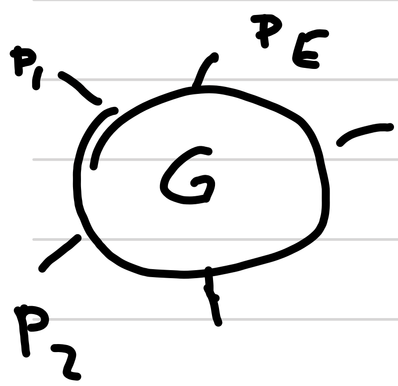
- **Power counting: basics**

- superficial degree of divergence of a graph G : $\delta(G)$



$E \equiv$ # external legs
$V =$ # vertices
$I =$ # propagators
$L =$ # loops

$$L = I - V + 1$$

$$G = \int \prod_{l=1}^{L} d\mu_l \prod_{a=1}^{I} \Delta(p_a) \prod_{\alpha=1}^{V} \underbrace{f_\alpha(p_{a(\alpha)})}_{\text{polynomial degree } d_\alpha}$$

considering the region of integration where all $k_l \to \infty$ together $k_l \propto k$

$$G \sim k^{LD} dk \; k^{-2I} \; k^{\sum_\alpha d_\alpha}$$

$$\sum_\alpha d_\alpha \hookrightarrow d_V$$

$$\delta(G) = LD + d_V - 2I$$

- In the case of $L=1$, $\delta(G)<0$ is the necessary and sufficient condition for the finiteness of G. For $L>1$ the situation is made more complicated by the existence of sub-divergences.

Elaborating

$$\delta = D(I-V+1) + \sum_\alpha d_\alpha - 2I$$

$$= D + (D-2)I + \sum_\alpha (d_\alpha - D)$$

$$2I + E = \sum_\alpha n_\alpha$$

$n_\alpha \longrightarrow$ legs ending in vertex α

$$I = -\frac{E}{2} + \frac{1}{2}\sum_\alpha n_\alpha$$

$$\delta = D - \frac{(D-2)}{2}E + \sum_\alpha \left(\frac{D-2}{2}n_\alpha + d_\alpha - D\right)$$

$$\delta(G) = D - \Delta_\phi E - \sum_\alpha \Delta_\alpha$$

$$g_\alpha \, \partial^{d_\alpha} \phi^{n_\alpha}$$

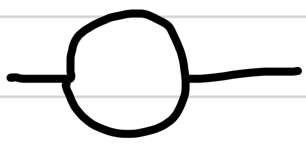
$$\Delta_\alpha \equiv [g_\alpha] = D - d_\alpha - \frac{D-2}{2} n_\alpha$$

if $\Delta_\alpha \geq 0$ for all vertices,

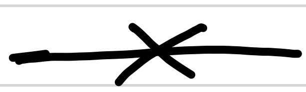
$\delta(G) < 0$ for sufficiently large E

• Towards the general theory of renormalization

2-loop in ϕ^3 in 6D

 $\frac{g^2\mu^{\epsilon}}{2}\left(\frac{1}{4\pi}\right)^{d/2}\frac{\Gamma(2-d/2)\,\Gamma(d/2-1)^2}{\Gamma(2)\,\Gamma(d-2)}(p^2)^{1+\frac{d}{2}-3}$

$$= \frac{g^2 p^2}{6(4\pi)^3}\left[\frac{1}{d-6}+\frac{1}{6}(3\gamma-8)+\frac{1}{2}\ln\frac{p^2}{4\pi\mu^2}\right]$$

CT =  $= -\frac{g^2p^2}{6(4\pi)^3}\frac{1}{d-6}$

$$\mathcal{L}_{CT} \Rightarrow \delta Z\frac{(\partial\phi)^2}{2} \Rightarrow \delta Z = \frac{g^2p^2}{6(4\pi)^3}\frac{1}{d-6}$$

• Notice that, as "d" varies, the singularities are determined by the numerators as Γ has poles but no zeroes in the complex plane

Two classes of singularities:

$\bullet$ UV $\rightarrow \Gamma(2-d/2) \leftarrow \int d^dk \frac{1}{k^2(k+p)^2}$

$$\int \frac{t^{d/2-1}\,dt}{t^2} \rightarrow \frac{d}{2}-2$$

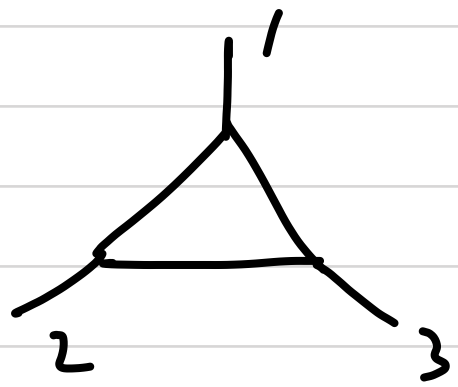
$\bullet$ IR $\rightarrow \left[\Gamma\left(\frac{d}{2}-1\right)\right]^2 \leftarrow \int d^dk_1 \frac{1}{k^2(k+p)^2}$

$$\int \frac{t^{d/2-1}\,dt}{t\,(\cdots)} \rightarrow 1-\frac{d}{2}$$

$$\frac{d}{2}-1 \leq 0 \qquad d=2 \qquad \Rightarrow \text{IR divergent}$$

Actually the denominator $\frac{1}{\Gamma(d-2)}$ ensures the IR div pole is a simple one

$\bullet$



$$-\frac{g^3\mu^{\frac{3}{2}\epsilon}}{(4\pi)^{d/2}}\Gamma(3-d/2) \times$$

$$\times \int \frac{\delta(1-x-y-z)\,dx\,dy\,dz}{(p_1^2xz+p_2^2yz+p_2^2xy)^{3-d/2}}$$

$$\rightarrow \text{pole part} = -\frac{g^3 \mu^{\varepsilon/2}}{64\pi^3}\frac{1}{6-d}$$

