

Perturbation Theory

$$S = \frac{1}{2} \phi^a K_{ab} \phi^b + S_{\text{int}}[\phi]$$

Ex single scalar

$$\frac{1}{2} \phi^a K_{ab} \phi^b = \frac{1}{2} \int d^4x d^4y \phi(x) K(x, y) \phi(y)$$

$$K(x, y) = \delta^4(x - y) (-\partial_y^2 + m^2)$$

$$Z[J] = \int \mathcal{D}\phi \, e^{-\frac{1}{2} \phi K \phi - S_{int}[\phi] + J \cdot \phi}$$

$$= e^{-S_{int}[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \, e^{-\frac{1}{2} \phi^a K_{ab} \phi^b + J \phi}$$

$$= e^{-S_{int}[\frac{\delta}{\delta J}]} \cdot \underbrace{e^{\frac{1}{2} J^a K_{ab}^{-1} J^b}}_{Z_0[J]}$$

$$K_{ab}^{-1} \equiv G_{ab} \equiv \text{propagator}$$

single scalar in momentum space $K \rightarrow (p^2 + m^2) \rightarrow G = \frac{1}{p^2 + m^2}$

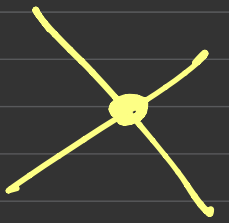
$$\langle \phi_{e_1} \dots \phi_{e_n} \rangle = \frac{\delta}{\delta J_{e_1}} \dots \frac{\delta}{\delta J_{e_n}} Z[J] \Big|_{J=0}$$

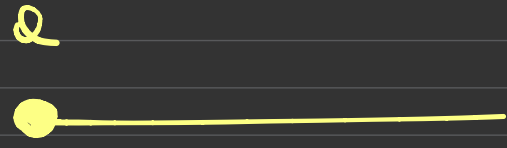
$$= \frac{\delta}{\delta J_{e_1}} \dots \frac{\delta}{\delta J_{e_n}} e^{-S'_{\text{int}}[\frac{\delta}{\delta J}]} e^{\frac{1}{2} J \cdot G \cdot J} \Big|_{J=0}$$

$\frac{\delta}{\delta J} \rightarrow$ vertices & external legs

$G_{\alpha\beta} \rightarrow$ propagators

$$\underline{\underline{Ex}} \quad + S_{int} = \frac{\lambda}{4!} \sum_a \left(\frac{\delta}{\delta J_a} \right)^4 \xrightarrow{a \rightarrow x} \frac{\lambda}{4!} \int d^4x \left(\frac{\delta}{\delta J(x)} \right)^4$$

• $-\frac{\lambda}{4!} \left(\frac{\delta}{\delta J_a} \right)^4 \rightarrow$  $-\lambda$ vertex

• $\frac{\delta}{\delta J_a} \rightarrow$  external line

• $\frac{\delta}{\delta J_a} \frac{\delta}{\delta J_b} \left(\frac{1}{2} J_c G_{cd} J_d \right) = G_{ab}$

  internal line

Expanding

$$\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{q_i}} \sum_{n_v} \frac{(-1)^{n_v}}{n_v!} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^{n_v} \cdot \sum_{n_p} \frac{1}{n_p!} \left(\frac{1}{2} J \cdot G \cdot J \right)^{n_p} \quad \Bigg|_{J=0}$$

at any particles order n_v, n_p

$$\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{q_i}} \frac{(-1)^{n_v}}{n_v!} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^{n_v} \cdot \frac{1}{n_p!} \left(\frac{1}{2} J \cdot G \cdot J \right)^{n_p}$$

Ex $S_{\text{int}} = \frac{\lambda}{m!} \sum_a \left(\frac{\delta}{\delta J_a} \right)^m$



m - lines end
in each vertex

$$\frac{\delta}{\delta J} \rightarrow n_e + n_v \cdot u = 2n_p$$

$$\left[\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{q_i}} \frac{(-1)^{n_v}}{n_v!} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^{n_v} \cdot \frac{1}{n_p!} \left(\frac{1}{2} J \cdot G \cdot J \right)^{n_p} \right]_{J=0}$$

$$= \sum_{\text{Diagrams}} N_D (\text{Diag})$$

$$N_D = ?$$

Ex

$$S_{int} = \sum_Q \frac{\lambda}{m!} \left(\frac{\delta}{\delta J_Q} \right)^m$$

