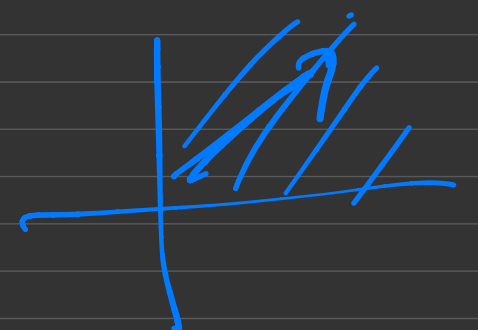
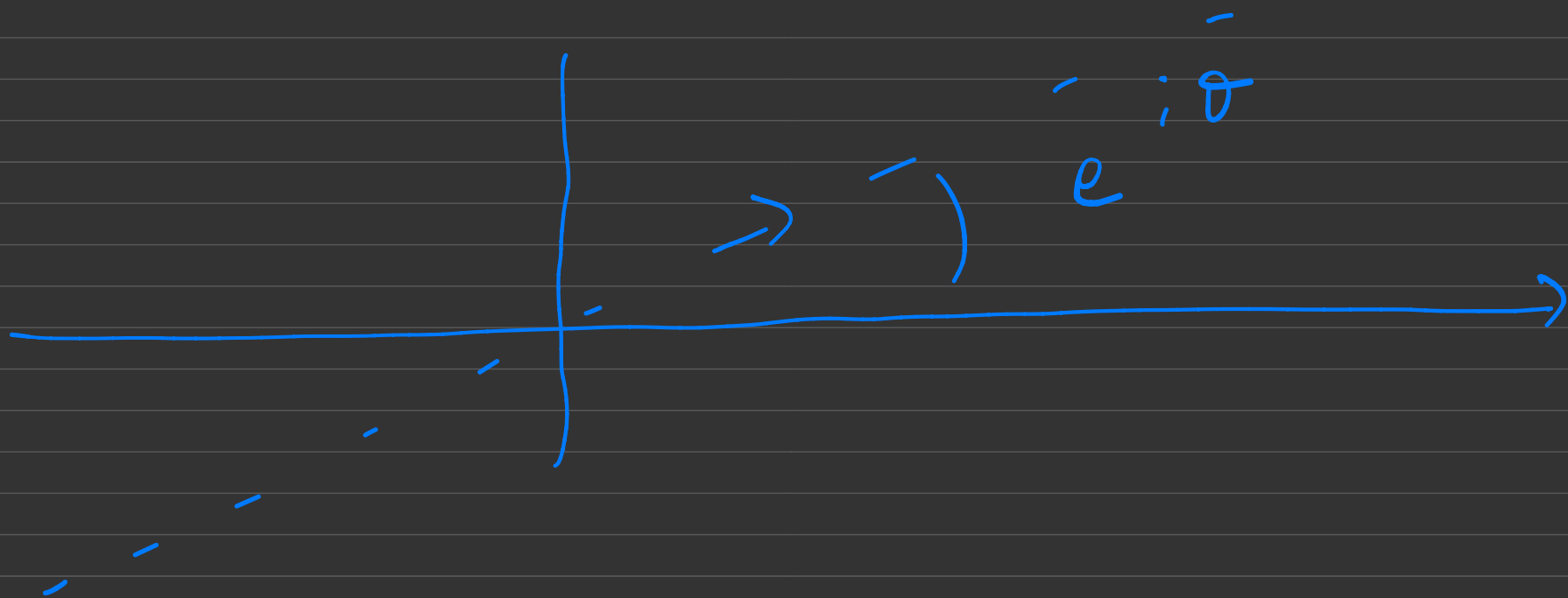


 Euclidean $\longleftrightarrow$ Minkowsky

$$Z[\mathcal{J}, \theta] = \int \prod_i dq_i e^{-\sum_i \frac{1}{2} \frac{(\Delta q_i)^2}{\delta} e^{i\theta} + V(q_i) \delta e^{i\theta} - \mathcal{J}_i q_i \delta}$$

$$\frac{t_i - t_f}{N} \equiv \delta e^{i\theta}$$



$$= \int \mathcal{D}q e^{-\int \left(\frac{e^{-i\theta}}{2} \dot{q}^2 + e^{i\theta} V(q) - \mathcal{J}q \right) dz}$$

$$0 < \theta < \frac{\pi}{2}$$

$\theta = 0 \implies$ Euclidean path integral

$$\theta \rightarrow \frac{\pi}{2}^- \equiv \text{Minkowsky (with } \varepsilon\text{-prescription)}$$

$$\theta = \frac{\pi}{2} - \varepsilon \quad e^{i\theta} = i(1 - i\varepsilon) \quad e^{-i\theta} = -i(1 + i\varepsilon)$$

$$\rightarrow \int \mathcal{D}q \, e^{i \left[\int \frac{\dot{q}^2}{2} - V(q) + \frac{i\varepsilon}{2} (\dot{q}^2 + V(q)) \right] - \int \mathcal{I} q}$$

$$\begin{aligned} \underline{\underline{\text{Ex}}} \quad V(q) = \frac{m^2}{2} q^2 &\Rightarrow \frac{1}{\omega^2 - m^2} \Rightarrow \frac{1}{\omega^2(1+i\varepsilon) - m^2(1-i\varepsilon)} \\ &\approx \frac{(1-i\varepsilon)}{\omega^2 - m^2 + 2i\varepsilon m^2 + i\varepsilon} \end{aligned}$$

① Analytic continuation

$$\underline{t_i} \ll 0 < \tau < \ll \underline{t_f}$$

Ex 2 pt function

$$\langle q(\tau) q(0) \rangle_\theta = \lim_{\substack{\delta \rightarrow 0 \\ n\delta = \tau}} \int \prod_i Dq_i \underbrace{q_n q_0}_e e^{-\sum_i \left(\frac{(\Delta q_i)^2}{\delta e^{i\theta}} + V(q_i) \delta e^{i\theta} \right)}$$

$$\frac{t_f - t_i}{N} = \delta$$

$$\lim_{\substack{\delta \rightarrow 0 \\ n\delta = \tau}} f(n, \delta e^{i\theta}) = f(n|\delta e^{i\theta}) = f(|\Delta\tau| e^{i\theta})$$

$$\langle \phi(\tau) \phi(0) \rangle_\varepsilon \equiv \langle \phi(\tau) \phi(0) \rangle_{\theta=0} = f(|\Delta\tau|) = G(\Delta\tau)$$

$$\langle \phi(\tau) \phi(0) \rangle_H = \langle \phi(\tau) \phi(0) \rangle_{\theta \rightarrow \pi/2} = f(|\Delta\tau| i)$$