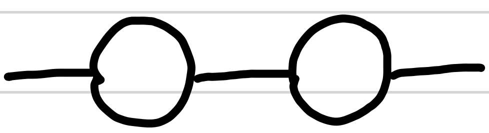
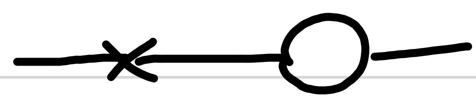
$$\Rightarrow \Delta\mathcal{L}_{CT} = -\left(\frac{g^3\mu^{\varepsilon/2}}{64\pi^3}\frac{1}{6-d}\right)\frac{\phi^3}{3!}$$

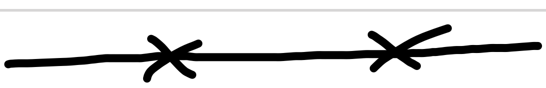
$$g \rightarrow \mu^{\varepsilon/2} g\left(1-\frac{g^2}{64\pi^3}\frac{1}{6-d}+\ldots\right)$$

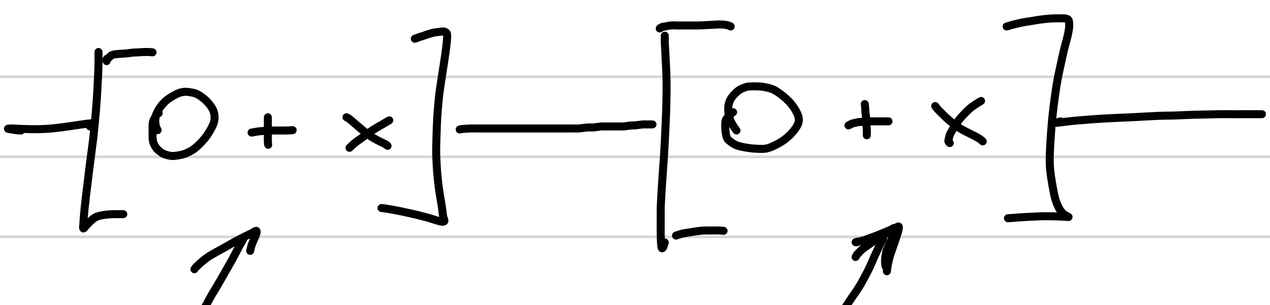
2-point function at 2-loops

Working with propagator we have

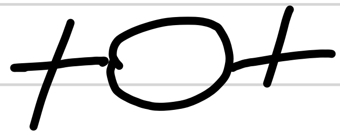
- non 1PI

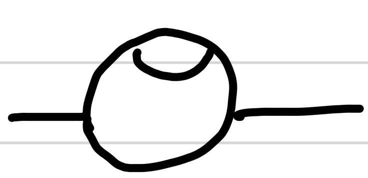
 + 

 + 

= 

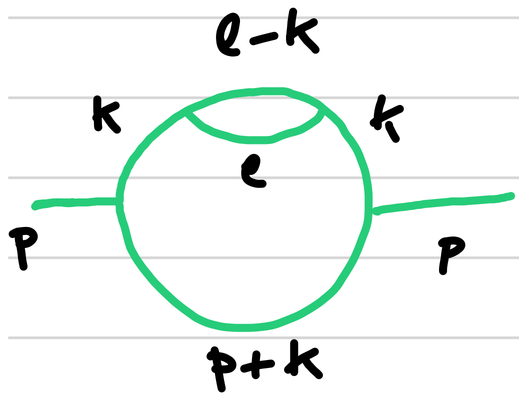
finite finite

- 1PI  +  = finite

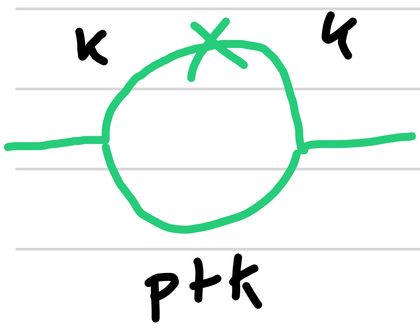
- Two classes of 1P1
  - nested 
  - overlapping 

$$d\sigma_k \equiv \frac{d^d k}{(2\pi)^d}$$

- We will now check that the nested diagram is renormalized by a <u>local</u> counterterm



$$\frac{g^4 \mu^{2\epsilon}}{2} \int d\sigma_k \, d\sigma_l \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \frac{1}{(l-k)^2} \frac{1}{l^2}$$



$$g^2 \mu^\epsilon \int d\sigma_k \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \left(-\frac{g^2}{6(4\pi)^3} \frac{k^2}{d-6}\right)$$

$$= g^4 \mu^\epsilon \int d\sigma_k \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \left[\left(\frac{\mu^\epsilon}{2} \int d\sigma_l \frac{1}{(l-k)^2 l^2}\right) - \frac{k^2}{6(4\pi)^3(d-6)}\right]$$

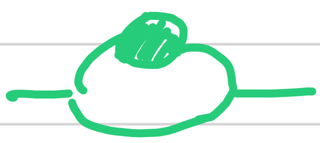
$= \Sigma(k) \equiv$ finite !

