Perturbation theory

$$S = +\frac{1}{2}\phi^a K_{ab}\phi^b + S_{int}(\phi)$$

- K_{ab} = free kinetic term
- $\langle \phi_a \phi_b \rangle\Big|_{free} = K_{ab}^{-1} \equiv G_{ab}$

$$Z_1[J] = \int D\phi \; e^{-\frac{1}{2}\phi K \phi - S_{int}(\phi) + J\phi}$$

$$= e^{-S_i\left[\frac{\delta}{\delta J}\right]} \int D\phi \; e^{-\frac{1}{2}\phi K\phi + J\phi}$$

$$= e^{-S_i\left[\frac{\delta}{\delta J}\right]} \underbrace{e^{\frac{1}{2}JK^{-1}J}}_{Z_0[J]}$$

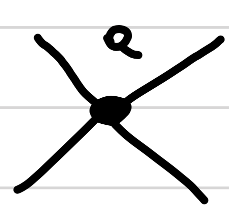
We then have

$$\langle \phi_{a_1} \cdots \phi_{a_{n_e}} \rangle = \prod_{i=1}^{n_e} \frac{\delta}{\delta J_{a_i}} \, e^{-S\left[\frac{\delta}{\delta J}\right]} \, e^{\frac{1}{2} J G J} \Big|_{J=0}$$

Expanding the exponents we obtain a series of terms which can be described by Feynman diagrams.

- $\frac{\delta}{\delta J} \rightarrow$ vertices and external legs
- $G_{ab} \rightarrow$ propagators

<u>Ex</u> $-S_{int} = -\frac{\lambda}{4!} \sum_a \left(\frac{\delta}{\delta J_a} \right)^4 \Longrightarrow \int \frac{\lambda}{4!} \phi^4$

- $-\frac{\lambda}{4!} \left(\frac{\delta}{\delta J_a} \right)^4 \longrightarrow$  $-\lambda$ <u>Vertex</u>

- $\frac{\delta}{\delta J_{a_i}} \longrightarrow$  <u>external line</u>

$$\frac{\delta}{\delta J_a}\frac{\delta}{\delta J_b}\left(\frac{1}{2} J_c G_{cd} J_d\right) = G_{ab}$$

$$\overset{a \qquad\qquad\qquad b}{\rule{4cm}{0.5pt}} = G_{ab}$$

- Setting $J=0$ eliminates all the terms for which not all propagators are connected to either a vertex or an external line: analogue of of full contractions in Wick's theorem

Now expanding

$$\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{a_i}} \sum_{n_v} \frac{(-1)^{n_v}}{n_v!}\left(S\left(\frac{\delta}{\delta J}\right)\right)^{n_v} \sum_{n_p} \frac{1}{n_p!}\left(\frac{1}{2} JGJ\right)^{n_p}$$

We can focus on each particular order

$$\prod_{i=1}^{n_e} \frac{\delta}{\delta J_{a_i}} \frac{(-1)^{n_v}}{n_v!}\left(S\left(\frac{\delta}{\delta J}\right)\right)^{n_v} \frac{1}{n_p!}\left(\frac{1}{2} JGJ\right)^{n_p}\Bigg|_{J=0}$$

(**)

Given n_e, n_v only a limited choice of n_p allows a non-vanishing result. For instance for $S = \left(\frac{\delta}{\delta J}\right)^m$ we need $n_e + n_v m = 2 n_p$

- The non-vanishing contractions can be grouped into classes of equivalent diagrams which are related by a trivial permutation of the $\frac{\delta}{\delta J}$ and J;

the contribution of each class is represented by the same diagram D with a coefficient that results from the multiplicity of the class times

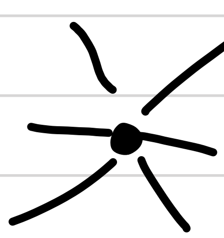
$\frac{1}{n_v!} \frac{1}{n_p!} \frac{1}{2^{n_p}}$

It is useful to spell out the rules to compute the numerical coefficient.