v

E

$\bullet 1$

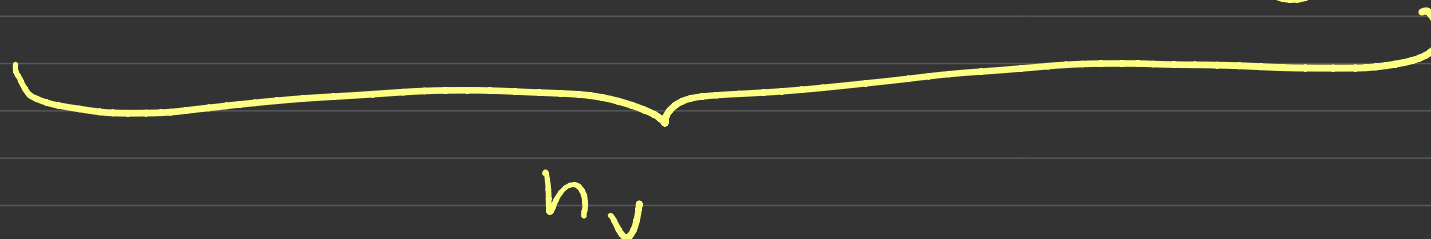
$\bullet$

$\vdots$

$\bullet n_e$



...

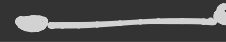


n_v

P



...



n_p

$$N_D = \left[(m!)^{n_v} n_v! n_p! 2^{n_p} \right] / S_D$$

31.10.2024

• permutation of equivalent $\frac{\delta}{\delta J}$ and equivalent J

$$\Rightarrow (u!)^{n_v} n_i! 2^{n_p} \Rightarrow G_E \equiv \text{equivalence group}$$

• given any contraction giving rise to a certain diagram D , acting with an element of G_E will give

either a) another contraction for the same D

or b) precisely the same contraction

the set (b) forms a subgroup H_D . Elements of G related by $f_1 = f_2 h$ will mutually identical contractions $\Rightarrow$ the number of independent

but equivalent contraction is the dimension
of the coset G_E/H_D

$$\dim \frac{G_E}{H_D} = \frac{\dim G_E}{\dim H_D} = \frac{(k!)^{n_v} n_v! n_p! 2^{n_p}}{S_D}$$

$\swarrow$

$\equiv$ symmetry factor
of D

$$N_D = \frac{(m!)^{n_v} n_v! n_p! z^{n_p}}{S_D}$$

$$\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{a_i}} \frac{(-1)^{n_v}}{n_v!} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^{n_v} \cdot \frac{1}{n_p!} \left(\frac{1}{2} J \cdot G \cdot J \right)^{n_p} \Big|_{J=0}$$

numerical coeff in front of diagram in the end

$$= N_D \cdot \frac{(-1)^{n_v}}{n_v!} \frac{1}{(n_e!)^{n_v}} \frac{1}{n_p!} \frac{1}{2^{n_p}}$$