- from the explicit computation $\epsilon \to 0$

$$\Sigma(k) = k^2 \left(A \ln k^2 + B\right)$$

$$\downarrow$$

$$-\frac{1}{\epsilon}\left(k^{-2\epsilon} - 1\right)$$

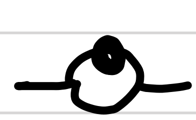
 $= g^4 \mu^{\varepsilon} \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right) \frac{1}{(p+k)^2} \left(A \ln k^2 + B\right)$

this integral is divergent for $d \to 6$ (in fact for $d \to 4, 6$)

However its $(\partial_k)^3$ derivative is a well defined integral as $d-$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(p+k)^5} \left(A \ln k^2 + B\right)$$

$$\overset{d\to 6}{\sim} \int_{k \gg p}^{\infty} \frac{dk}{k^2} \ln k^2 \longrightarrow \text{finite}$$

Therefore we conclude that the divergent part of  must be a polynomial of degree 2 in p. By Lorentz inv and dimensional analysis $(m=0)$ it can only be $\propto k^2$.

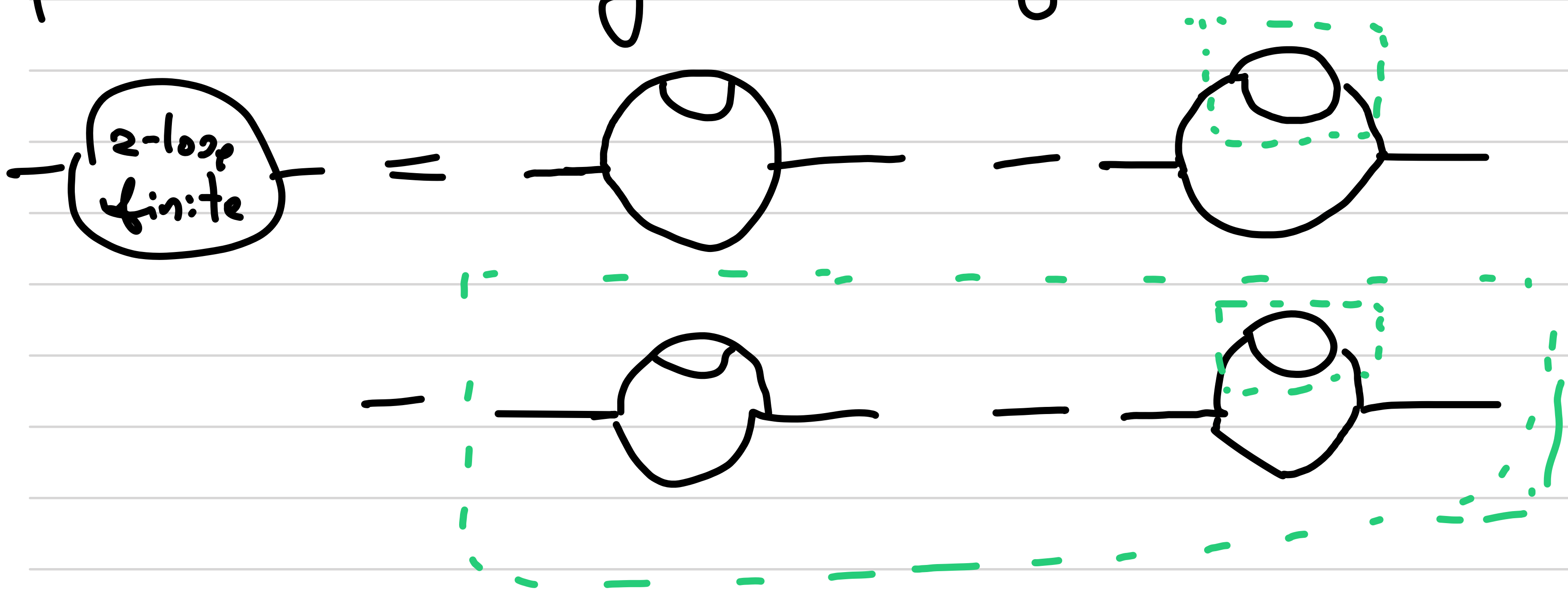
· A few comments on the final result are in order

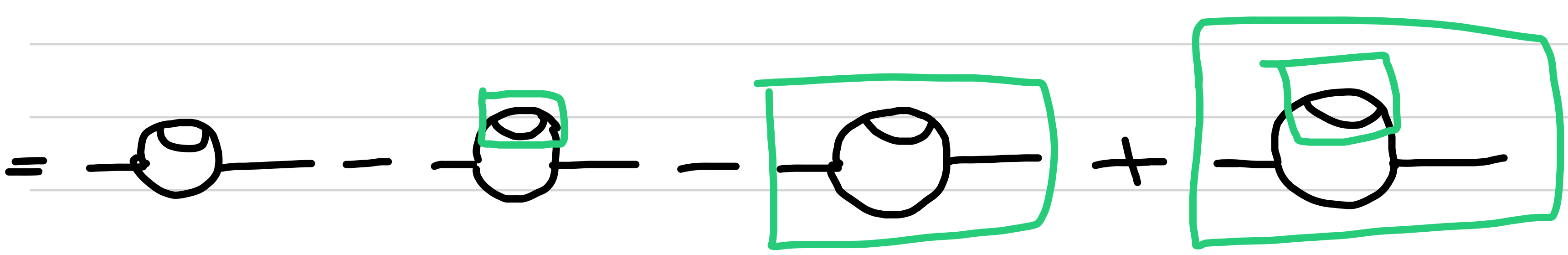
- The 2-loop counterterm contains single as well double $1/\varepsilon$ poles. That is indeed the lowest iteration of a general property of n-loop diagrams: the corresponding counterterms are polynomials of degree n in $1/\varepsilon$.
Had we used a regulator like P.V. the n-loop counterterm would have been a degree-n polynomial in $\ln \Lambda/\mu$. The $(\ln\Lambda)^n$ originates from the integration region where the momenta of the n-loops are hierarchically ordered $k_1 \gg k_2 \gg \dots \gg k_n$, for subdiagrams satisfying $\gamma_1 \subset \gamma_2 \subset \gamma_3 \dots \subset \gamma_n$

- Along with $1/\varepsilon^2$ et $1/\varepsilon$ there are $(\ln K)^2$ et $(\ln K)$ terms in the finite part. These go clearly along with $(\ln \lambda)^2$ et $(\ln \lambda)$ we would get in P.V. This feature also generalizes to n-loops where there appear $(\ln K)^n$ terms with "K" here indicating a combination of external momenta.