$$\Delta\tau > 0 \quad f(\Delta\tau) \rightarrow f(\Delta\tau \cdot i)$$

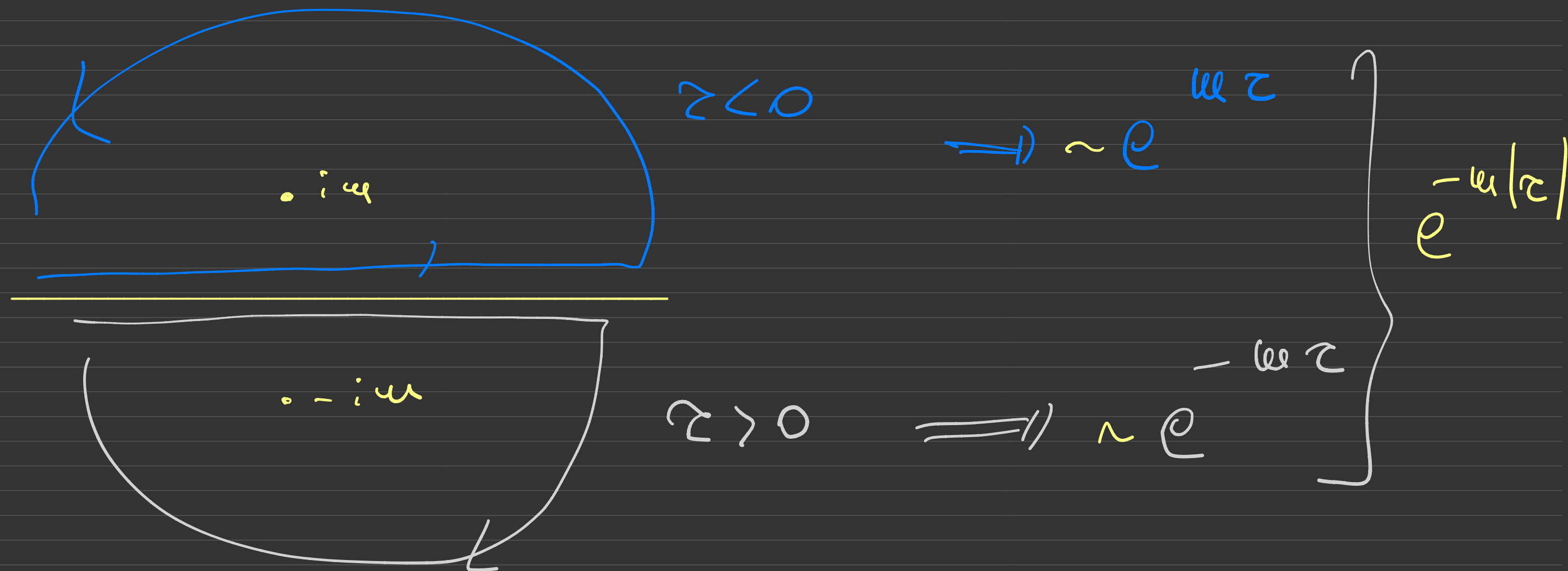
$$\Delta\tau < 0 \quad f(-\Delta\tau) \rightarrow f(-\Delta\tau \cdot i)$$

$$\Delta\tau \rightarrow \Delta\tau \cdot i \equiv \underline{\text{Wick rotation}}$$

Ex Harmonic oscillator

$$\mathcal{L}_{\bar{E}} = \frac{\dot{q}^2}{2} + \frac{m^2}{2} q^2$$

$$G(\tau) = \frac{1}{2m} e^{-m|\tau|} = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega^2 + m^2}$$

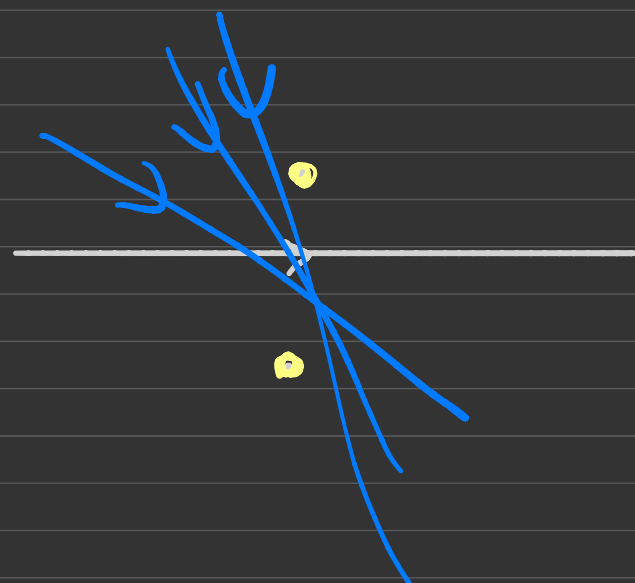


$$G_{\theta}(\tau) = \frac{1}{2u} e^{-u|\tau|e^{i\theta}}$$

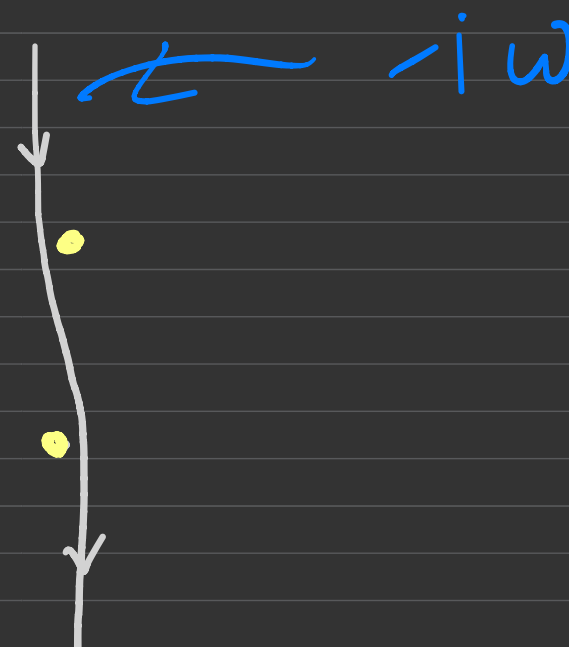
$\equiv$ same poles with
no contribution at ∞

$$\Rightarrow \int d\omega \frac{e^{-i\omega\tau}}{\omega^2 + u^2}$$

$$\begin{aligned}\tau &\rightarrow e^{i\theta}\tau \\ \omega &\rightarrow e^{-i\theta}\omega\end{aligned}$$



$\Rightarrow$



$$\begin{aligned}&= \int \frac{-i d\omega}{2\pi} \frac{e^{-i\omega\tau}}{-\omega^2 + u^2 - i\varepsilon} = \int \frac{d\omega}{2\pi} \frac{i e^{-i\omega\tau}}{\omega^2 - u^2 + i\varepsilon}\end{aligned}$$

- Field theory

$$G_E(x) = \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-i q x}}{q^2 + m^2} \implies G_H(x) = \int \frac{d^4 q}{(2\pi)^4} \frac{ie^{-i q x}}{q^2 - m^2 + i\varepsilon}$$

$$|\nabla \phi|^2 \Rightarrow k^2 \phi^2$$

$$m^2 \rightarrow m^2 + \vec{k}^2$$

$$\frac{1}{\omega^2 - m^2} \rightarrow \frac{1}{\omega^2 - m^2 - \vec{k}^2} = \frac{1}{p^2 - m^2}$$

S-matrix

- 1) Compute euclidean correlators by path integral
- 2) Wick rotate to Minkowsky and use LSZ

⊙ Ground state projection

$$\langle \psi_f | e^{-H(\tau_f - \tau_n)} \mathcal{O}_n(x_n) e^{-H(\tau_n - \tau_{n-1})} \dots \mathcal{O}_1(x_1) e^{-H(\tau_1 - \tau_i)} | \psi_i \rangle$$

$$\Downarrow \tau_f - \tau_i \rightarrow \infty$$

$$\underbrace{\langle \psi_f | 0 \rangle \langle 0 | \psi_i \rangle}_{= \langle \psi_f | e^{-H(\tau_f - \tau_i)} | \psi_i \rangle} e^{-E_0(\tau_f - \tau_i)} \left(\langle 0 | T(\mathcal{O}_n(x_n) \dots \mathcal{O}_1(x_1)) | 0 \rangle + o\left(e^{-(\bar{E} - E_0)(\tau_f - \tau_i)}\right) \right)$$

$$= \langle \psi_f | e^{-H(\tau_f - \tau_i)} | \psi_i \rangle$$

$$\Rightarrow \langle 0 | T(\mathcal{O}_n(x_n) \dots \mathcal{O}_1(x_1)) | 0 \rangle = \lim_{\substack{\tau_i \rightarrow -\infty \\ \tau_f \rightarrow +\infty}} \left[\frac{\int \mathcal{D}\varphi \mathcal{O}_n(x_n) \dots \mathcal{O}_1(x_1) e^{-S}}{\int \mathcal{D}\varphi e^{-S}} \right]_{\psi_i, \psi_f}$$