$$= \frac{(-1)^{n_v}}{S_D}$$

Ex

$$\frac{\lambda \phi^4}{4!}$$

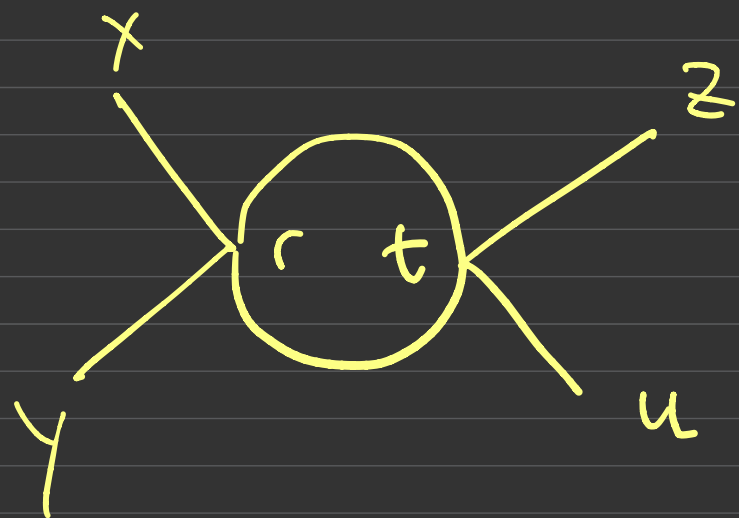


-1



$$= \int \frac{\lambda}{2} G(x, z) G(z, y) G(z, z) dz$$

$$\frac{4 \cdot 3 \cdot 1}{4!} = \frac{1}{2}$$



$$= \frac{\lambda}{2} \int G(x, r) G(y, r) G(z, t) G(u, t) [G(r, t)]^2 dr dt$$

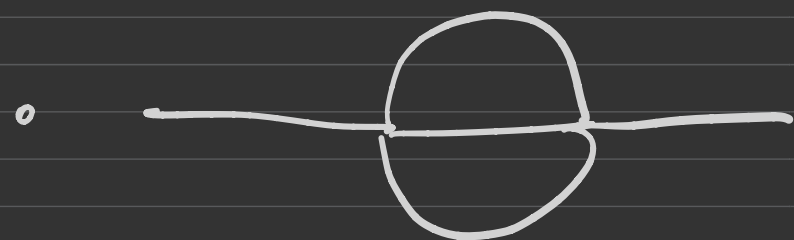
4.3

4.3

$$\frac{1}{2} \frac{\phi^4(r)}{4!} \frac{\phi^4(t)}{4!} \phi(x) \phi(y) \phi(z) \phi(u)$$

$$\frac{1}{2} \left(\frac{1}{4!} \right)^2 (4 \cdot 3)^2 \underbrace{\phi(r)\phi(x)} \underbrace{\phi(r)\phi(s)} \underbrace{\phi(t)\phi(z)} \underbrace{\phi(t)\phi(u)} \\ \cdot 2 \left(\underbrace{\phi(r)\phi(t)} \right)^2$$

$$= \left(\frac{1}{4!} 4 \cdot 3 \right)^2 = \frac{1}{4} \times 2 \text{ (exchange of } r \text{ t)}$$

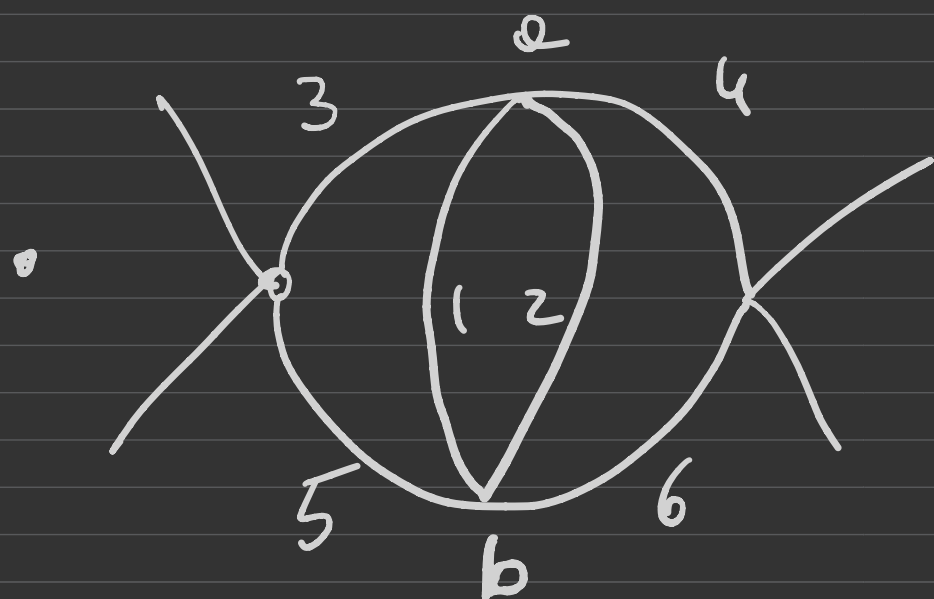


$$S = 3!$$



$$S = 4! \cdot 2$$

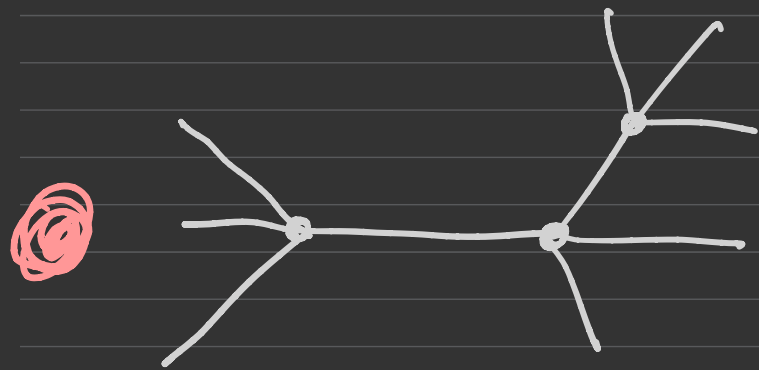
$$\frac{1}{2(4!)^2} \underbrace{\phi^4(x) \phi^4(y)}_{4!} \Rightarrow \frac{1}{2 \cdot 4!}$$