For definiteness let us consider the case

$$S_{int} = \sum_a \frac{\lambda}{m!}\left(\frac{\delta}{\delta J_a}\right)^m$$

- When contracting each vertex with a chosen set of J's (m of them) we have of course $m!$ possible contractions from whence the Feynman rule



$= -\lambda$

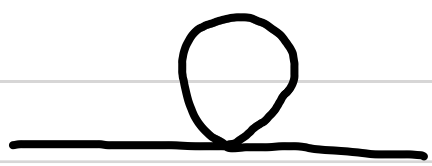
m-lines

- We have then to some over all possible permutations of the role of the n_v vertices, of the n_p propagators and of each of their 2 endpoints. Superficially this seems to give a multiplicity factor $n_v!\, n_p!\, 2^{n_p}$ which exactly

compensates the denominator in (**)
However, in general, a subset of these permutations are equivalent to permutations of the m-lines in a vertex, which we have already counted.
Indicating by S_D the number of such equivalent permutations we have that the total multiplicity is $\frac{n_V!\, n_P!\, 2^{n_P}}{S_D}$ so that in the end the diagram comes with a coefficient $\frac{1}{S_D}$.

- S_D is also know as symmetry factor as it counts the number of permutations that map the diagram back to itself.

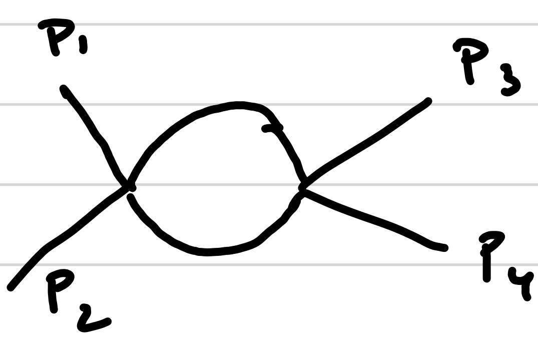
- Some example in ϕ^4

• 

$S=2$

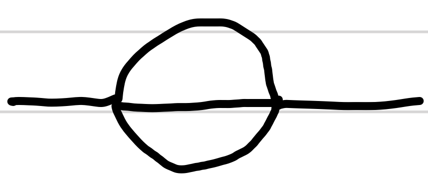
$$\Rightarrow \frac{1}{2}\lambda \sum G_{ee}$$

$$\to \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}$$

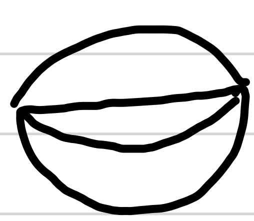
• 

$$\to \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}\,\frac{1}{(p_1+p_2+p)^2+m^2}$$

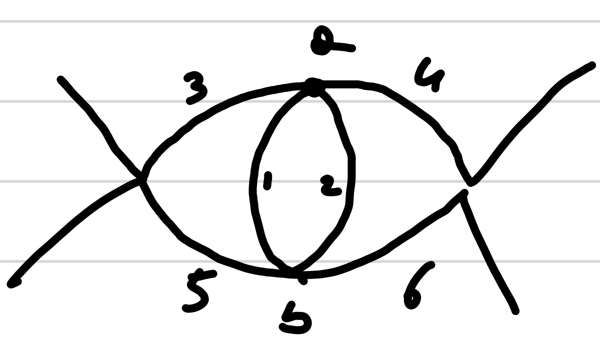
$S=2$

• 

$S=3!$

• 

$S=4!\cdot 2$

• 

$$S=\underbrace{2}_{\text{lines } 1-2}\times\underbrace{2}_{3\,a\,4 \,\leftrightarrow\, 5\,b\,6}=4$$

- **Loop diagrams** in $\lambda\phi^4$: **first encounter with UV divergences and renormalization**

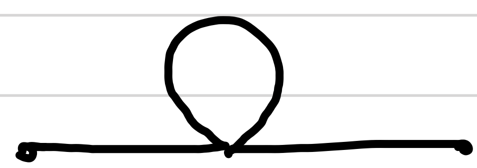
$$S = \int \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

given $n_e, n_v \to \underset{\substack{\downarrow \\ \text{internal legs}}}{n_I} = \frac{4n_v - n_e}{2}$

$$\underset{\substack{\downarrow \\ \text{loops}}}{n_\ell} = n_I - n_v + 1 = n_v - \frac{n_e}{2} + 1$$