Functional Methods

• partition function: $Z[J] = \int \mathcal{D}\phi \, e^{-S + \int \phi J}$

$\downarrow$

$\equiv \int \frac{e^{-S(x)}}{e} \phi(x) \frac{d^4 x}{e}$

$$\left. \frac{1}{Z[J=0]} \frac{\delta}{\delta J_1(x_1)} \dots \frac{\delta}{\delta J_n(x_n)} Z[J] \right|_{J=0} = \frac{\int \mathcal{D}\phi \, \phi(x_n) \dots \phi(x_1) e^{-S}}{\int \mathcal{D}\phi \, e^{-S}} = \langle 0 | T(\phi(x_n) \dots \phi(x_1)) | 0 \rangle$$

• Connected correlators

• definition

$$\langle \phi_{q_1}(x_1) \dots \phi_{q_n}(x_n) \rangle \equiv \overline{\sum} \langle \dots \rangle_c \langle \dots \rangle_c \dots$$

All partitions

$$\bullet \langle \phi_{q_1}(x_1) \rangle = \langle \phi_{q_1}(x_1) \rangle_c$$

$$\bullet \langle \phi_{q_1}(x_1) \phi_{q_2}(x_2) \rangle = \langle \phi_{q_1}(x_1) \phi_{q_2}(x_2) \rangle_c + \langle \phi_{q_1}(x_1) \rangle_c \langle \phi_{q_2}(x_2) \rangle_c$$

$$\langle \phi_a \phi_b \rangle_c = \langle \phi_a \phi_b \rangle - \langle \phi_a \rangle \langle \phi_b \rangle$$

More compactly $\phi_a(x) : (x, a) \longrightarrow a$

$$\sum_a \int d^4x \longrightarrow \sum_a$$

$$\langle \phi_a \phi_b \phi_c \rangle = \langle \phi_a \phi_b \phi_c \rangle_c + \langle \phi_a \rangle \langle \phi_b \phi_c \rangle_c + \langle \phi_b \rangle \langle \phi_a \phi_c \rangle_c \\ + \langle \phi_c \rangle \langle \phi_a \phi_b \rangle_c$$

$$+ \langle \phi_a \rangle \langle \phi_b \rangle \langle \phi_c \rangle$$

$$\cancel{Z[J]} \langle \phi_{q_1} \dots \phi_{q_n} \rangle = \frac{\delta}{\delta J_{q_1}} \dots \frac{\delta}{\delta J_{q_n}} Z[J] \Big|_{J=0}$$

$$\langle \phi_{q_1} \dots \phi_{q_n} \rangle_c = \frac{\delta}{\delta J_{q_1}} \dots \frac{\delta}{\delta J_{q_n}} \ln Z[J] \Big|_{J=0}$$

$$W \equiv \ln Z$$

• $Z[J] \equiv e^{W[J]} \longrightarrow$ generator of connected correlators

• $Z[0] = 1 \implies W[0] = 0$

•
$$\langle \phi_1 \dots \phi_n \rangle = \frac{\delta}{\delta J_1} \dots \frac{\delta}{\delta J_n} Z \Big|_{J=0} = \frac{\delta}{\delta J_1} \dots \frac{\delta}{\delta J_n} e^{W[J]} \Big|_{J=0}$$

$= \left(\sum_{\text{all partitions}} \frac{\delta}{\delta J} W \right)$

Ex $\langle \phi_1 \phi_2 \rangle = \frac{\delta}{\delta \phi_1} \frac{\delta}{\delta \phi_2} W + \frac{\delta W}{\delta \phi_1} \frac{\delta W}{\delta \phi_2}$

▲ Cluster property

• In euclidean QFT : $\langle \phi_1(x_1) \dots \phi_n(x_n) \phi_{n+1}(x_{n+1}+a) \dots \phi_{n+m}(x_{n+m}+a) \rangle$

$$\xrightarrow{a \rightarrow \infty} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \langle \phi_{n+1}(x_{n+1}) \dots \phi_{n+m}(x_{n+m}) \rangle$$

• by induction $\Rightarrow$

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \phi_{n+1}(x_{n+1}+a) \dots \phi_{n+m}(x_{n+m}+a) \rangle_c \xrightarrow{a \rightarrow \infty} 0$$

2-pt | $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle_c + \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$

cluster $\Rightarrow \lim_{x_2 \rightarrow \infty} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$

3-pt | $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle_c +$
 $+ \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle_c + \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_1(x_1) \mathcal{O}_3(x_3) \rangle_c + \langle \mathcal{O}_3 \rangle \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle_c$
 $+ \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \rangle$

$$\langle \mathcal{O}_1 \rangle \left(\langle \mathcal{O}_2 \mathcal{O}_3 \rangle_c + \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \rangle \right) = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \mathcal{O}_3 \rangle$$

- $\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle_c \rightarrow 0$ when any $|x_i - x_j| \rightarrow \infty$

$\Rightarrow$ Fourier transf "well behaved at" $p_i \rightarrow 0$

$$= \int e^{ip_1 x_1 \dots ip_n x_n} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle d^4 x_1 \dots d^4 x_n$$

$$= (2\pi)^4 \delta^4(p_1 + \dots + p_n) G_c(p_2, \dots, p_n)$$

$$G_c(\phi_1 x_2, \dots, x_n) = \int e^{-ip_2 x_2 \dots -ip_n x_n} \underbrace{G_c(p_2, \dots, p_n)}_{\text{}} d^4 p_2 \dots d^4 p_n$$

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle_c +$$

$$+ \langle \phi_1(x_1) \phi_2(x_2) \rangle_c \langle \phi_3(x_3) \dots \phi_n(x_n) \rangle_c + \dots$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \delta^4(p_1 + p_2) & \delta^{(4)}(p_3 + \dots + p_n) \end{array}$$

$$\equiv \underbrace{\delta^{(4)}(p_1 + \dots + p_n)}_{\text{}} \underbrace{\delta^{(4)}(p_1 + p_2)}_{\text{}}$$

ex 2 pt function $m^2=0$

$$G_c(p^2) = \frac{1}{p^2}$$

$$G_c(0, x) = \int e^{-ipx} \frac{1}{p^2} d^4p \sim \frac{1}{x^2}$$

if $G_c(p) = \delta^4(p)$

$$G_c(0, x) \rightarrow \text{const}$$

④ Effective action (sketch)

$$\int \mathcal{D}\phi \, e^{-S[\phi] + J\phi} = Z[J]$$

$$-\frac{\delta S}{\delta \phi} + J = 0 \quad \Rightarrow \quad \phi[J] = \phi_c$$