$$S = 2 \cdot 2 = 4$$

① Loops in $\lambda\phi^4$: UV divs and renormalization

$$S = \int \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$



$$\boxed{6h_v = n_e + 2n_I} \Rightarrow n_I = \frac{4h_v - n_e}{2}$$

$$3 \cdot 4 = 8 + 2 \cdot 2 = 12$$

$$\textcircled{2} \quad n_L = n_I - (n_v - 1) = n_v - \frac{n_e}{2} + 1$$

↓
loops

$$\boxed{n_L = n_v - \frac{n_e}{2} + 1}$$

expression in λ
 $\equiv$ loop expansion

LO leady = tree

NLO

next-to-leady = 1-loop

NNLO next-to-next-to-lead = 2-loop

$$\int \frac{\lambda}{4!} \phi^4 d^4x \rightarrow \sum_a \frac{\lambda}{4!} \phi_a^4 \rightarrow \sum_a \frac{\lambda}{4!} \left(\frac{\delta}{\delta J_a} \right)^4$$

$$\text{diagram} \rightarrow \sum_b G_{ab} G_{bc} G_{bb} \left(-\frac{i}{2} \right)$$

$$G_{bb} \Rightarrow \langle \phi(x) \phi(x) \rangle = \langle \phi(0) \phi(0) \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2}$$

② 2-point function $\langle \phi(p_1) \phi(p_2) \rangle = (2\pi)^4 \delta^4(p_1 + p_2) W^{(2)}(p_1)$

$$W^{(2)} = \text{---} + \text{---} \bigcirc \text{---} + \dots$$

$$= \frac{1}{p^2 + m^2} + \left(\frac{1}{p^2 + m^2} \right)^2 \left(\underbrace{-\frac{\lambda}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2}}_I \right) + \dots$$

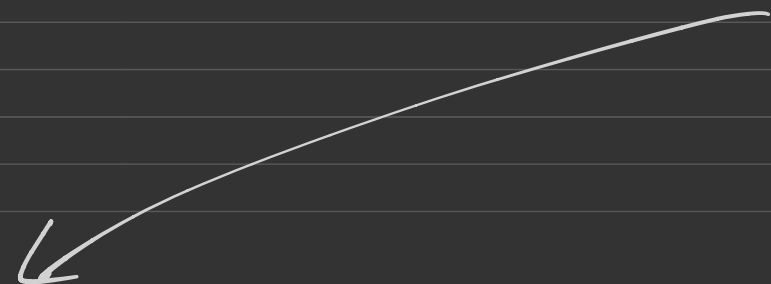
$$= \frac{1}{p^2 + m^2} + \left(\frac{1}{p^2 + m^2} \right)^2 \left(-\frac{\lambda}{2} I \right) + \dots = \frac{1}{p^2 + m^2 + \frac{\lambda}{2} I + \dots}$$

$$\Gamma^{(2)} = W^{(2)-1} = \underbrace{p^2 + m^2 + \frac{\lambda}{2} I}_{\text{1-loop "renormalizes" the mass}} + \dots \xrightarrow{\quad} \mathcal{O}(\lambda^2)$$

1-loop "renormalizes" the mass

$$m^2 \rightarrow m_{\Delta\text{-loop}}^2 = m^2 + \frac{\lambda}{2} I$$

$$\frac{d^4 p}{(2\pi)^4} = \frac{d^3 p \, p^3 dp}{(2\pi)^4} = \frac{(2\pi^2) p^3 dp}{(2\pi)^4} = \frac{p^2 dp^2}{16\pi^2}$$



● rough approach: cut-off integral at some UV scale Λ

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2} = \frac{1}{16\pi^2} \int \frac{p^2 dp^2}{p^2 + m^2} \longrightarrow \frac{1}{16\pi^2} \int_0^{\Lambda^2} \frac{p^2 dp^2}{p^2 + m^2}$$

$$= \frac{1}{16\pi^2} \left[\int_0^{\Lambda^2} dp^2 - m^2 \int_0^{\Lambda^2} \frac{dp^2}{p^2 + m^2} \right]$$

$$= \frac{1}{16\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2} \right]$$

$$\sim \frac{1}{16\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right]$$

$$\underbrace{\hspace{15em}}_I$$

at face value

$$m^2 \rightarrow m^2 + \frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right]$$

$$\underbrace{\hspace{15em}}_{\Lambda \rightarrow \infty}$$

is not a small perturbation
of tree level result!!
..