These last two properties and their n-loop generalizations where put into rigorous form by Weinberg in the 60's. They pass under the name of "Weinberg's Theorem".

The procedure by which we obtained a finite result by performing subtractions on our original integral can also be presented in algorithmic form:





(*)

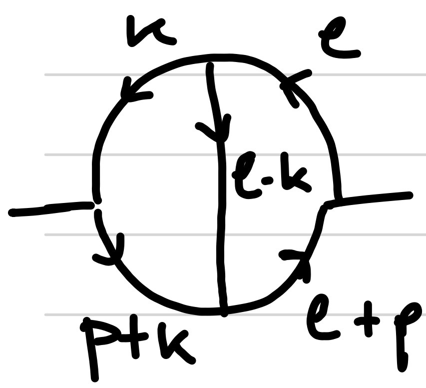
where by the green box we indicate the pure pole parts of the enclosed integrals. The last two subtractions are separately non-local. Their sum, as we have shown, is however local and corresponds to a 2-loop counterterm.

$$\Delta\mathcal{L}^{(2)} = -\frac{1}{4}\frac{g^4}{(4\pi)^6}\left(\frac{1}{18}\frac{1}{(d-6)^2} + \frac{11}{216}\frac{1}{d-6}\right)(\partial\phi)^2$$

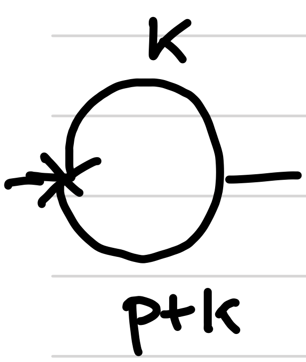
$$\delta Z^{(2)} = -\frac{1}{2}\frac{g^4}{(4\pi)^6}\left(\frac{1}{18}\frac{1}{(d-6)^2} + \frac{11}{216}\frac{1}{d-6}\right)$$

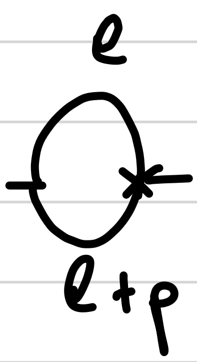
The diagrammatic eq. $(*)$ is an instance of the so-called <u>forest formula</u> that establishes an algorithmic procedure for subtracting divergences by starting from the inner ones and moving, recursively, to the superficial ones.

- The case we just considered belongs to the simplest class of <u>nested</u> subdivergent diagrams. They are characterized by the absence of divergent subdiagrams with lines in common. When instead the last instance happens we have the so-called <u>overlapping</u> divergences. The simplest such example is given by the other 2-loop diagram contributing to the 2-point function in ϕ^3 theory. Let us study it and let us show that after subtraction of subdivergences, we are left with purely local superficial divergences.



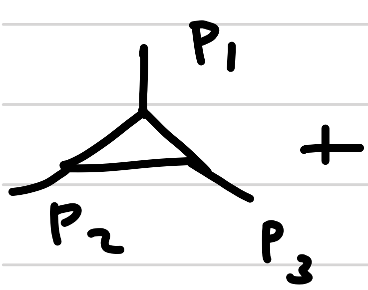
$$= \frac{g^4 \mu^{2\epsilon}}{2} \int d\sigma_k \, d\sigma_l \, \Delta(k)\Delta(p+k)\Delta(l-k)\Delta(l) \times \Delta(l+p)$$

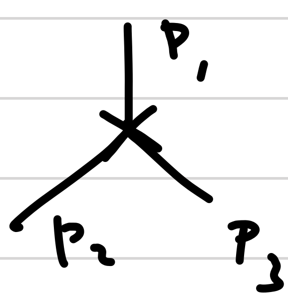


$+$


$$= \frac{g \, \delta_g \, \mu^{\varepsilon}}{2} \left\{ \int d\sigma_k \, \Delta(k)\Delta(p+k) + \int d\sigma_l \, \Delta(l)\Delta(l+p) \right\}$$

$$\delta_g = \frac{g^3}{(4\pi)^3} \frac{1}{d-6}$$



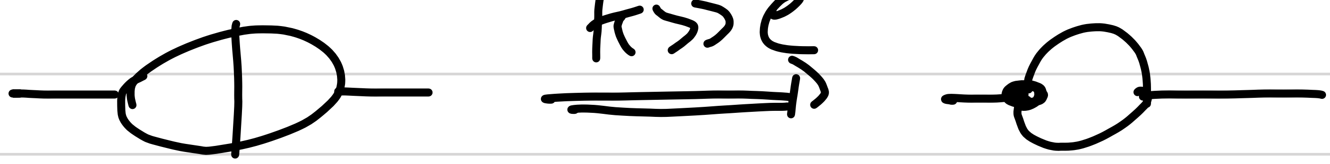
$+$


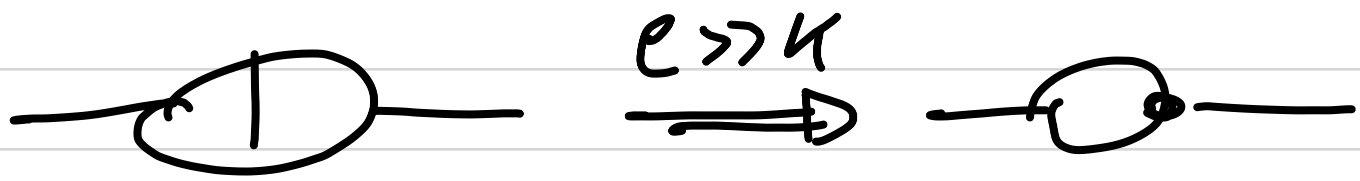
$\propto A \ln p + B$

$\Gamma_{1\text{-loop}} + \delta_g$

combination of P:

The difficulty with respect to the previous example is that we have two independent subdivergences: one corresponding to $\ell \ll K \to \infty$ and the others to $K \ll \ell \to \infty$. This can be depicting by "shrinking" the correspon= ding loops to a point



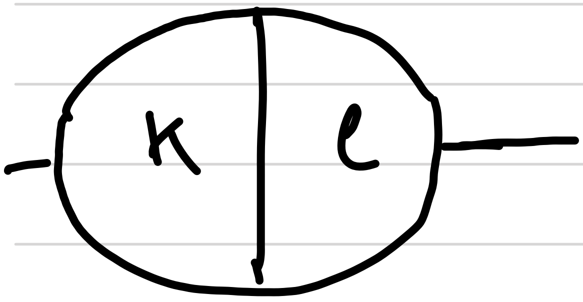


The two counterterm diagram, which are actually numerically identical, are each meant to subtract the corresponding subdivergence. More concretely we can write the  integral as the sum of integrals over dijoint regions

 $$= \int_{|K|>|\ell|} d\sigma_K \, d\sigma_\ell \, (\cdots) + \int_{|\ell|\geq|K|} d\sigma_K \, d\sigma_\ell$$

We can then combine the integral on each of these two regions with the corresponding counterterm and prove that the resulting expression possesses purely local divergences. Let us see that explicitly. We only need to focus on one of the two sets, given they are identical up to $k \Leftrightarrow \ell$.

The discussion is actually made more clear and more physical by working with a hard momentum or PV cut off. But it can be easily extended to Dim Reg.



- Consider region $l \gg k, p$

Expand propagators

$$\int d\sigma_k \; \frac{1}{k^2} \frac{1}{(p+k)^2} \underbrace{\int_{l \gg k,p} d\sigma_l \; \frac{1}{l^6}}_{\propto \ln \frac{\Lambda}{\max(k,p)}}$$

- Consider $\underbrace{k \gg p}_{A}$ et $\underbrace{k \lesssim p}_{B}$

A) expand $\displaystyle\int_p^\Lambda d^6k \, \frac{1}{k^4}\left(1 - 2\frac{pk}{k^2} - \frac{p^2}{k^2} \cdots\right) \ln\frac{\Lambda}{k}$ $(*)$ $|$

$$= \int_p^\Lambda dk \left(k \ln\frac{\Lambda}{k} - \frac{p^2}{k} \ln\frac{\Lambda}{k} \right)$$

$\displaystyle\int_0^1 x \ln x = \frac{1}{2}$ $\qquad \displaystyle\int d\ln x \, \ln x = \frac{1}{2}(\ln x)^2$

$$= \Lambda^2 + \frac{1}{2} p^2 \left[\left(\ln \frac{\Lambda^2}{p^2} \right)^2 \right] \Rightarrow \text{non-local divergence}$$

Now the vertex counterterm simply replaces $\ln \frac{\Lambda}{k} \to \ln \frac{\mu}{k}$ in eq. $*$, so that the integral becomes

$$\int_p^\Lambda \frac{d^6 k}{k^4} \left(1 - \frac{2pk}{k^2} - \frac{p^2}{k^2} + \dots \right) \ln \mu/k$$

$$p^2 \int_p^\Lambda \frac{dk}{k} \ln \frac{\mu}{k} \qquad t \equiv \ln k/\Lambda$$

$$= p^2 \int_{\ln p/\Lambda}^0 dt \left(\ln \frac{\mu}{\Lambda} + \ln \frac{\Lambda}{k} \right)$$

$$= p^2 \int_{\ln p/\Lambda}^0 dt \left(\ln \mu/\Lambda - t \right)$$

$$= p^2 \left[- \ln p/\Lambda \ln \mu/\Lambda + \frac{1}{2} \left(\ln p/\Lambda \right)^2 \right] =$$

$$= p^2 \left[\frac{1}{2}\left(\ln \frac{p}{\mu}\right)^2 - \frac{1}{2}\left(\ln \frac{\mu}{\lambda}\right)^2 \right]$$

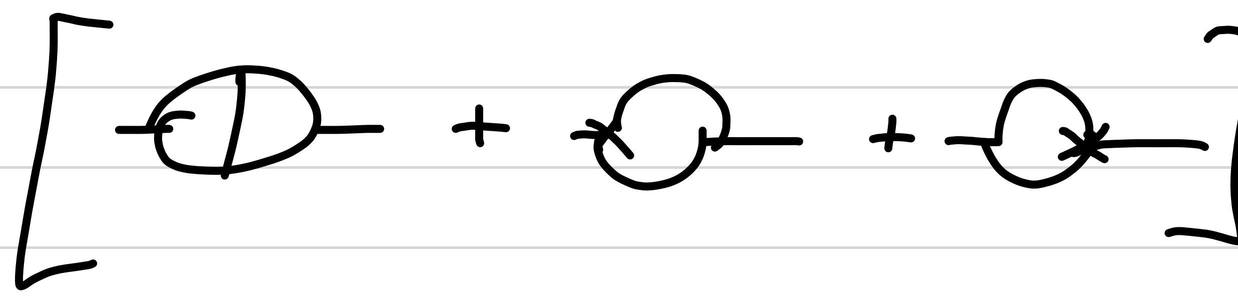
$\Rightarrow$ The non-local divergence has been cancelled.