$$J = J[\phi_c]$$

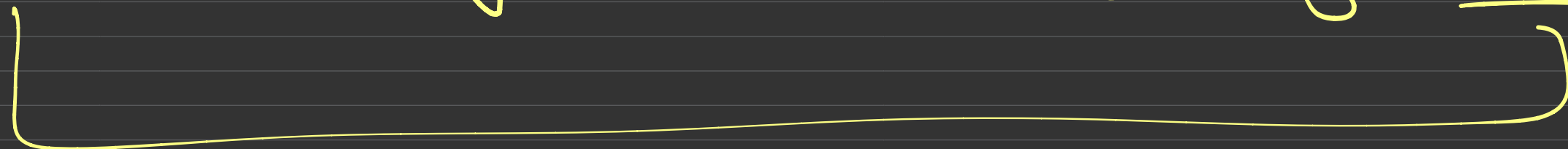
$$-S[\phi[J]] + J\phi[J]$$

$$\sim e^{W[J]} \quad \left(1 + \dots\right) = Z[J] = e^{W[J]}$$

Semiclassical approx $W[J] = -S[\phi[J]] + J\phi[J] \dots$



$$W[\perp] \equiv \text{Legendre transform of } \underline{S[\phi]}$$



$$W[\perp]$$

$$\text{define } \underline{T[\phi]} \equiv \text{Legendre transform of } W[\perp]$$

$$\eta \rightarrow 0$$

$$-S[\phi[\eta]] + \eta \phi[\eta] = \omega[\eta]$$

$$-S[\phi_c] + \eta[\phi_c] \phi_c = \omega[\eta[\phi_c]]$$

$$S[\phi_c] = -\omega[\eta[\phi_c]] + \eta[\phi_c] \phi_c$$

Effective Action

$$W[J] \xrightarrow{\text{Legendre}} \tilde{\phi}_a = \frac{\delta W}{\delta J_a} \quad \tilde{\phi}_a = \tilde{\phi}_a[J]$$

$$\underline{\underline{\text{def}}} \quad \Gamma[\tilde{\phi}] = -W[J[\tilde{\phi}]] + J[\tilde{\phi}] \tilde{\phi}$$

↓
quantum effective action

$$\begin{aligned} \bullet \quad \frac{\delta \Gamma}{\delta \tilde{\phi}_b} &= - \frac{\delta W}{\delta J_a} \frac{\delta J_a}{\delta \tilde{\phi}_b} + J_b + \tilde{\phi}_a \frac{\delta J_a}{\delta \tilde{\phi}_b} \\ &= - \cancel{\frac{\delta W}{\delta J_a} \frac{\delta J_a}{\delta \tilde{\phi}_b}} + J_b + \cancel{\frac{\delta W}{\delta J_a} \frac{\delta J_a}{\delta \tilde{\phi}_b}} = J_b[\tilde{\phi}] \end{aligned}$$

$$\bullet \quad \tilde{\phi}_a = \frac{\delta \omega[\tilde{J}]}{\delta \tilde{J}_a} = \langle \phi_a \rangle_{\tilde{J}} \xrightarrow{\tilde{J}=0} \langle 0 | \phi_a | 0 \rangle$$

$$\bullet \quad \tilde{J}_a = \frac{\delta \Gamma[\tilde{\phi}]}{\delta \tilde{\phi}_a} \xrightarrow{\tilde{J}=0} \left. \frac{\delta \Gamma}{\delta \tilde{\phi}_a} \right|_{\tilde{\phi}_a = \langle \phi_a \rangle} = 0$$

$$\frac{\delta \Pi}{\delta \phi_0} = 0$$

in the absence of
sources

• notice $\phi_0 \Big|_{J=0} = \frac{\delta W}{\delta J_0} \Big|_{J_0=0} = \langle \phi_0 \rangle$

① Γ -generator of proper vertices

2-point function

$$\left. \frac{\delta^2 \Gamma}{\delta \phi_a \delta \phi_b} \right|_{\phi = \langle \phi \rangle} = \frac{\delta}{\delta \phi_a} \frac{\delta \Gamma}{\delta \phi_b} = \frac{\delta J_b}{\delta \phi_a} = \left(W^{(2)-1} \right)_{ba} \Big|_{J=0}$$

$$\delta_{bc} = \frac{\delta J_b}{\delta \phi_a} \frac{\delta \phi_a}{\delta J_c} = \frac{\delta J_b}{\delta \phi_a} \cdot \frac{\delta^2 W}{\delta J_a \delta J_c}$$

$$= \frac{\delta J_b}{\delta \phi_a} \cdot W^{(2)}_{ac}$$

$$\left(W^{(2)-1} \right)_{ba} = \frac{\delta J_b}{\delta \phi_a}$$

$$\Gamma^{(2)}_{ab} = \left(W^{(2)} \right)_{ab}$$

$$\left(W^{(2)} \right)_{as} \neq \frac{1}{W^{(2)}_{sa}}$$

$$\Gamma_{ab}^{(2)} W_{bc}^{(2)} = \delta_{ac}$$

Ex: free field $a, b, c, \dots \longrightarrow p \equiv \text{momenta}$

$$W_{bc} \longrightarrow \underbrace{\delta^4(p_1 - p_2) G(p_1)}_{\equiv G(p_1, p_2)}$$

$$\Gamma_{as}^{(2)} \longrightarrow \Gamma(p_1, p_2)$$

$$\int \Gamma(p_1, p_2) G(p_2, p_3) d^4 p_2 = \delta^4(p_1 - p_3)$$

$$\Rightarrow \Gamma(p_1, p_2) = \delta^4(p_1 - p_2) G'(p_1)$$

$$\int G^{-1}(p_1) \delta^4(p_1 - p_2) \delta^4(p_2 - p_3) G(p_3) d^4 p_2$$

$$\int G^{-1}(p_1) \delta^4(p_1 - p_3) G(p_3) = \delta(p_1 - p_3)$$

$$\left. \begin{aligned} \frac{\delta J_b}{\delta \phi_a} \cdot \frac{\delta \phi_a}{\delta J_c} &= \delta_{bc} \\ \frac{\delta J_b}{\delta \phi_a} \cdot W^{(2)}_{ac} &= \delta_{bc} \end{aligned} \right\} \Rightarrow \frac{\delta J_b}{\delta \phi_a} = \left(W^{(2)} \right)^{-1}_{ba}$$

3 pt function $\frac{\delta}{\delta \phi_c} \frac{\delta^2 \Gamma}{\delta \phi_a \delta \phi_b}$

$$\frac{\delta^3 \Gamma}{\delta \phi_a \delta \phi_b \delta \phi_c} = \frac{\delta}{\delta \phi_c} \left(\frac{\delta^2 W}{\delta J_a \delta J_b} \right)_{ab} = \frac{\delta}{\delta \phi_c} \frac{\delta^2 W}{\delta J_{a'} \delta J_{b'}}$$

$$= - \left[(W^{(2)})^{-1} \frac{\delta W^{(2)}}{\delta \phi_c} (W^{(2)})^{-1} \right]_{ab}$$

$$\Gamma_{abc}^{(3)} = - (W^{(2)})^{-1}_{aa'} (W^{(2)})^{-1}_{bb'} (W^{(2)})^{-1}_{cc'} W_{a'b'c'}^{(3)}$$