Basic remark : the split

$$S = \underbrace{\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2}_{S_{\text{free}}} + \frac{1}{4!} \int \phi^4 d^4x$$

may not be the right one
to do perturbation theory

Consider a different split

$$S_0 = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2$$

$$S_{\text{int}} = \underbrace{\frac{1}{4!}\phi^4 + \frac{\delta m^2}{2!}\phi^2}_{\text{small perturbation}}$$

Feynman rules

$$\begin{array}{l} \text{---} \times \text{---} \quad -i \\ \text{---} \times \text{---} \quad -\delta m^2 \end{array}$$

correction to mass

$$\begin{array}{c} \text{loop} \\ \hline \frac{1}{p^2 + m^2} \end{array} \quad \frac{-\lambda I}{2} \quad \frac{1}{p^2 + m^2} \quad + \quad \frac{1}{p^2 + m^2} \quad \frac{-\delta m^2}{\hline} \quad \frac{1}{p^2 + m^2} = \Delta \omega^{(2)}$$

$$\frac{\lambda I}{2} + \delta m^2 = \Delta T^{(2)}$$

• could even imagine a choice such that

$$\frac{\lambda I}{2} + \delta m^2 = 0$$

this choice corresponds to
 $\Rightarrow$

$$\delta u^2 = -\frac{\lambda}{32\pi^2} \left[\Lambda^2 - u_R^2 \ln \frac{\Lambda^2}{u_R^2} \right]$$

$$u^2 \longrightarrow u^2 + \frac{\lambda}{32\pi^2} \left[\Lambda^2 - u^2 \ln \frac{\Lambda^2}{u^2} \right] = u_R^2$$

Regularization & Renormalization

• Regularization $\rightarrow$ make loop integrals mathematically well defined

Ex

1.) discrete space-time $\equiv$ lattice



$p \lesssim \frac{1}{a}$ are cut-off

effectively $\lambda = \frac{1}{a}$

$$(\partial_\mu \phi \partial_\mu \phi)^2 \rightarrow |\phi_i - \phi_{i+1}|^2 + \text{other directions}$$

finite volume V
lattice length a $\left. \vphantom{\begin{matrix} V \\ a \end{matrix}} \right\} \implies \# \text{ variables} \sim \frac{V}{a^d}$

path integral $\equiv$ ordinary integral
with $N \sim \frac{V}{a^d}$ variables

• Advantage = it works beyond perturbation theory

• disadvantage = Poincaré symmetry is broken

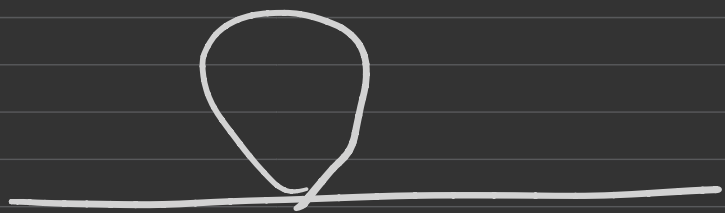
= Poincaré is not funcl. but
only emergent at $K \ll \frac{1}{2}$

2. Pauli-Villars: modify $\mathcal{L}$ to make propagators
decay faster at $K \rightarrow \infty$

Ex in ϕ^4

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial\phi)^2}_{p^2} + \underbrace{\frac{(\Box\phi)^2}{2\Lambda_1^2}}_{p^4} - \underbrace{\frac{(\Box\phi)\Box(\Box\phi)}{2\Lambda_2^4}}_{p^6} + \underbrace{\frac{m^2}{2}\phi^2}_{p^0} + \frac{\lambda}{4!}\phi^4$$

$$\frac{1}{p^2 + m^2} \longrightarrow \frac{1}{p^2 + \frac{p^4}{\lambda_1^2} + \frac{p^6}{\lambda_2^4} + m^2} \stackrel{p \rightarrow \infty}{\sim} \frac{1}{p^6}$$

Ex 

$$\longrightarrow \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + \frac{p^4}{\lambda_1^2} + \frac{p^6}{\lambda_2^4} + m^2}$$

$$\sim \int \frac{p^3 dp^3}{p^6} \sim \int \frac{dp^3}{p^3} \sim \sqrt{\frac{dt}{t^2}}$$

- Advantage: Poincaré manifest at all stages
- Disadvantage: cannot be used to respect non-relativistic gauge symmetry

- PU regulator is unphysical
because of ghosts

3) Dimensional Regularization: define QFT in
arbitrary real (complex) dimension d

ex
$$\int \frac{d^4 p}{p^2 + m^2} \rightarrow \int \frac{d^d p}{p^2 + m^2} = \frac{S_d}{(\cdot)^\#} \int \frac{p^{d-1} dp}{p^2 + m^2}$$

Advantages : Symmetry friendly
gauge symmetries in
particular

Disadvantage : Not "physical", very formal

4) String Theory : potentially the real
thing !!

