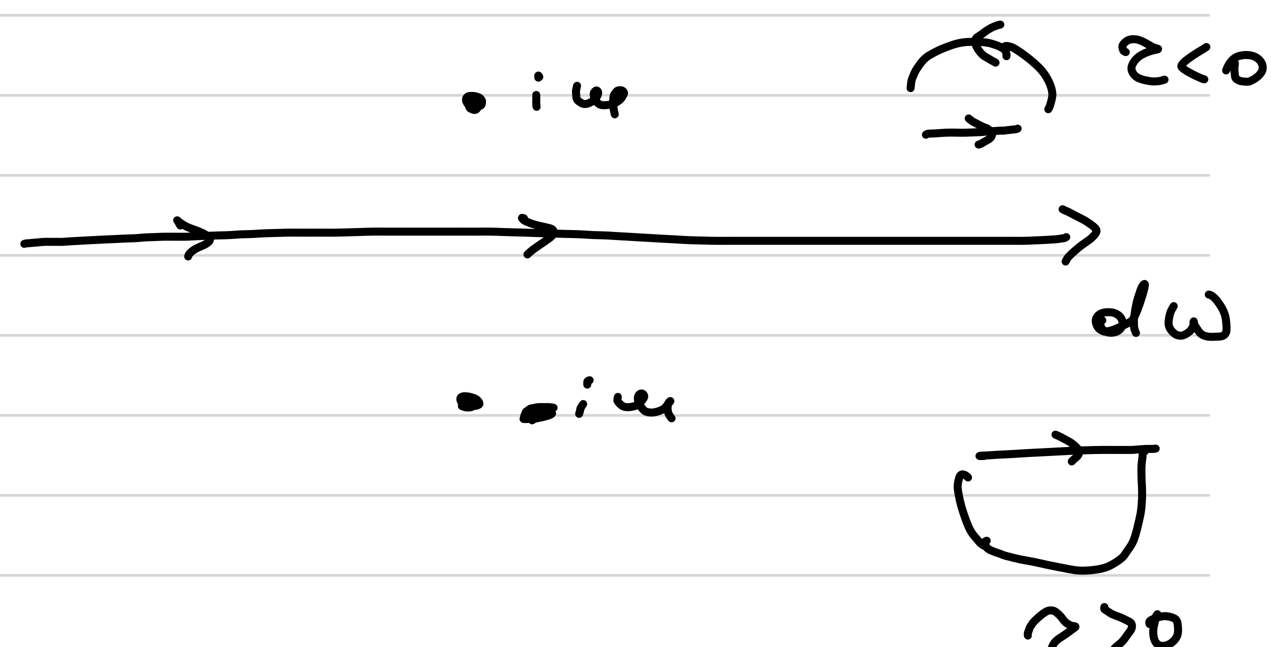
• $\displaystyle\int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega^2+m^2} = \frac{1}{2m} e^{-m|\tau|} = G(\tau)$

$$(-\partial_\tau^2 + m^2)\, G(\tau) = \int \frac{d\omega}{2\pi}\, e^{-i\omega\tau} = \delta(\tau)$$

indeed $\displaystyle -\partial_\tau^2\, e^{-m|\tau|} = +\partial_\tau \left[m \,\mathrm{sgn}\,\tau \; e^{-m|\tau|} \right]$

$$= -m^2 e^{-m|\tau|} + 2m\,\delta(\tau)$$

• $G(\tau)$



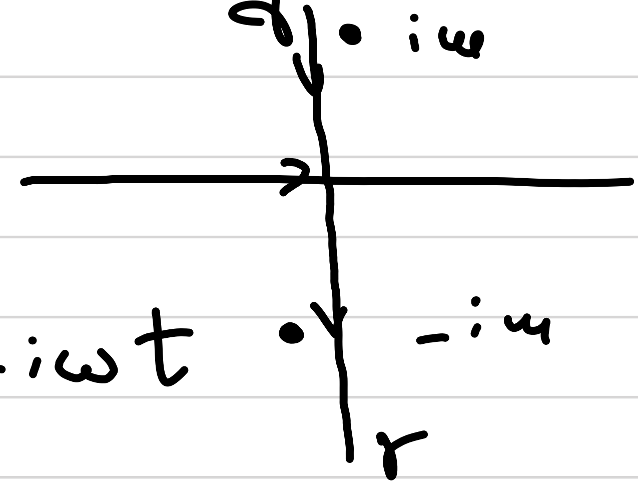
• Continuation to real time is done by continuous counterclockwise rotation of τ