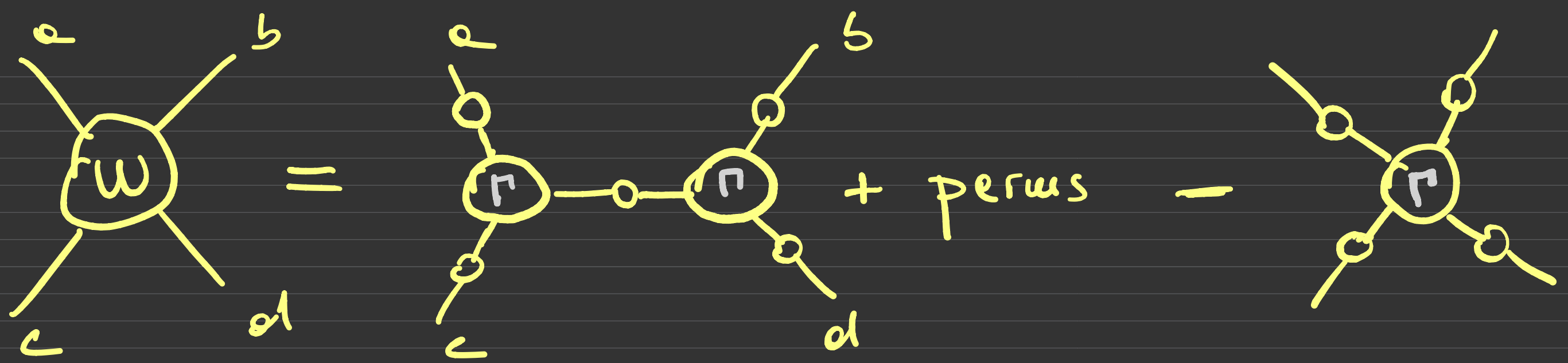
$$W_{abc}^{(3)} \equiv \frac{\delta^3 W}{\delta J_a \delta J_b \delta J_c}$$

$$W_{abc}^{(3)} = - W_{aa'}^{(2)} W_{bb'}^{(2)} W_{cc'}^{(2)} \Gamma_{a'b'c'}^{(3)}$$



4-pt $\Rightarrow$

$$W_{abcd}^{(4)} \equiv \frac{\delta}{\delta J_d} W_{abc}^3 = - \left[W_{dqq'}^{(3)} W_{bb'}^{(2)} W_{cc'}^{(2)} \Gamma_{e's'c'}^{(3)} + \text{perms} \right. \\ \left. + W_{qq'}^{(2)} W_{bb'}^{(2)} W_{cc'}^{(2)} W_{dd'}^{(2)} \Gamma_{e's'c'd'}^{(4)} \right]$$



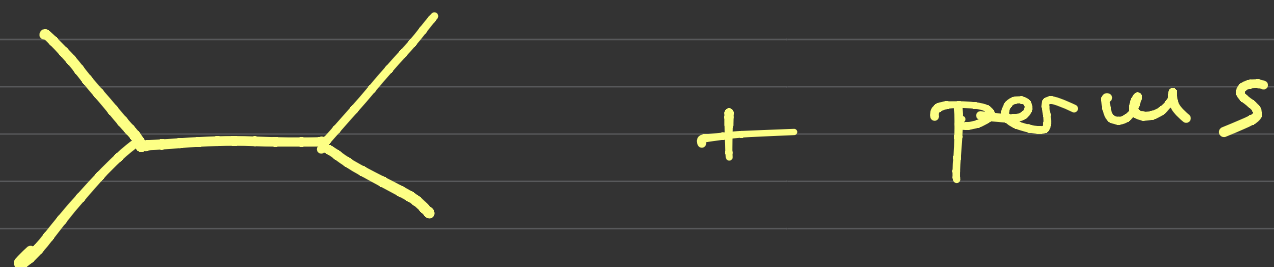
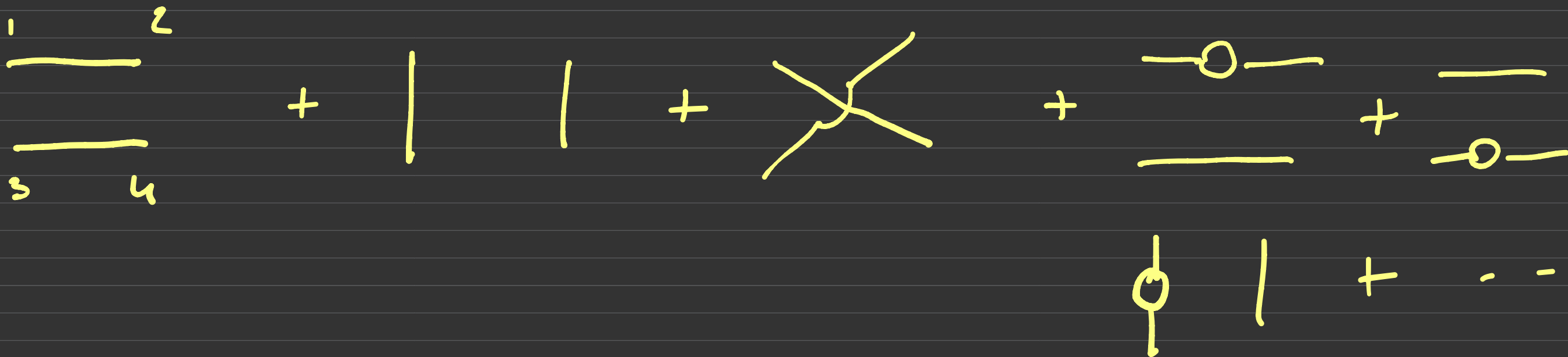
$$2 \rightarrow 2 \quad W^{(4)} \cdot W^{(2)-1} W^{(2)-1} W^{(2)-1} W^{(2)-1}$$

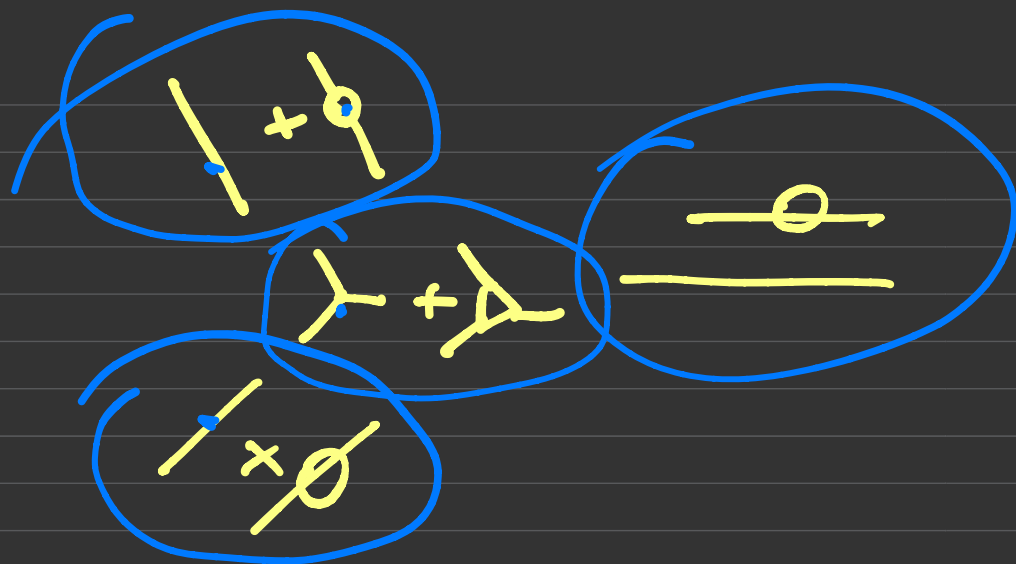
$$\delta(2 \rightarrow 2) = \text{Diagram 1} + \text{perms} - \text{Diagram 2}$$

Diagram 1: Two circles labeled Γ connected by two internal lines. External lines are labeled 1, 2, 3, 4.

Diagram 2: A single circle labeled Γ with four external lines labeled 1, 2, 3, 4.