Renormalization

basic idea

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

↓
renormalized

finite

↓
counter-terms

depends on
regulator

$\mathcal{L}_{CT}$ is constructed order by order such as
to eliminate the UV divergences $\equiv$
terms that diverge as $\left\{ \begin{array}{ll} a \rightarrow 0 & \text{lattice} \\ \Lambda \rightarrow \infty & \text{P.V.} \\ d \rightarrow 4 & \text{dim. reg} \end{array} \right.$

• non-trivial fact : this cancellation can be achieved by local $\mathcal{L}_{CT}$

• when $\mathcal{L}_{CT}$ contains only finitely operators (ex $(\partial\phi)^2$, ϕ^2 , ϕ^4) theory is said to be renormalizable

~~1.1~~ Special role of logarithmic div

$$\Rightarrow \mathcal{L}_{CT} = \sum_i P_i(\lambda_i) \mathcal{O}_i$$

↓ polynomials in the couplings

$$P(\lambda, \ln \lambda/\mu)$$

what interpretation?

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{c.T.} = \mathcal{L}_r' + \mathcal{L}_{c.T.}'$$

$$\mathcal{L}_r \neq \mathcal{L}_r'$$

Useful formulae

$$\bullet \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\bullet \Gamma(n) = \frac{1}{n} - \gamma + \frac{1}{12} (6\gamma^2 + \pi^2) n + O(n^2)$$

around $n=0$

$$\Gamma(n) \Big|_{n \text{ integer}} = (n-1)!$$

$$n \Gamma(n) = \Gamma(n+1)$$

$$\bullet \prod_{i=1}^n \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\sum \alpha_i)}{\prod_i \Gamma(\alpha_i)} \int \frac{\prod_j x_j^{\alpha_j-1} dx_j \delta(1-\sum x_j)}{\left[\sum_k A_k x_k \right]^{\sum \alpha_i}}$$

$$\bullet \int d^d k = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int k^{d-1} dk$$

$$\bullet \int_0^\infty \frac{dt t^{u-1}}{(t+a)^n} = \frac{\Gamma(u)\Gamma(n-u)}{\Gamma(n)} a^{n-u}$$

← Feynman trick

⊙ 1-loop in massive $\lambda \phi^4$ in 4D

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$

$$= \text{tree} + \text{1-loop bubble} + \text{crossed} + O(\lambda^3)$$

• dim reg: $S \rightarrow \int \mathcal{L} d^d x$ $d = 4 - \varepsilon$

$$[(\partial \phi)^2] = 2 + 2\Delta_\phi = d \Rightarrow \Delta_\phi = \frac{d-2}{2}$$

$$[\lambda] = d - 4\Delta_\phi = d - 2d + 4 = 4 - d = \varepsilon$$

$$\Rightarrow \lambda \rightarrow \lambda \mu^{4-d}$$

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{CT}$$

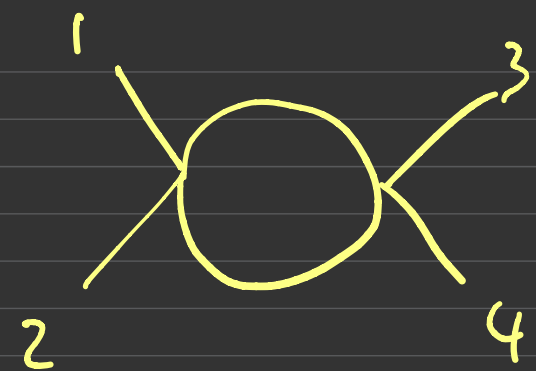
$$= \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda \mu^{\epsilon}}{4!} \phi^4 + \mathcal{L}_{CT}$$

$$\Gamma\left(\frac{\epsilon}{2}\right) (f)^{-\epsilon} = \left(\frac{2}{\epsilon} - \gamma\right) (1 - \epsilon \ln f + o(\epsilon^2))$$

$$e^{-\epsilon \ln f} = \left(\frac{2}{\epsilon} - \gamma - 2 \ln f + o(\epsilon)\right)$$

s-channel

1



$$= \frac{\lambda^2}{2} \mu^{2\epsilon} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \int dx \left[u^2 + x(1-x)s \right]^{\frac{d}{2}-2}$$

$$S = (p_1 + p_2)^2$$

expand around $\epsilon = 0$ ($d=4$)

$$= \frac{\lambda^2 \mu^\epsilon}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma - \int dx \ln \frac{u^2 + x(1-x)s}{4\pi\mu^2} \right]$$

$$\equiv A\left(\frac{u^2}{\mu^2}, \frac{s}{\mu^2}\right) \equiv A(s)$$

$$\bullet s \oplus t \oplus u \rightarrow \frac{\lambda^2 \mu^\epsilon}{32\pi^2} \left[\frac{6}{\epsilon} + A(s) + A(t) + A(u) \right]$$