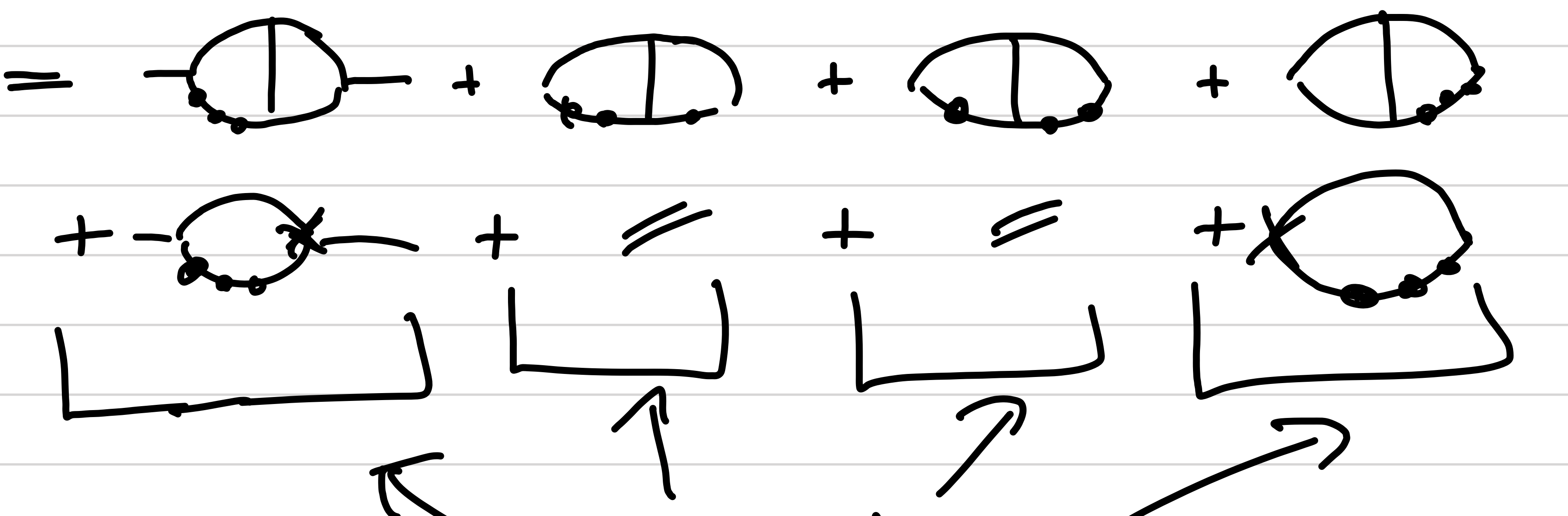
B) this is easier since integrating in the region $k \lesssim p$ does not give rise to UV divergencies.

Collins has a very convenient diagrammatic way to represent this result: present the

$$\partial_p \frac{}{(p+k)^2} = \text{—•—} \qquad \partial_p^2 \text{——} = \text{—••—}$$

etc.

then ∂_p^3 

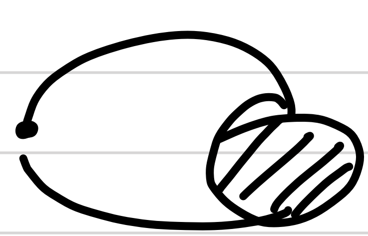


finite

This procedure generalizes to all higher loops. The structure of the necessary counterterms can be determined by induction through simple power counting. Let us assume starting from 1-loop that all graphs up to loop $(n-1)$ have been renormalized by suitable counter-terms. What additional counter-terms are needed at loop n? Consider then a n-loop graph G_n. We can always write the corresponding integral as

$$G_n = \int d\mu_{p_n} \Pi(\Delta_p)\, G_{n-1}(P_n\,,\; P_E)$$

External



G_n

Schematically

by dimensional analysis (refined by Weinberg's theorem) the convergence of the integral will be controlled by the superficial degree of divergence $\delta(G)$. Two cases

1) $\delta(G) < 0 \Rightarrow G_n \equiv$ finite, no c.t. needed

2) $\delta(G) > 0 \Rightarrow G_n$ has a divergence given by a degree $\delta(G)$ polynomial in p_E

⚠ The structure of needed c.t.'s is thus determined by the $\delta(G)$ of the various n-point functions.

We previously found

$$\delta(G) = D - \frac{(D-2)}{2} E + \sum_\alpha \left(\frac{D-2}{2} n_\alpha + d_\alpha - D \right)$$

$$\delta(G) = D - \Delta_\phi E - \sum_\alpha \Delta_\alpha$$

$$g_\alpha \, \partial^{d_\alpha} \phi^{n_\alpha}$$

- The crucial distinction is then $\Delta_\alpha \gtrless 0$

1. $\boxed{\Delta_\alpha \geq 0}$ in this case

$$\delta(G) \geq 0 \iff E \leq \frac{D}{\Delta_\phi}$$

$$\text{C.T.} \sim \partial^{\delta(G)} \phi^E \, c \qquad E \leq \frac{2D}{D-2}$$

$$[c] = 0$$

<u>Ex</u> in $|\partial\phi|^2 + m^2\phi^2 + g\phi^3$ in 6D

$$E \leq \frac{6}{4} = 3$$

Theories with $\Delta_\alpha \geq 0$ are said to be <u>renormalizable</u>, cause they can be made finite by introducing a finite number of counterterm types

(2) $\Delta_\alpha < 0$ for some vertex

In this case, as we increase the # of insertions of this vertex, the dimensionality $\delta(G) + \Delta_\phi E$ of the required counterterm becomes arbitrarily high. To fully renormalize the theory we thus need an infinite class of counterterms, each