Ex $\langle \phi(x_1) \dots \phi(x_n) \rangle$ in $\lambda \phi^3$ in pert. theory





$$\Gamma = \text{diagram 1} + \text{diagram 2} + \text{diagram 3}$$

The equation shows the sum of three Feynman diagrams, each with blue 'x' marks on its external lines, representing a total quantity Γ .

$$\begin{array}{c} \diagup \\ \text{---} \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \triangle \\ \diagdown \end{array} + \begin{array}{c} \circ \\ \diagup \\ \text{---} \\ \diagdown \end{array} + \text{perms}$$

$$\begin{array}{c} \circ \\ \diagup \\ \text{---} \\ \diagdown \\ \circ \end{array} + \begin{array}{c} \circ \\ \diagup \\ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array} +$$

$$\begin{array}{c} \circ \\ \diagup \\ \triangle \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \triangle \\ \text{---} \circ \text{---} \end{array}$$

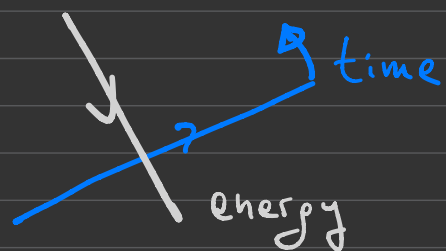
$$\begin{array}{c} \text{---} \circ \text{---} \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array} = \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \end{array}$$

$$\text{---} \circ \text{---} = \lambda + \begin{array}{c} \diagup \\ \triangle \\ \diagdown \end{array} + \dots$$

▲ S-matrix from Wick rotation (scalar field)

$$G_E(x_1, \dots, x_n) = \int d^4 p_1 \dots d^4 p_n e^{-i p_1 x_1 - \dots - i p_n x_n} \delta^{(4)}(p_1 + \dots + p_n) \hat{G}_E(p_2, \dots, p_n)$$

$$= \int d^4 p_2 \dots d^4 p_n e^{-i p_2(x_2 - x_1) - \dots - i p_n(x_n - x_1)} \hat{G}_E(p_2, \dots, p_n) e^{-i(p_0 x_0 + p_1 x_1 + \dots)}$$



Wick rotate

$$p_0 \rightarrow -i p_0$$

$$\Rightarrow \int (-i)^{n-1} d^4 p_2 \dots d^4 p_n e^{-i p_2(x_2 - x_1) - \dots - i p_n(x_n - x_1)} \hat{G}_E(\hat{p}_2, \dots, \hat{p}_n)$$

$p_\mu x^\mu$

$$\hat{p}_\mu = (-i p_0, \vec{p})$$

$$\blacksquare \quad G_M^{(n)}(p_2, \dots, p_n) = (-i)^{n-1} G_E^{(n)}(\hat{p}_2, \dots, \hat{p}_n)$$

$$\blacksquare \quad G_M^{(2)}(p) = -i G_E^{(2)}(\hat{p}) = \frac{-i}{(-ip_0)^2 + \vec{p}^2 + m^2}$$

$$= \frac{i}{p_0^2 - \vec{p}^2 + m^2}$$

$$\blacksquare \quad \text{LSZ} \Rightarrow (\text{for } \sqrt{z}=1)$$

$$S(p_1, \dots, p_n) = (2\pi)^4 \delta(p_1 + \dots + p_n) G_M^{-1}(p_1) \dots G_M^{-1}(p_n) G_M^{(n)}(p_2, \dots, p_n)$$

$$= (-i)^{-1} \cdot (2\pi)^4 \delta(p_1 + \dots + p_n) G_E^{-1}(\hat{p}_1) \dots G_E^{-1}(\hat{p}_n) G_E^{(n)}(\hat{p}_2, \dots, \hat{p}_n)$$

$$\text{with } p_i^2 \rightarrow m^2$$

• Connected part of S : $G^{(n)} \rightarrow G_c^{(n)}$

$$S_c(p_1, \dots, p_n) = -i \Gamma_E(\hat{p}_1, \dots, \hat{p}_n) + o.o$$

▣ Ex $\lambda \phi^4 \Rightarrow \Gamma_E^{(4)} = (2\pi)^4 \delta^4(p_1 + \dots + p_4) \lambda + O(\lambda^2)$

$$\Rightarrow S(p_1, \dots, p_4) = - (2\pi)^4 \delta^4(p_1 + \dots + p_4) (i\lambda) + O(\lambda^2)$$

in agreement with Feynman diagrams

for $\mathcal{L} = \frac{(\partial\phi)^2}{2} - \frac{\lambda}{4!} \phi^4$

$2 \rightarrow 2$ scattering

$$S(p_1, p_2, p_3, p_4) = i G_E(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) G_E^{-1}(\hat{p}_1) \dots G_E^{-1}(\hat{p}_4)$$

$$= -i \Gamma_E^{(4)}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) + i \left[\Gamma_E^{(3)}(\hat{p}_1, \hat{p}_2, \hat{q}) G_E(\hat{q}) \Gamma_E^{(3)}(\hat{q}, \hat{p}_3, \hat{p}_4) + \text{permutations} \right]$$

$$= -i \Gamma_E^{(4)} + \left[(-i \Gamma_E^{(3)}) (-i G_E(\hat{q})) (-i \Gamma_E^{(3)}) + \text{perms} \right]$$

$$= i \Gamma_M^{(4)} + \left[i \Gamma_M^{(3)} G_M(\hat{q}) i \Gamma_M^{(3)} + \text{perms} \right]$$

where I defined $\Gamma_M^{(n)}(p_1, \dots, p_n) \equiv -\Gamma_E^{(n)}(\hat{p}_1, \dots, \hat{p}_n)$

▲ Eqs. of motion

$$\int D\phi e^{-S + i\phi} = Z[i]$$

$$\int D\phi \frac{\delta}{\delta\phi_a} \left(e^{-S + i\phi} \right) = 0$$

no boundary

$$\int D\phi \left(-\frac{\delta S}{\delta\phi_a} + i \right) e^{-S + i\phi} = 0$$

terms in ϕ
space

$$\left\langle -\frac{\delta S}{\delta\phi_a} + i \right\rangle = 0$$

$$\downarrow \quad \frac{\delta S}{\delta \phi_a} = \mathcal{F}(\phi_a) \rightarrow \mathcal{F}\left(\frac{\delta}{\delta J_a}\right)$$

$$\left[-\frac{\delta S}{\delta \phi_a} \left[\frac{\delta}{\delta J} \right] + J_a \right] \sum_i [J] = 0$$

act with $\frac{\delta}{\delta J_a}$

$$\left. \frac{\delta S}{\delta \phi_a} \left[\frac{\delta}{\delta J} \right] \frac{\delta}{\delta J_{b_1}} \dots \frac{\delta}{\delta J_{b_n}} \sum_i \right|_{J=0} = \left(\delta_{b_1 a} \frac{\delta}{\delta J_{b_2}} \dots \frac{\delta}{\delta J_{b_n}} + \delta_{b_2 a} \frac{\delta}{\delta J_{b_1}} \frac{\delta}{\delta J_{b_3}} \dots \right) \sum_i \Big|_{J=0}$$

$$\frac{\delta}{\delta J_{b_1}} \dots \frac{\delta}{\delta J_{b_n}} \Big|_{J=0} \Rightarrow \text{infinite set of identities}$$

$Q \rightarrow (Q, x)$

$$\left\langle \frac{\delta S}{\delta \phi_a} \phi_{b_1} \dots \phi_{b_n} \right\rangle = \delta_{ab_1} \langle \phi_{b_2} \dots \phi_{b_n} \rangle + \delta_{ab_2} \langle \phi_{b_1} \phi_{b_3} \dots \rangle + \dots \delta_{ab_n} \langle \phi_{b_1} \dots \phi_{b_{n-1}} \rangle$$

$$\begin{aligned} \left\langle \frac{\delta S}{\delta \phi_{\beta_1}(x)} \phi_{\beta_1}(x_1) \dots \phi_{\beta_n}(x_n) \right\rangle &= \delta_{\alpha\beta_1} \delta(x-x_1) \langle \phi_{\beta_2}(x_2) \dots \phi_{\beta_n}(x_n) \rangle \\ &+ \dots \delta_{\alpha\beta_n} \delta(x-x_n) \langle \phi_{\beta_1}(x_1) \dots \phi_{\beta_{n-1}}(x_{n-1}) \rangle \end{aligned}$$