- Given a certain n_e-point function the expansion in powers of λ coincide with a loop-expansion
- leading order : tree
- next-to-leading : 1-loop
- and so on
- Let us now consider 1-loop effects

- 2-point function $\langle \phi \phi \rangle$

 $+$ 

$$\frac{1}{p^2+m^2} + \left(\frac{1}{p^2+m^2}\right)^2 \left(\frac{-\lambda}{2} \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2}}_{I} \right)$$

$$\Gamma^{(2)} = W^{(2)\,-1}$$
$$= (p^2+m^2) + \frac{\lambda}{2} I$$

- I appears to "renormalize" the mass
$$m^2 \longrightarrow m^2 + \frac{\lambda}{2} I$$
- There is however a problem: I is not a well defined integral, at face value it is ∞, with the infinity arising from the region $k \to \infty$. We thus this an ultraviolet (UV) divergence.

- The most obvious approach to this problem, would be to simply cut-off the integral at some UV scale Λ:

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2} \Rightarrow \frac{1}{16\pi^2} \int_0^{\Lambda^2} \frac{p^2 dp^2}{p^2+m^2} =$$

$$= \frac{1}{16\pi^2}\left[\int_0^{\Lambda^2} dt - m^2 \int_0^{\Lambda^2} \frac{dt}{t+m^2}\right] =$$

$$= \frac{1}{16\pi^2}\left[\Lambda^2 - m^2 \ln \frac{\Lambda^2+m^2}{m^2}\right]$$

$$\overset{\Lambda \gg m}{\simeq} \frac{1}{16\pi^2}\left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2}\right]$$

- If we take this result at face value we have that 1-loop the mass becomes

$$m^2 \longrightarrow m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2}\right]$$

- no matter how small λ, if we take $\Lambda \to \infty$, the 1-loop correction beats the supposedly leading order result. But what about higher loops then: it seems we cannot use perturbation theory. The deep origin of this problem is the presence of an infinity of length scales in QFT: decomposing ϕ into Fourier mode, we see that long wavelength modes are coupled (via ϕ^4) modes with arbitrary short wavelengths. It is the quantum fluctuations of the latter, that are responsible for the UV divergence of I.

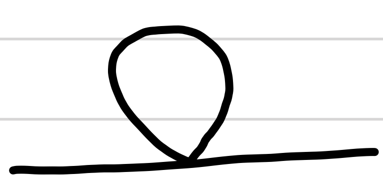
But there is luckily a way out.

The basic remark is that the split $S = S_0 + S_{int}$ $\quad S_{int} = \frac{\lambda \phi^4}{4!}$ may not be the one suitable to carry out perturbation theory, given what we are computing, which in this case is the mass. More generally we can consider a split of the form

$$S_0 = \frac{1}{2}(\partial\phi)^2 + \frac{m_R^2}{2}\phi^2$$

$$S_{int} = \frac{\lambda}{4!}\phi^4 + \frac{\delta m^2}{2}\phi^2$$

the correction to $\Gamma^{(2)}$ is now

 + δm^2 

$$\frac{\lambda}{2} I \quad + \quad \delta m^2$$

in principle we could imagine a split where trivially $\delta m^2 + \frac{\lambda}{2} I = 0$, which corresponds to choosing m_R^2 to be the physical mass

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m_R^2 \ln\frac{\Lambda^2}{m_R^2}\right]$$

- For just $\Gamma^{(2)}$ at just 1-loop the situation is pretty trivial. It gets more involved when considering $\Gamma^{(4)}$ and much more involved when considering higher. The procedure to make sense of the resulting seemingly ill defined computation goes under the name of <u>Renormalization</u>

It will keep us busy for the next few lectures.

Loops & Renormalization : Generalities

With the computation of loop diagrams the true nature of QFT manifests itself: the virtual existence of an infinity of degrees of freedom associated to shorter and shorter length scales leads to divergent expressions.
The well developed formalism to deal with this problem consists of two steps, Regularization and Renormalization.
While the deeper physical meaning of the procedure will become evident later on (ex renormalization group & Wilson approach) and through practice, we can limit

ourselves here to a few general comments.