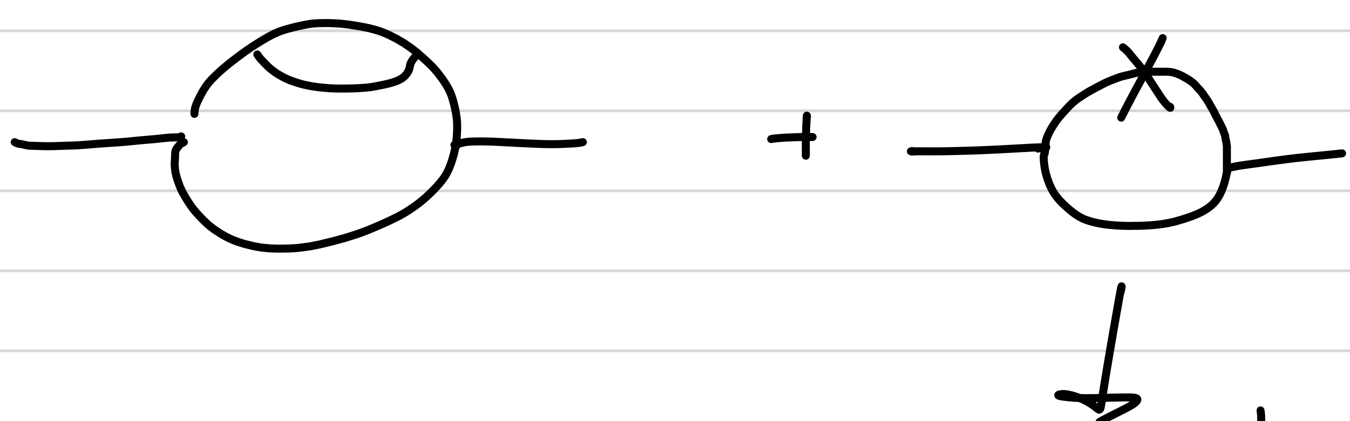
associated to an in principle free renormalized coupling (of negative di= mension). Because of the infinity of parameters these theories are termed <u>non-renormalizable</u>. From the modern perspective this is an improper terminology dating back to the days when the physical meaning of renor= malization was not fully appreciated. The modern perspective came with the works of Ken Wilson.
According to the modern interpretation non-renormalizable theories should be interpreted as low energy effective theories

valid below some physical UV scale M, which is also controlling the mass scaling of the couplings with negative dimensionality. Generically we expect

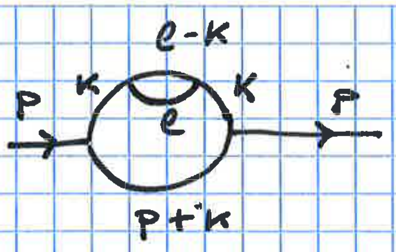
$$\frac{\int_x^E \Theta^{\delta(G)} \phi^E}{M^{D-\delta(G)-\Delta_\phi E}}$$

$\Rightarrow$ at energies $\sqrt{s} \ll M$ the effects of this infinite tower of terms is suppressed by powers of $\left(\sqrt{s}/M\right)$.

The following 5 pages present the details of the computa-tion of the graphs



counterterm insertion



$$\frac{g^4 \mu^{2\epsilon}}{\underset{\text{symm}}{2}} \int d\ell_\ell \, d\ell_k \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \frac{1}{(\ell-k)} \frac{1}{\ell^2}$$

$$= \frac{g^4 \mu^{2\epsilon}}{2} \int d\ell_\ell \, d\ell_k \int dx \left(\frac{1}{k^2}\right)^2 \frac{1}{(p+k)^2} \frac{1}{\left[\ell^2 + x(1-x)k^2\right]^2}$$

$$= \frac{g^4 \mu^{2\epsilon}}{2} \int d\ell_k \left(\frac{1}{(k)^2}\right)^2 \frac{1}{(p+k)^2} \int dx \, \frac{1}{\Gamma(d/2)} \frac{1}{(4\pi)^{d/2}} \int \frac{d\ell^2 (\ell^2)^{\frac{d}{2}-1}}{(\ell^2 + x(1-x)k^2)^2}$$

$$= \frac{g^4 \mu^{2\epsilon}}{2} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int d\ell_k \, dx \, (x(1-x))^{\frac{d}{2}-2} \times$$

$$\times \left(k^2\right)^{-2+\frac{d}{2}-2} \frac{1}{(p+k)^2}$$

$$\int dx \, x^{\frac{d}{2}-2} (1-x)^{\frac{d}{2}-2} = \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)}$$

$$= \frac{g^4 \mu^{2\epsilon}}{2(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)} \cdot \underbrace{\int d\ell_k \frac{1}{(p+k)^2 (k^2)^{4-d/2}}}_{I(p)}$$

$$I(p) = \frac{\Gamma(5-d/2)}{\Gamma(4-d/2)} \int d\ell_k \, dw \, \frac{w^{3-d/2}}{\left(k^2 + (1-w)(2pk+p^2)\right)^{5-\frac{d}{2}}}$$

$$= \frac{\Gamma(5-d/2)}{\Gamma(4-d/2)} \int d\ell_k \int dw \, \frac{w^{3-d/2}}{\left[k^2 + w(1-w)p^2\right]^{5-d/2}}$$

$$= \left(\frac{1}{4\pi}\right)^{d/2} \frac{\Gamma(5-d/2)}{\Gamma(4-d/2)} \cdot \frac{\Gamma(5-d)}{\Gamma(5-d/2)} \int dw \, w^{3-\frac{d}{2}} \left[w(1-w)\right]^{d-5} (p^2)^{d-5}$$

$$= \left(\frac{1}{4\pi}\right)^{d/2} \frac{\Gamma(5-d)}{\Gamma(4-d/2)} \int dw \, w^{-2+\frac{d}{2}} (1-w)^{d-5} (p^2)^{d-5}$$

$$= \left(\frac{1}{4\pi}\right)^{\frac{d}{2}} \frac{\Gamma(5-d)}{\Gamma(4-d/2)} \frac{\Gamma(d/2-1)\,\Gamma(d-4)}{\Gamma(\frac{3}{2}d-5)} (p^2)^{d-5}$$