⊙ Wick rotation $-S \rightarrow iS$

▲ Ward identities

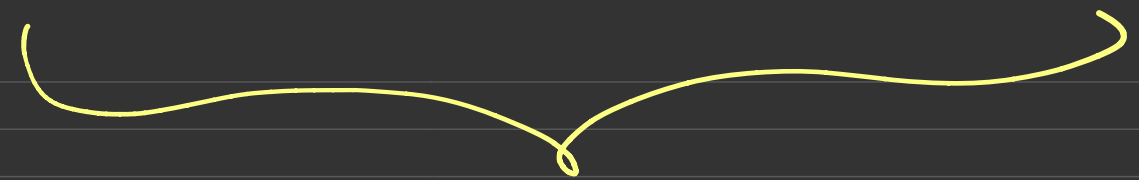
■ Classical case (Noether): $S = \int \mathcal{L}$

$\phi_a \longrightarrow \phi_a + \Delta_i \phi_a \cdot \varepsilon_i$ is a symmetry if $i=1 \dots N \equiv \text{Lie parameters}$

$$\frac{\partial \mathcal{L}}{\partial \phi} \Delta_i \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \Delta_i \phi = \partial_\mu K_i^\mu \quad i=1 \dots N$$

$$\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right) \Delta \phi = - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \partial_\mu K^\mu$$

$$= \partial_\mu \left(K^\mu - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right)$$



μ

Quantum case: path integral

• change of variables $\phi_a = \phi'_a + \varepsilon \cdot (\Delta\phi)_a + o(\varepsilon^2)$

$$Z[\hat{J}] = \int \mathcal{D}\phi e^{-S + \hat{J}\phi} = \int \mathcal{D}(\phi' + \varepsilon \Delta\phi) e^{-S[\phi' + \varepsilon \Delta\phi] + \hat{J}(\phi' + \varepsilon \Delta\phi)}$$

$$= \int \mathcal{D}\phi' \left(1 + \int d^4x \varepsilon \frac{\delta \Delta\phi_a(x)}{\delta \phi_a(x)} \right) \left(1 - \int \frac{\delta S}{\delta \phi} \varepsilon \cdot \Delta\phi + \hat{J} \varepsilon \Delta\phi \right) e^{-S + \hat{J}\phi'} + o(\varepsilon^2)$$

$$\int \mathcal{D}\phi \int d^4x \left(\varepsilon \frac{\delta \Delta\phi_a}{\delta \phi_a} - \frac{\delta S}{\delta \phi} \varepsilon \Delta\phi + \hat{J} \varepsilon \Delta\phi \right) e^{-S + \hat{J}\phi} = 0$$

$$\bullet \int \frac{\delta S}{\delta \phi} \varepsilon \Delta \phi \equiv \int \frac{\partial \mathcal{L}}{\partial \phi_a} \varepsilon \Delta \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu (\varepsilon \Delta \phi_a) =$$

$$\int \varepsilon \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \Delta \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu \Delta \phi_a \right) + \partial_\mu \varepsilon \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta \phi_a$$

$$\stackrel{\text{hp}}{=} \int \varepsilon \partial_\mu K^\mu + \partial_\mu \varepsilon \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta \phi_a$$

$$= \int \varepsilon \partial_\mu \left(K^\mu - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \Delta \phi_a \right) \equiv - \int \varepsilon \partial_\mu \tilde{J}^\mu$$

$$\int \mathcal{D}\phi \int d^4x \left(\varepsilon(x) \frac{\delta \Delta \phi_a(x)}{\delta \phi_a(x)} + \varepsilon \partial_\mu \tilde{J}^\mu(x) + \varepsilon \hat{\tilde{J}}_a(\Delta \phi)_a \right) e^{-S + \hat{\tilde{J}}\phi} = 0$$

$$\int \mathcal{D}\phi \left(\frac{\delta \Delta \phi_a(x)}{\delta \phi_a(x)} + \partial_\mu \vec{J}^\mu(x) + \hat{J}_a \cdot \Delta \phi_a(x) \right)$$

$$\times e^{-S + \mathcal{D}\phi} \Rightarrow$$

$$\vec{J}_\mu \neq \vec{J}_a$$

$$\pi d p_a$$

$$\frac{\partial p_a}{\partial p'_b} = |\delta_{ab} + \varepsilon f_{ab}|$$

$$p_a = p'_a + \varepsilon f_a(p')$$

$$= 1 + \varepsilon \sum_a f_{aa}$$

$$\frac{\partial p_a}{\partial p'_b} = \delta_{ab} + \varepsilon \frac{\partial f_a}{\partial p'_b} = \underbrace{\delta_{ab} + \varepsilon f_{ab}}$$

$$\pi d p_a = \pi d p'_a \left| \frac{\partial p_a}{\partial p'_a} \right|$$

$$\prod_{x,a} \mathcal{D}\phi_a(x)$$

$$\sum_p f_{aa} \rightarrow \int_x f_{aa}$$

$$\phi'_a(x) = \phi_a(x) + \varepsilon_i (\Delta; \phi)_a(x)$$

$$\text{Jacobian} = \left| \frac{\delta \phi_a(x)}{\delta \phi_b(x')} \right| = \left| \delta_{ab} \delta(x-x') + \varepsilon_i \frac{\delta (\Delta; \phi)_a(x)}{\delta \phi_b(x')} \right|$$

$$= \underline{1} + \sum_a \int_x \varepsilon_i(x) \frac{\delta (\Delta; \phi)_a(x)}{\delta \phi_a(x)}$$

● Notice

$$\frac{\delta \Delta \phi_a(x)}{\delta \phi_a(x)} = \delta^{(4)}(0) \mathcal{F}(\phi)$$

$$\begin{array}{l} \text{Ex } \Delta \phi_a = \phi_a \\ \frac{\delta \phi_a(x)}{\delta \phi_b(y)} = \delta_{ab} \delta(x-y) \end{array}$$

- proper definition requires "regulator" of $\delta^4(0)$

A) in most cases of interest $\mathcal{F}(\phi) = 0$

Ex phase rotations

B) If symmetry can be extended to $d \neq 4$ then in dimensional regularization $\delta^d(0) = 0$