Regularization. This is the procedure to render the loop integrals (indeed the path integral) mathematically well defined. There are various options with advantages and disadvantages.

- Lattice: replace the continuum space (-time) by a lattice of spacing a (corresponding to energy scale $E \sim 1/a$). If we also put our QFT in a finite volume this would reduce the path integral to an ordinary integral (in $\propto V/a^d$ variables).

Advantage → works non-perturbatively so it can be used by simulating strongly coupled QFT on a computer

Disadvantage → Poincaré invariance is broken by short distance effects and is only an emergent symmetry at long distance

- Pauli-Villars : Modify the Lagrian so as to make all loop integrals converge. In practice this is achieved by making the propagator decrease faster than $\frac{1}{p^2}$.

For instance in ϕ^4 in 4D

$$\mathcal{L} \to \frac{|\partial\phi|^2}{2} + \frac{(\Box\phi)^2}{2\Lambda_1^2} - \frac{(\Box\phi)\Box(\Box\phi)}{2\Lambda_2^4} + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

the propagator is modified into

$$\frac{1}{p^2} \longrightarrow \frac{1}{p^2 + \frac{p^4}{\Lambda_1^2} + \frac{p^6}{\Lambda_2^4} + m^2} \xrightarrow{p\to\infty} \frac{1}{p^6}$$

Such behaviour is sufficient to make all loops converge.

- Advantages: Poincaré and other global symmetries can be kept exact

Disadvantages: Cannot be straightforwardly made to respect non-abelian gauge symmetries.

Secondarily: cannot be viewed as "physical" given it always implies negative norm states (ghosts).

<u>Dim Reg</u>: Define QFT in arbitrary dimension by analytic continuation, starting from d small enough to eliminate all UV divergences

<u>Advantages</u>: Respects most symmetries including non-abelian local symms!

<u>Disadvantages</u>: Not "physical", very formal.

<u>Strings</u>: physical regulator of QFT; could be the true thing! Conceptually important to know such thing exists, but of no or little practical relevance
(≡ cannon to kill a fly)

<u>Renormalization</u>: procedure to consistently subtract UV divergences order by orders in perturbation theory and compute finite correlators and S-matrix. The basic idea is that the full Lagrangian can

be written as

$$\mathcal{L} = \underbrace{\mathcal{L}_R}_{\substack{\text{renormalized} \\ \text{Lagrangian} \\ \equiv \text{finite}}} + \underbrace{\mathcal{L}_{CT}}_{\substack{\text{counter-term} \\ \text{Lagrangian} \\ \Rightarrow \text{explicitly depends} \\ \text{on regulator}}} \qquad (*)$$

$\mathcal{L}_{CT}$ is constructed order by order in perturbation theory so as to exactly cancel the terms that diverge as the regulator is removed, i.e. as

$\Lambda \to \infty$		PV
$a \to 0$		lattice
$d \to 4$		DimReg

- It is non-trivial (that) this cancellation can be achieved by a local $\mathcal{L}_{CT}$
- When $\mathcal{L}_{CT}$ contains only a finite set of structures (ex $(\partial\phi)^2$, ϕ^4) the theory is said to be <u>renormalizable</u>.

- Special role played by logarithmic divergences (Technically one can even define a basis of operator where these are the only ones)
- $\mathcal{L}_{CT}$ will be a series in coupling g (or λ) and depend on $\ln \Lambda$
- by dim analysis $\ln \Lambda \to \ln \Lambda/\mu$
- The presence of an explicit finite scale μ in $\mathcal{L}_{CT}$ offers the interpretation of the split in $(*)$: in order to perform perturbation theory the split of $\mathcal{L}$ is not unique, but if we want to compute correlators or S matrix elements characterized by an external momentum scale $p \sim \mu$, that the split in $(*)$ is optimal.

The basic message underlying renormalization is that, in a system with ∞ many length scales, perturbation theory is not a given and that the split between leading and subleading terms must be adapted to the quantity we are focussing on (or, more specifically, the overall energy or length scale associated to our quantity).

Having made these general remarks which will be fully appreciated later on, we can now plunge into the technology of loop computation.