Finally

$$\frac{g^4 \mu^{2\epsilon}}{2(4\pi)^d} \frac{\Gamma(2-d/2)\,\Gamma(\frac{d}{2}-1)^3\,\Gamma(5-d)\,\Gamma(d-4)}{\Gamma(d-2)\,\Gamma(4-\frac{d}{2})\,\Gamma(\frac{3d}{2}-5)} (p^2)^{d-5}$$

Expanding around $d = 6$ next page.

$$\cancel{\frac{g^4 \mu^{2(6-d)}}{2(4\pi)^6} \; \frac{p^2 (p^2)^{d-6}}{(4\pi)^{d-6}} \left[\frac{1}{18}\frac{1}{(d-6)^2} + \frac{1}{216}\right]}$$

$$\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left(\frac{p^2}{\mu^2 4\pi}\right)^{d-6}\left[\frac{1}{18}\left(\frac{1}{d-6}\right)^2+\frac{1}{d-6}\,\frac{1}{216}\left(-43+12\gamma\right)+\text{finite}\right]$$

G = number

$$=\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left\{\left[\frac{1}{18}\left(\frac{1}{d-6}\right)^2+\frac{1}{d-6}\left[\frac{1}{18}\ln\frac{p^2}{4\pi\mu^2}+\frac{1}{216}\left(-43+12\gamma\right)\right]\right]\right.$$

$$\left.+\left[\frac{1}{216}\left(-43+12\gamma\right)\ln\frac{p^2}{4\pi\mu^2}+\frac{1}{36}\ln^2\left(\frac{p^2}{4\pi\mu^2}\right)+G\right]\right\}$$

The divergent part contains a non-local piece associated with the subdivergence

- Let us check it is cancelled by insertion of 1-loop counterterm

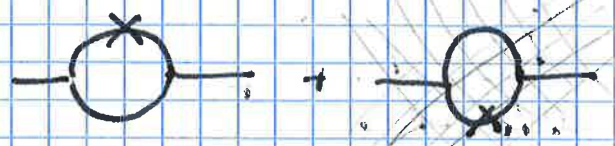
$$=\frac{g^2\mu^\epsilon}{2}\left(\frac{1}{4\pi}\right)^{d/2}\frac{\Gamma(2-d/2)}{\Gamma(2)}\,\frac{\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma(d-2)}\,(p^2)(p^2)^{\frac{d}{2}-3}$$

$$=\frac{g^2\,p^2}{2(4\pi)^3}\cdot\left(\frac{p^2}{4\pi\mu^2}\right)^{\frac{d}{2}-3}\,\frac{\Gamma(2-d/2)\,\Gamma(d/2-1)^2}{\Gamma(2)\,\Gamma(d-2)}$$

$$=\frac{g^2}{(4\pi)^3}\,\frac{p^2}{2}\left[\frac{1}{3}\,\frac{1}{d-6}+\frac{1}{18}\left(3\gamma-8\right)+\frac{1}{6}\ln\frac{p^2}{4\pi\mu^2}\right]$$

$$\text{C.T.}=-\frac{g^2}{(4\pi)^3}\,\frac{p^2}{6}\,\frac{1}{d-6}$$

The C.T. "2-loop" diags ~~are~~ is then ..



$$g^2\mu^\epsilon \cdot \left(-\frac{8^2}{(4\pi)^3}\,\frac{1}{6(d-6)}\right)\cdot$$

$$\cdot\left(\frac{1}{4\pi}\right)^{d/2}\frac{\Gamma(2-d/2)\,\Gamma(d/2-1)^2}{\Gamma(2)\,\Gamma(d-2)}\;p^2\,(p^2)^{\frac{d}{2}-3}$$

$$= -\frac{g^4}{(4\pi)^6}\left[\frac{1}{6(d-6)}\right]p^2\left[\frac{1}{3}\,\frac{1}{d-6}+\frac{1}{6}\ln\frac{p^2}{4\pi\mu^2}+\cdots\right]$$

$$= -\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left[\frac{1}{9}\,\frac{1}{(d-6)^2}+\frac{1}{18(d-6)}\ln\frac{p^2}{4\pi\mu^2}+\cdots\right]$$

This exactly cancels the two loop non local divergence

More precisely

$$= -\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left[\frac{1}{9}\,\frac{1}{(d-6)^2}+\frac{1}{d-6}\left(\frac{1}{18}\ln\frac{p^2}{4\pi\mu^2}+\frac{1}{54}(3\gamma-8)\right)\right]$$

When added we get

$$\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left\{-\frac{1}{18(d-6)^2}-\frac{11}{216}\,\frac{1}{d-6}+\text{finite}\right\}$$

notice transcendental term γ cancelled out along with $\ln 4\pi$

see next page

more precisely

$$\frac{g^4}{(4\pi)^6}\,\frac{p^2}{2}\left\{-\frac{1}{18}\,\frac{1}{(d-6)^2}-\frac{11}{216}\,\frac{1}{d-6}+\frac{(791-324\gamma+36\gamma^2-324\,\mathrm{Log}+72\,\gamma\,\mathrm{Log}+36\,\mathrm{Log}^2)}{2582}\right\}$$

where $\mathrm{Log} \equiv \ln \dfrac{p^2}{4\pi\mu^2}$