$$\tau \to e^{i\theta}\tau \qquad \theta : 0 \to \pi/2$$

• when τ is so rotated the domain of ω plane where $\mathrm{Im}(\omega\tau) < 0$

also rotates, but clockwise, by $e^{-i\theta}$

In the end, the contour of ω integration becomes



$$\Rightarrow G_{real}(t) = \int_{\gamma} \frac{-i d\omega}{2\pi} \frac{e^{-i\omega t}}{-\omega^2 + m^2}$$

$$\equiv \int \frac{d\omega}{2\pi} \frac{i e^{-i\omega t}}{\omega^2 - m^2 + i\varepsilon} \equiv G(it)$$

· Similarly for field theory

$$\langle T(\varphi(x)\varphi(y))\rangle_E = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + m^2}$$

$$\Rightarrow \langle T(\varphi(x)\varphi(y))\rangle_M = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

- For the case of the 2-point function in free field theory the situation is clear and simple. But it also holds true for all higher point functions in the theories of interest. This is guaranteed by important results in constructive (axiomatic) QFT: Osterwalder-Schrader theorem.

- S-matrix elements can then be computed in a two step procedure

1) Compute euclidean correlators by using euclidean path integral

2) Analytically continue to Minkowsky and use LSZ.

— o —

Ground state projection

$$\langle \psi_f | e^{-H(\tau_f-\tau_n)} \varphi_n(x_n) e^{-H(\tau_n-\tau_{n-1})} \varphi_{n-1}(x_{n-1}) \cdots \varphi_1(x_1) e^{-H(\tau_1-\tau_{in})} |\psi_i\rangle$$

$$\langle \psi_f|0\rangle\langle 0|\psi_i\rangle e^{-E_0(\tau_f-\tau_i)} \Big(\langle 0| T\big(\varphi_n(x_n) \cdots \varphi_1(x_1)\big)|0\rangle + O\big(e^{-(E_1-E_0)(\tau_f-\tau_i)}\big)\Big)$$

$$\Downarrow$$

$$\lim_{\substack{\tau_f\to\infty \\ \tau_i\to-\infty}} \frac{\int_{\psi_i,\psi_f} \mathcal{D}\phi \; \varphi_n(x_n)\cdots\varphi_1(x_1)\, e^{-S}}{\int_{\psi_i\psi_f} \mathcal{D}\phi \; e^{-S}} =$$

$$= \langle 0 | T\left(\varphi_n(x_n) \cdots \varphi_1(x_1)\right) | 0 \rangle$$

Formally I could include the denominator in the measure $D\phi$ and simply write

$$\langle 0 | T(\phi_n(x_n) \cdots \phi_1(x_1)) | 0 \rangle = \int D\phi \, \phi_n(x_n) \cdots \phi_1(x_1) e^{-S}$$

Functional methods

$$Z_1[J] = \int D\phi \, e^{-S + J \cdot \phi}$$

$$J\phi \equiv \int d^4x \, J_a \phi_a$$

$$\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_n} Z_1[J]\Big|_{J=0} = \int D\phi\, \phi_1 \cdots \phi_n\, e^{-S}$$

$$= Z_1[0] \langle 0 | T(\phi_1 \cdots \phi_n) | 0 \rangle$$

in the normalization of previous page $Z_1[0] = 1$

$$\Rightarrow \prod_{i=1}^{N} \frac{\delta}{\delta J_i} Z_1[J]\Big|_{J=0} = \langle 0 | T(\phi_1 \cdots \phi_n) | 0 \rangle$$