Useful formulae

- $$\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\sum \alpha_i)}{\prod_i \Gamma(\alpha_i)} \cdot \int \frac{\prod_j^n x_j^{\alpha_j - 1} dx_j \cdot \delta(1 - \sum x_j)}{\left[\sum_k A_k x_k\right]^{\sum \alpha_i}}$$

- $$\int d^d k = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int k^{d-1} dk$$

- $$\int_0^\infty \frac{dt\, t^{m-1}}{(t+a)^n} = \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} \; a^{n-m}$$

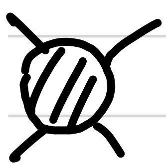
- $$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

- $$\Gamma(n) = \frac{1}{n} - \gamma + \frac{1}{12}\left(6\gamma^2 + \pi^2\right) n + O(n^2)$$

  around $n = 0$

- 1-loop in massive $\lambda\phi^4$ in 4D

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$



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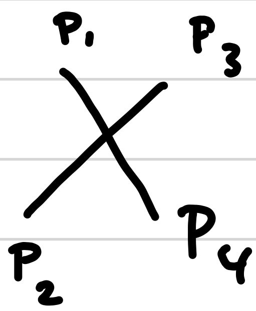

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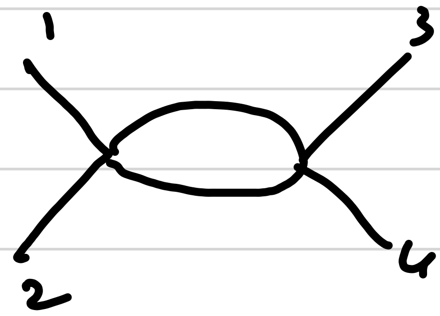
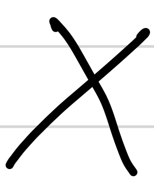
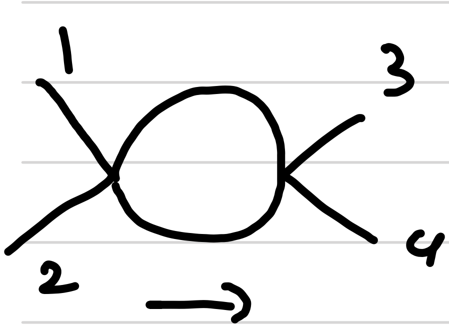


$+$ crossed

$+ O(\lambda^2)$

-  $= -\lambda$

- dim reg $\begin{cases} \mathcal{L} \to \mathcal{L} + \mathcal{L}_{cT} \\ \lambda \to \lambda\mu^{4-d} \end{cases}$ $\qquad 4-d \equiv \varepsilon$



$$= \frac{\lambda^2\mu^{2\varepsilon}}{2}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int dx\left[m^2 + x(1-x)s^2\right]^{\frac{d}{2}-2}$$

$s = (p_1+p_2)^2$

which expended around $d=4$ becomes

$$\frac{\lambda^2\mu^\epsilon}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma-\int dx \ln\frac{m^2+x(1-x)s}{4\pi\mu^2}\right]$$

$$-\gamma-\int dx \ln\frac{m^2+x(1-x)s}{4\pi\mu^2} \equiv A\left(\frac{m^2}{\mu^2},\frac{s}{\mu^2}\right).$$

Considering the three channels we have

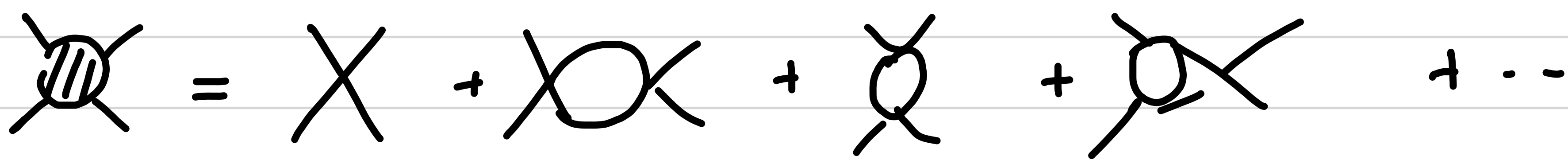
$$\frac{\lambda^2\mu^\epsilon}{32\pi^2}\left[\frac{6}{\epsilon}+A(s)+A(t)+A(u)\right]$$

the c.T. eliminating the $1/\epsilon$ is then

$$\delta\mathcal{L}^{(1)} = \frac{\lambda^2\mu^\epsilon}{16\pi^2}\frac{3}{\epsilon}\frac{\phi^4}{4!}$$

$$\lambda \to \lambda\Big(1+\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}+\underset{\text{2-loops}}{\underbrace{O(\lambda^2)}}\Big)$$

