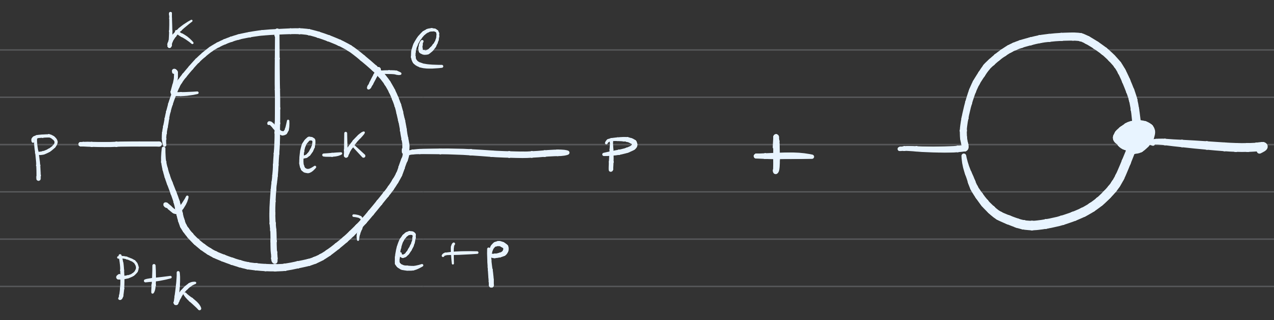
$$\sim \int k \, dk \left(1 - \frac{p^2}{k^2} + \cdots\right)\left(\ln \frac{\Lambda}{k} + \frac{k^2}{\Lambda^2} + \cdots\right)$$

$$\sim \int^{\Lambda} k \ln \frac{\Lambda}{k}\, dk \quad - \quad p^2 \int_{k \gtrsim p}^{\Lambda} \frac{dk}{k} \ln \frac{\Lambda}{k}$$

$$\# \Lambda^2 \quad + \quad \# p^2 \left(\ln \Lambda^2/p^2 \right)^2$$

$$p^2 \left(\ln \Lambda^2 - \ln p^2 \right)^2$$

$$p^2 \left((\ln \Lambda^2)^2 + (\ln p^2)^2 - 2 \ln \Lambda^2 \ln p^2 \right)$$



$$\sim \int d\sigma_{k_1} \left(\frac{1}{k^2}\right)^2 \left(1 - \frac{2p\cdot k}{k^2} - \frac{p^2}{k^2} + \dots\right) \times \ln \frac{\mu}{k}$$

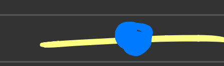
$$\Downarrow$$

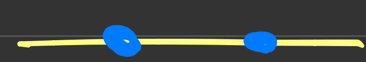
$$p^2 \int_{p}^{\Lambda} \frac{dk}{k} \ln\frac{\mu}{k} = p^2 \int_{\ln p/\mu}^{\ln \Lambda/\mu} \left(d \ln k/\mu\right) \ln k/\mu$$

$$= p^2 \left[\frac{1}{2}\left(\ln\frac{k}{\mu}\right)^2\right]_{\ln p/\mu}^{\ln \Lambda/\mu} = \frac{p^2}{2}\left[\left(\ln\frac{\Lambda}{\mu}\right)^2 - \left(\ln\frac{p}{\mu}\right)^2\right]$$

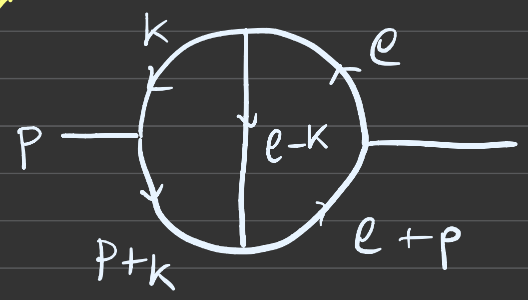
The derivative trick

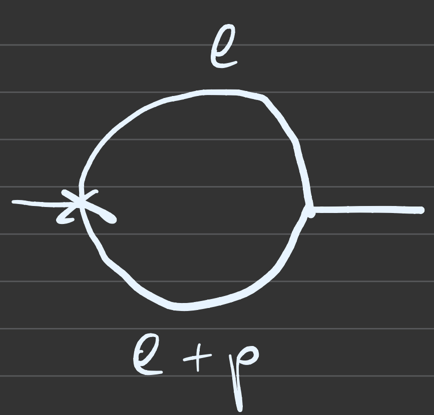
Idea

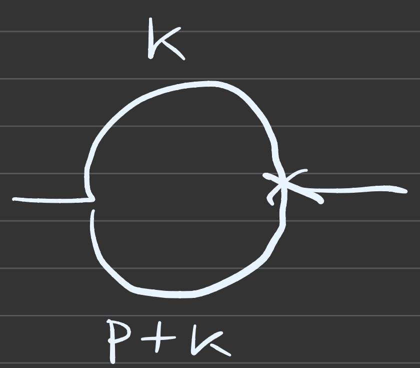
- $\partial_p \dfrac{1}{p^2} \sim \dfrac{1}{p^3} =$ 

- $\partial_p^2 \dfrac{1}{p^2} \sim \dfrac{1}{p^4} =$ 

- ∂_p^n improves UV convergence

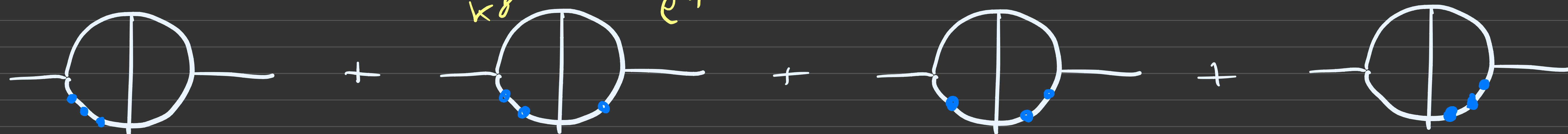
$\partial_p^3 \Big($ 

$+$ 

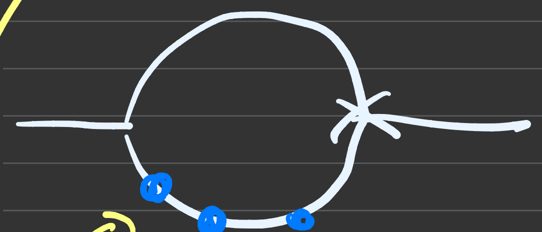
$+$ 

$\Big) =$

$\frac{d^6k}{k^8}$ $\frac{d^6\ell}{\ell^7}$



even negleting vertical propagator



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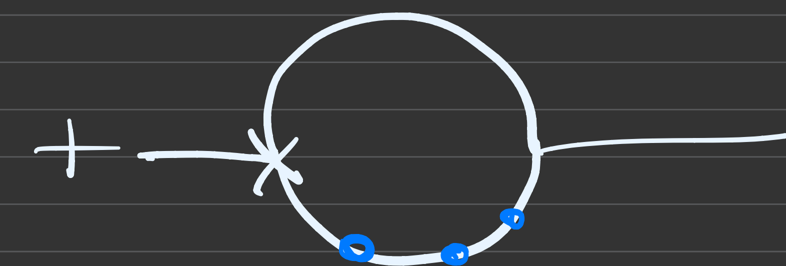

$\frac{d^6k}{k^7}$

finite finite finite finite

$\Rightarrow$ divergent part $\equiv$ quadratic polynomial in p $\equiv$ <u>local</u>

In terms of forest formula