$\Rightarrow$ counter-term $\delta \mathcal{L}^{(1)} = \frac{\lambda^2 \mu^\epsilon}{16\pi^2} \frac{3}{\epsilon} \frac{\phi^4}{4!}$

$$\lambda \rightarrow \lambda_0 = \lambda \mu^\epsilon \left[1 + \frac{3\lambda}{16\pi^2} \frac{1}{\epsilon} + \dots \right]$$

$\downarrow$
2-loop

- With the $\frac{1}{\epsilon}$ divergence subtracted away
we can safely take $\epsilon \rightarrow 0$ limit

$\hat{W}_4$ without ext legs

$$\hat{W}_4 = \text{diagram with 4 external legs and a shaded circle} = \text{diagram with 4 external legs and a cross} + \text{diagram with 4 external legs and a circle with an 's' above it} + \text{diagram with 4 external legs and a circle with a 't' above it} + \text{diagram with 4 external legs and a circle with a 'u' above it} + \dots$$

$$= -\lambda \left[1 - \frac{\lambda}{32\pi^2} \left(A\left(\frac{u^2}{\mu^2} \mid \frac{s}{u^2}\right) + (s \rightarrow t) + (s \rightarrow u) \right) + \dots \right]$$

• Analytic continuation to Minkowsky:

• $s \rightarrow -s$ $t \rightarrow -t$ $u \rightarrow -u$

$$p_0^2 + p_i^2 \rightarrow -p_0^2 + p_i^2$$

$$p_0 \rightarrow -ip_0$$

$$\begin{array}{c} \text{---} \rightarrow E \\ \downarrow M \quad \downarrow \text{wick} \end{array}$$

$$p^2 \rightarrow -p^2$$

• $\hat{G}_4 \rightarrow i G_4 = S_4$

$$S = 1 + (2\pi)^4 \delta^4(P_i - P_f) i/\mu$$

$$\mathcal{M}_{2 \rightarrow 2} = -\lambda \left[1 - \frac{\lambda}{32\pi^2} \left(A\left(\frac{s}{\mu^2}, \frac{-s}{\mu^2}\right) + \text{crossed} \right) \right]$$

1. At tree level $\mu = -\lambda \Rightarrow$ no angular dep.
 • no energy dep.
 $\frac{d\sigma}{d\Omega} = \text{const}$
 $\sigma_{\text{tot}} \propto \frac{\lambda^2}{s} \phi$
 $\phi \downarrow \text{phase space} \rightarrow \frac{1}{s_{11}}$

• At 1-loop, specific dependence on both

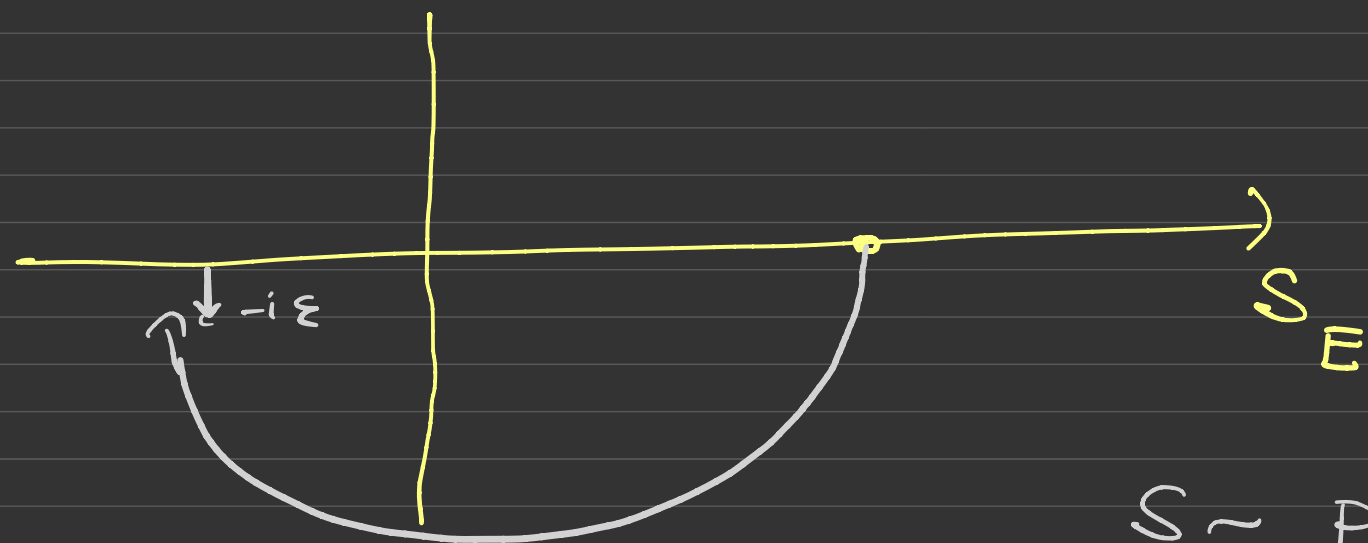
• dependence controlled by λ

2. $A\left(\frac{m^2}{\mu^2}, \frac{-s}{\mu^2}\right)$ possesses an imaginary part