With the divergence subtracted we can now safely take the $d \to 4$ finding for the truncated 4-pt function



$$-\lambda\left[1-\frac{\lambda}{32\pi^2}\left(A\left(\frac{m^2}{\mu^2},\frac{s}{\mu^2}\right)+A\left(\frac{m^2}{\mu^2},\frac{t}{\mu^2}\right)+A\left(\frac{m^2}{\mu^2},\frac{u}{\mu^2}\right)\right)\cdots\right]$$

$$s=(p_1+p_2)^2 \quad t=(p_1-p_3)^2 \quad u=(p_1-p_4)^2$$

analytic continuation to Minkowsky corresponds to $s \to -s$ $\quad t \to -t \to u \to -u$

$S_4 \sim i\, G_4 \qquad S = 1 + iT$

$\qquad\qquad + i(2\pi)^4\delta^4(\ )\mathcal{M}$

$$\Rightarrow \mathcal{M} = -\lambda\left[1-\frac{\lambda}{32\pi^2}\left(A\left(-s/\mu^2\right)+\text{crossed}\right)\right]$$

1. Tree level amplitude did not depend on kinematic $\Longrightarrow$ $\dfrac{d\sigma}{d\Omega} = \text{const}$

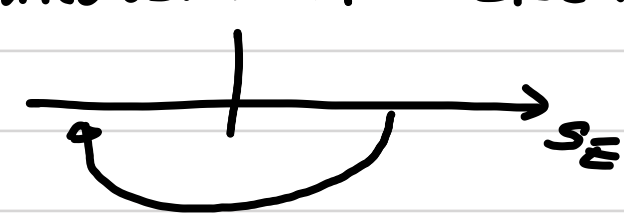
↳ solid angle

- At 1-loop we <u>predict</u> a specific dependence on angle θ (u, t)
- The dependence in weaker the smaller λ

2. $A(-s/\mu^2)$ possesses an imaginary part

$$\int dx \ln\left(m^2 + x(1-x) s_E\right)$$

must be continued to Minkowski via clockwise rotation of $(p_1+p_2)^0$ $\Longrightarrow$



$$\Longrightarrow \quad s_E \to -s - i\varepsilon$$

$$\int dx \ln\left(m^2 - x(1-x)s - i\varepsilon\right)$$

$$\int dx \left(\ln \left| \frac{m^2 - x(1-x)s}{4\pi\mu^2} \right| - i\pi\, \theta\left(x(1-x)s - m^2\right) \right)$$

$s \gg m^2$

$$\sim \ln \frac{s}{4\pi\mu^2} + \underbrace{\int_0^1 dx \ln x(1-x)}_{-2} - i\pi$$

- $t, u < 0$ in Minkowsky so no imaginary part from there

In the end in the relativistic regime

$$\mathcal{M} = -\lambda \left[1 + \frac{\lambda}{32\pi^2} \left(3\gamma - 6 + \ln \frac{stu}{(4\pi\mu^2)^3} - i\pi \right) \cdots \right]$$

- $\mathrm{Im}\, \mathcal{M} = + \dfrac{\lambda^2}{32\pi} > 0$ optical theorem

3. At tree level $\mathcal{M}$ is energy independent, corresponding to the scale invariance of $\lambda\phi^4$ in 4D.

This is no longer true at 1-loop.
Now $\mathcal{M}$ depends on s,t,u. In particular $|\mathcal{M}|$ features a slow (λ is hp small) logarithmic growth with energy.
Notice that this logarithmic growth and the presence of imaginary part have a common analytic origin