$Z_1[J] \equiv$ partition function

$$Z_1[J] \equiv e^{W[J]}$$

generator of connected correlators

$$\langle \phi_1 \cdots \phi_n \rangle_c \equiv \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} W[J]$$

Connected and disconnected

· Definition (by induction)

$$\langle q_{a_1} \cdots q_{a_n} \rangle = \langle q_{a_1} \cdots q_{a_n} \rangle_c + \sum_{\substack{\text{All} \\ \text{partitions}}}{}' \langle \quad \rangle_c \langle \quad \rangle_c$$

$$\equiv \sum_{\substack{\text{All} \\ \text{partitions}}} \langle \quad \rangle_c \langle \quad \rangle_c \cdots$$

$$\langle q_a \rangle = \langle q_a \rangle_c$$

<u>Ex</u> $\langle q_a q_b \rangle = \langle q_a q_b \rangle_c + \langle q_a \rangle_c \langle q_b \rangle_c$

$$\langle q_a q_b q_c \rangle = \langle q_a q_b q_c \rangle_c + \langle q_a \rangle \langle q_b q_c \rangle_c$$
$$\langle q_b \rangle \langle q_a q_c \rangle_c + \langle q_c \rangle \langle q_a q_b \rangle_c$$
$$+ \langle q_a \rangle \langle q_b \rangle \langle q_c \rangle$$

$$\langle q_a q_b q_c \rangle = \langle q_a q_b q_c \rangle_c + \langle q_a \rangle \langle q_b q_c \rangle$$
$$+ \cdots$$
$$- 2 \langle q_a \rangle \langle q_b \rangle \langle q_c \rangle$$

$$\langle q_a q_b q_c \rangle_c = \langle q_a \cdots q_c \rangle - \langle q_a \rangle \langle q_b q_c \rangle - \cdots$$
$$+ 2 \langle q_a \rangle \langle q_b \rangle \langle q_c \rangle$$

$$Z[J] = e^{W[J]} = \int \mathcal{D}\phi \, e^{-S + J\phi} \qquad e^{W(0)} = 1$$

$$\langle \phi_1 \cdots \phi_n \rangle = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} e^{W(J)}$$

$$= \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} W + \sum_{\substack{\text{all} \\ \text{partitions}}}{}'$$

Ex

$$\langle \phi_1 \phi_2 \rangle = \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} W + \frac{\delta W}{\delta J_1} \frac{\delta W}{\delta J_2} =$$

$$= \langle \phi_1 \phi_2 \rangle_c + \langle \phi_1 \rangle \langle \phi_2 \rangle$$

Effective Action

$$W[J] \to \begin{cases} \Gamma[\phi] = -W[J] + J\cdot\phi \\ \phi_a = \dfrac{\delta W}{\delta J_a} \to J = J[\phi] \end{cases}$$

$$\frac{\delta \Gamma}{\delta \phi_b} = \left(-\cancel{\frac{\delta W}{\delta J_a}} \cancel{\frac{\delta J_a}{\delta \phi_b}} + \cancel{\frac{\delta J_a}{\delta \phi_b}} \phi_a + J_b \right) = J_b$$

$\Gamma \equiv$ generator of proper vertices

- 2-point function

$$\frac{\delta^2 \Gamma}{\delta \phi_a \delta \phi_b} = \frac{\delta J_b}{\delta \phi_a} = \frac{\delta J_a}{\delta \phi_b}$$

$$= \left(\frac{\delta^2 W}{\delta J_a \delta J_b} \right)^{-1} \equiv \left(W^{(2)} \right)^{-1}_{ab}$$

$$\Gamma^{(2)}_{ab} = \left(W^{(2)} \right)^{-1}_{ab}$$

- 3point function

$$\frac{\delta^3 \Gamma}{\delta\phi_a \delta\phi_b \delta\phi_c} = \frac{\delta}{\delta\phi_c}\left(\frac{\delta^2 W}{\delta J_a \delta J_b}\right)^{-1}$$