Euclidean

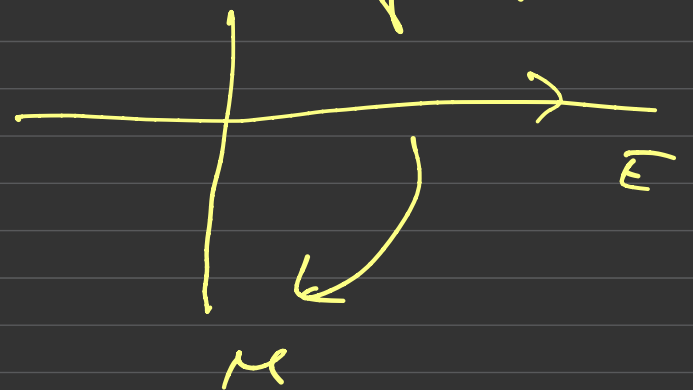
$$A \propto \int dx \ln \left(\frac{m^2 + x(1-x)s}{4\pi\mu^2} \right)$$

$s \rightarrow -s$ by Wick rotation



clockwise rotation of $P_1^0 + P_2^0$

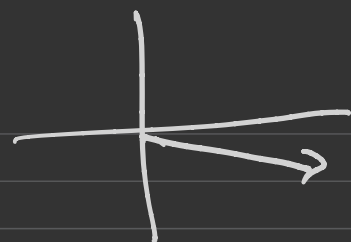
$$P_0 \rightarrow -iP_0$$



$$s_E = P_0^2 + P_i^2 \xrightarrow{\text{Wick}} -P_0^2 - i\varepsilon + P_i^2 = -s_H - i\varepsilon$$

• s-channel

$$u^2 - x(1-x)s > 0$$



$$\int dx \ln \left(\frac{u^2 - x(1-x)s - i\varepsilon}{4\pi\mu^2} \right) =$$

$$u^2 - x(1-x)s < 0$$



$$= \int dx \left\{ \ln \left| \frac{u^2 - x(1-x)s}{4\pi\mu^2} \right| - i\pi \theta(x(1-x)s - u^2) \right\}$$

$s \gg u^2$

$$\sim \ln \frac{s}{4\pi\mu^2} + \overbrace{\int_0^1 \ln x(1-x) dx}^{-2} - i\pi$$

• u, t channels: $u, t < 0 \Rightarrow$ same form without $i\pi$

$$\sim \ln \frac{-u}{4\pi\mu^2} - 2$$

Ex t -channel

$$t < 0$$

$$\int dx \ln \left(\frac{u^2 - x(1-x)t - i\varepsilon}{4\pi\mu^2} \right) =$$

$$u^2 - x(1-x)t > 0 \implies \text{no Im part}$$

• In the end

$$\lambda \left(1 + \frac{\lambda}{32\pi^2} (3\gamma - 6) \right) \equiv \bar{\lambda}$$

$$\begin{aligned} \mathcal{M} &= -\lambda \left[1 + \frac{\lambda}{32\pi^2} \left(3\gamma - 6 + \ln \frac{stu}{(4\pi\mu^2)^3} - i\pi \right) + \dots \right] \\ &= -\bar{\lambda} \left(1 + \frac{\bar{\lambda}}{32\pi^2} \left(\ln \frac{stu}{(4\pi\mu^2)^3} - i\pi \right) + \dots \right) \end{aligned}$$

3. $\mathcal{M}_{\text{tree}}$ energy indep. $\Leftrightarrow$ classical scale invariance
of $\lambda\phi^4$ in 4D

$\mathcal{M}_{\text{1-loop}} \equiv$ energy dep. $\Rightarrow$ scale invariance
explicitly broken

$$S_e = e^{2i\delta_e} = 1 + i\mu$$

$$\mu = -i(e^{2i\delta} - 1) = i(1 - e^{2i\delta})$$

$$= i\left(1 - 1 - 2i\delta - \frac{(2i\delta)^2}{2!} + \dots\right)$$

$$= i(-2i\delta + 2\delta^2 + \dots)$$

$$= \underbrace{2\delta} + \underbrace{2i\delta^2}$$