C) In some situations no regulator where $\delta^{(4)}(0) \mathcal{F}(\phi) = 0$ exists $\Rightarrow$ quantum anomaly

$$\Delta\phi = i\alpha\phi$$

$$\Delta\phi^* = -i\alpha\phi^*$$

$$\phi = \phi_1 + i\phi_2$$

Measure

$$\underline{\Delta\phi} \underline{\Delta\phi^*} = \Delta\phi_1 \Delta\phi_2$$

(A) et (B) $\equiv$ ordinary situation $\Rightarrow$

$$\int \mathcal{D}\phi \left(\partial_\mu \vec{J}^\mu(\phi(y)) + \vec{J}_q \cdot (\Delta \phi_q)(\phi(y)) \right) e^{-S + \hat{J} \cdot \phi}$$

$$\int \mathcal{D}\phi \left(\partial_\mu \vec{J}^\mu \left(\frac{\delta}{\delta \vec{J}(y)} \right) + \vec{J}_q \cdot (\Delta \phi_q) \left(\frac{\delta}{\delta \vec{J}(y)} \right) \right) e^{-S + \hat{J} \cdot \phi}$$

$$\boxed{\phi_q \sim \frac{\delta}{\delta \vec{J}_q}}$$

$$\left\{ \partial_\mu \vec{J}^\mu \left(\frac{\delta}{\delta \hat{J}(y)} \right) + \hat{J}_q(y) \Delta \phi_q \left(\frac{\delta}{\delta \hat{J}(y)} \right) \right\} Z[\hat{J}] = 0$$

$$\langle \phi_{q_1}(x_1) \cdots \phi_{q_n}(x_n) \rangle = \frac{\delta}{\delta \hat{J}_{q_1}(x_1)} \cdots \frac{\delta}{\delta \hat{J}_{q_n}(x_n)} Z[\hat{J}] \Big|_{\hat{J}=0}$$

$$\left. \frac{\delta}{\delta \hat{J}_{q_1}(x_1)} \dots \frac{\delta}{\delta \hat{J}_{q_n}(x_n)} \left\{ \partial_\mu \hat{J}^\mu \left(\frac{\delta}{\delta \hat{J}(y)} \right) + \hat{J}(y) \Delta \phi_q \left(\frac{\delta}{\delta \hat{J}(y)} \right) \right\} Z[\hat{J}] \right|_{\hat{J}=0} = 0$$

$$\begin{aligned} & \partial_\mu \langle \hat{J}^\mu(y) \phi_{q_1}(x_1) \dots \phi_{q_n}(x_n) \rangle + \delta_{q_1 q} \delta(x_1 - y) \langle \Delta \phi_q(y) \phi_{q_2}(x_2) \dots \phi_{q_n}(x_n) \rangle \\ & + \dots + \delta_{q_n q} \delta(x_n - y) \langle \Delta \phi_q(y) \phi_{q_1}(x_1) \dots \phi_{q_{n-1}}(x_{n-1}) \rangle \\ & = 0 \end{aligned}$$

Ward identity

functional		operator
$\hat{J}^\mu \left(\frac{\delta}{\delta \hat{J}(y)} \right)$	$\longrightarrow$	$\hat{J}^\mu(y)$
$\frac{\delta}{\delta \hat{J}_{q_1}(x_1)}$	$\longrightarrow$	$\phi_{q_1}(x_1)$

I) Linearly realized symmetry

$$\langle \mathcal{J}^H(y) \phi_{a_1}(x_1) \dots \phi_{a_n}(x_n) \rangle \xrightarrow{y \rightarrow \infty} 0 \quad (\text{fast enough})$$
$$\langle \mathcal{J}^H(y) \phi_{a_1}(x_1) \dots \phi_{a_n}(x_n) \rangle < O\left(\frac{1}{y^3}\right)$$

$$\langle \Delta \phi_{a_1} \phi_{a_2} \dots \phi_{a_n} \rangle + \langle \phi_{a_1} \Delta \phi_{a_2} \dots \phi_{a_n} \rangle + \dots + \langle \phi_{a_1} \dots \phi_{a_{n-1}} \Delta \phi_{a_n} \rangle = 0$$

$$\langle \Delta(\phi_{a_1} \dots \phi_{a_n}) \rangle = 0$$

precisely expected when $\exists$ a symmetry generator in Hilbert space

$$\text{When } \int_{\mathbb{R}} \mathbb{Q} \left\{ \begin{array}{l} \bullet \ i \int_{\mathbb{R}} \mathbb{Q}, \phi_{\mathbb{Q}}(x) = \Delta \phi_{\mathbb{Q}}(x) \\ \bullet \ \mathbb{Q} | 0 \rangle = 0 \end{array} \right.$$

$$U = e^{i \alpha \mathbb{Q}}$$

$$\langle 0 | i \int_{\mathbb{R}} \mathbb{Q}, \phi_{\mathbb{Q}_1} \dots \phi_{\mathbb{Q}_n} | 0 \rangle = \langle 0 | i \mathbb{Q} (\phi_{\mathbb{Q}_1} \dots \phi_{\mathbb{Q}_n}) - (\phi_{\mathbb{Q}_1} \dots \phi_{\mathbb{Q}_n}) i \mathbb{Q} | 0 \rangle$$

$$= 0$$

↓

$$\langle 0 | \Delta \phi_{\mathbb{Q}_1} \phi_{\mathbb{Q}_2} \dots \phi_{\mathbb{Q}_n} + \dots + \phi_{\mathbb{Q}_1} \dots \phi_{\mathbb{Q}_{n-1}} \Delta \phi_{\mathbb{Q}_n} \rangle = 0$$

II) Non-linearly realized symmetry

$$\langle j^\mu(y) \phi_{q_1}(x_1) \dots \phi_{q_n}(x_n) \rangle \stackrel{y \rightarrow \infty}{\propto} \partial^\mu \frac{1}{y^2} \propto \frac{1}{y^3}$$

$\Rightarrow$ boundary term cannot be dropped !!

$$\langle 0 | \Delta(\phi_{q_1} \dots \phi_{q_n}) | 0 \rangle \neq 0$$

• $\exists$ conserved current $j^\mu \Leftrightarrow$ Ward id.

~~• $\exists$ Q such that~~

$$\begin{cases} i[\bar{Q}, \phi] = \Delta \phi \\ Q|0\rangle = 0 \end{cases}$$

$$\int_{\mu} \partial \langle \partial^{\mu}(\eta) \phi(x) \rangle + \int \langle \Delta \phi(x) \rangle \delta(\eta - x) = 0$$

$$\partial \langle \partial^{\mu}(\eta) \phi(x) \rangle = -F \delta(\eta - x)$$

$$\langle \Delta \phi \rangle \neq 0$$

$$\nexists Q \text{ existed } Q(0) = 0$$

$$\int e^{ip\eta} \langle \partial^{\mu}(\eta) \phi(0) \rangle d^4_{\eta} = G^{\mu}(p) \quad \langle 0 | : [Q, \phi] | 0 \rangle = 0$$

$$\int e^{ip\eta} \langle 0 | \underbrace{U(\lambda)^{-1} \partial(\eta) U(\lambda)} \underbrace{U(\lambda)^{-1} \phi(0) U(\lambda)} | 0 \rangle \quad \langle \Delta \phi \rangle \equiv F$$

$$\int e^{ip\eta} \lambda^{\mu}_{\nu} \langle \partial^{\nu}(\lambda^{-1}\eta) \phi(0) \rangle d^4_{\eta}$$

$$= \Lambda^\mu_\nu \int e^{i(\Lambda'^\mu P)(\Lambda'^\nu y)} \langle \tilde{J}^\nu(\Lambda'^\nu y) \phi(0) \rangle d^4 y$$

$$\Lambda'^\nu y \equiv y' \quad d^4 y = d^4 y'$$

$$= \Lambda^\mu_\nu \int e^{-i(\Lambda'^\mu P)y'} \langle \tilde{J}^\nu(y') \phi(0) \rangle d^4 y'$$

$$= \underbrace{\Lambda^\mu_\nu G^\nu(\Lambda'^\mu P)}_{\Downarrow} = G^\mu(P)$$

$$G^\mu = P^\mu f(P^2)$$

Prove!

$$\partial_\mu \langle \partial^\mu(y) \phi(x) \rangle = -F \delta(y-x)$$

$$i p_\mu G^\mu(p) = -F$$

$$i p_\mu p^\mu f(p^2) = -F$$

$$f(p^2) = \frac{i F}{p^2}$$

↙ Goldstone pole