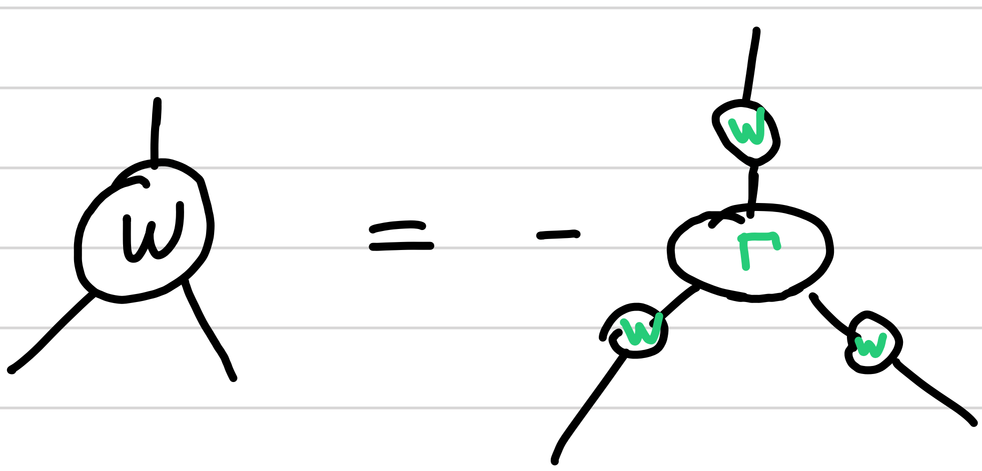
$$= -\left[\left(W^{(2)}\right)^{-1}\left(\frac{\delta W^{(2)}}{\delta\phi_c}\right)\left(W^{(2)}\right)^{-1}\right]_{ab}$$

$$= -\left(W^{(2)}\right)^{-1}_{aa'}\left(W^{(2)}\right)^{-1}_{bb'}\left(W^{(2)}\right)^{-1}_{cc'} W^{(3)}_{a'b'c'}$$

$$W^{(3)}_{abc} \equiv \frac{\delta^3 W}{\delta J_a \delta J_b \delta J_c}$$

or equivalently

$$W^{(3)}_{abc} = -\left(\Gamma^{(2)}\right)^{-1}_{aa'}\left(\Gamma^{(2)}\right)^{-1}_{bb'}\left(\Gamma^{(2)}\right)^{-1}_{cc'} \Gamma^{(3)}_{a'b'c'}$$



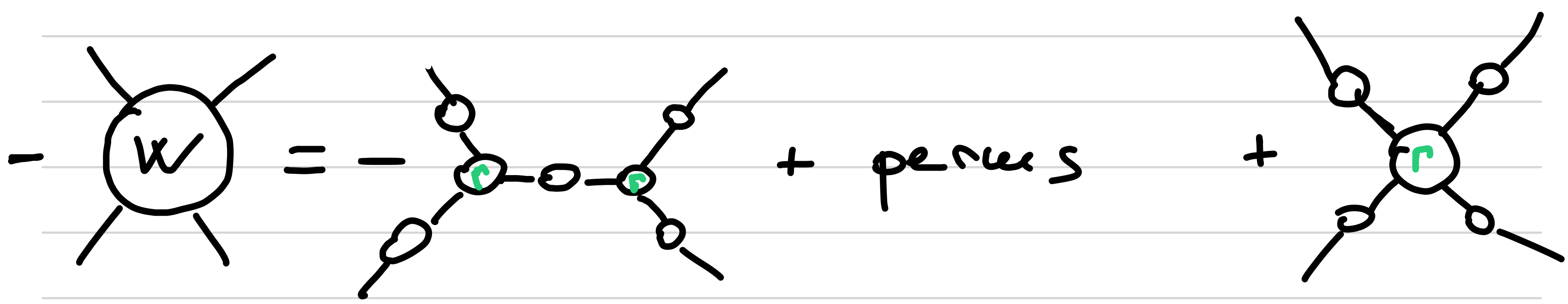
By writing $W^3_{abc} = -W^2_{aa'} W^2_{bb'} W^2_{cc'} \Gamma_{a'b'c'}$

we can immediately generalize to n-point

__Ex__

$$\frac{\delta W_{abc}}{\delta J_d} = \left(- \frac{\delta W_{aa'}}{\delta J_d} W_{bb'} W_{cc'} \Gamma_{a'b'c'} + \text{perms} \right)$$

$$- W_{aa'} W_{bb'} W_{cc'} W_{dd'} \Gamma_{a'b'c'd'}$$



Notice: in the relation between $\Gamma^{(n)}$ and $W^{(n)}$ there is a $(-)$ except for $n=2$

S-matrix from Wick rotation

$$G_E(x_1 \dots x_n) = \int d^4p_1 \dots d^4p_n \, \delta^4(p_1 + \dots p_n) \, e^{-ip_1x_1 \dots -ip_nx_n} \, \hat{G}_E(p_1, \dots p_n)$$

$$= \int d^4p_2 \dots dp_n \, e^{-ip_2(x_2-x_1) \dots -ip_n(x_n-x_1)} \, \hat{G}_E(p_1, \dots p_n)$$

$$p_1 = -p_2 \dots -p_n$$

$$\Big\Downarrow \text{ wick rotate}$$

$$= \int (-i)^{n-1} \underbrace{d^4p_2 \dots dp_n}_{\text{Minkowsky}} \, e^{-ip_1(x_2-x_1) - -} \, \hat{G}_E(\hat{p}_1 \dots \hat{p}_n)$$

$$\hat{p}_\mu = (-i p_0, \vec{p})$$

$$\Downarrow$$

$$G_M^{(n)}(p_1 \dots p_n) = (-i)^{n-1} G_E(\hat{p}_1 \dots \hat{p}_n)$$

with and without the $\delta(p_1 \dots p_n)$

$$G_M^{(2)}(p) = (-i)\, G_E(\hat{p})$$

By LSZ (and assuming $\sqrt{Z}=1$ for simplicity)

$$S = G_M^{(n)}(p_1, \dots p_n)\, G_M^{(2)}(p_1)^{-1} \cdots G_M^{(2)}(p_n)^{-1}$$

$$= (-i)^{-1}\, G_E^{n}(\hat{p}_1, \dots \hat{p}_n)\, G_E^{(2)}(\hat{p}_1)^{-1} \cdots G_M^{(2)}(\hat{p}_n)^{-1}$$

focussing on the connected part

$$S(p_1 \dots p_n) = -i\, \Gamma_E(\hat{p}_1, \dots \hat{p}_n)$$

Ex in $\lambda\phi^4$ $\Gamma = (2\pi)^4 \delta^4(p_1 \dots p_4)\, \lambda + O(\lambda^2$

$$\Rightarrow S(p_1 \dots p_4) = -(2\pi)^4 \delta^4(p_1 \dots p_4)(i\lambda)$$

in agreement with e^{iS}

$$S = \frac{(\partial\phi)^2}{2} - \frac{\lambda\phi^4}{4!}$$

Equations of motion and Ward identity

In differential calculus we are familiar with the vanishing of the integral of the total derivative of a function that vanishes at the boundaries, for instance a function that vanishes at ∞.

The same holds true in functional integrals:

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta \phi_a} e^{-S + J\cdot\phi} = 0$$

given $e^{-S} \to$ at large field values

Expanding we get

$$\int \mathcal{D}\phi \left(\frac{\delta S}{\delta \phi_a} - J_a \right) e^{-S + J\phi} = 0$$

$$\Rightarrow \left(\frac{\delta S}{\delta \phi_a}\left(\frac{\delta}{\delta J}\right) - J_a \right) Z[J] = 0$$

Which implies an infinite set of identities among correlators

. Taking derivative $\frac{\delta}{\delta J_{b_1}} \cdots \frac{\delta}{\delta J_{b_n}} \Big|_{J=0}$

we get

$$\left\langle \frac{\delta S}{\delta \phi_a} \phi_{b_1} \cdots \phi_{b_n} \right\rangle = \delta_{ab_1} \langle \phi_{b_2} \cdots \phi_{b_n} \rangle$$

$$+ \delta_{ab_2} \langle \phi_{b_1} \phi_{b_3} \cdots \phi_{b_n} \rangle + \cdots \delta_{ab_n} \langle \phi_{b_1} \cdots \phi_{b_{n-1}} \rangle$$

making the notation more explicit: $a = (\alpha, x)$
$b_i = (\beta_i, y_i)$

$$\left\langle \frac{\delta S}{\delta \phi_\alpha(x)} \phi_{\beta_1}(y_1) \cdots \phi_{\beta_n}(y_n) \right\rangle = \delta_{\alpha\beta_1} \delta(x-y_1) \langle \phi_{\beta_2}(y_2) \cdots \phi_{\beta_n}(y_n) \rangle$$

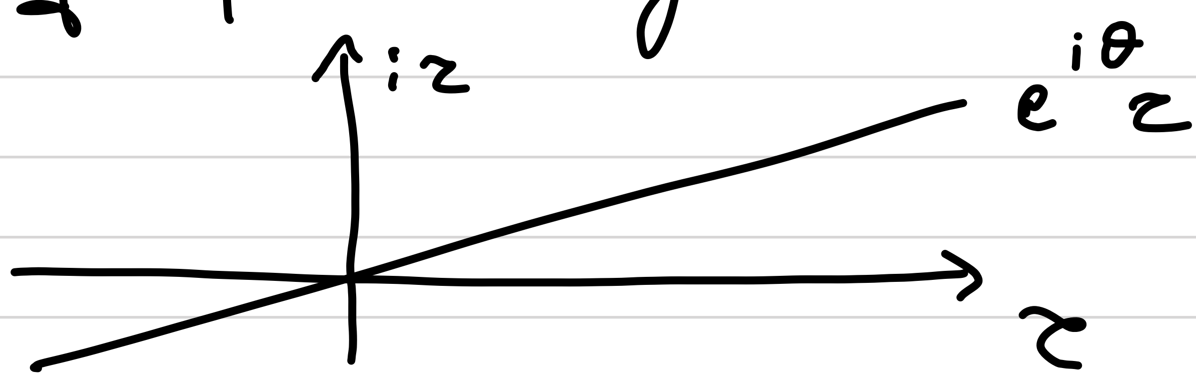
$$+ \cdots \delta_{\alpha\beta_n} \delta(x-y_n) \langle \phi_{\beta_1} \cdots \phi_{\beta_{n-1}}(y_{n-1}) \rangle$$

<u>Notice</u>: Wick rotating to Minkowski is equivalent to $-S \to iS$, hence a $(-i)$ appears on the left-hand side of the eq.

<u>Notice</u>: This the realization of Ehrenfest theorem on correlators. Indeed when points are not coincident this is precisely Ehrenfest's operatorial relation. At coinciding points things are more subtle because of Time ordering and because all time and space derivatives act outside the time ordering by construction in the path integral definition of the correlators. Working directly with the path integral makes the end result quite obvious. (See Chapt 2.5 in Collins)

It is important to notice that the corresponding relation in real time possesses additional factor of $i \equiv e^{i\pi/2}$

Indeed the continuation from euclidean to minkowski can be done by defining a θ-dependent family of path integrals



$$\int \prod_i Dq_i \, e^{-\sum_i \left(\frac{1}{2} \frac{(\Delta q_i)^2}{\varepsilon e^{i\theta}} + V(q)\varepsilon e^{i\theta} - J_i q_i \varepsilon \right)}$$

$\varepsilon \equiv \Delta\tau$

$$= \int Dq \; e^{-\int \left(\frac{e^{-i\theta}}{2} \dot{q}^2 + e^{-i\theta} V(q) - Jq \right) d\tau}$$

which for $\theta = 0$ gives euclidean

partition function and for $\theta \Rightarrow \pi/2$ gives the Minkowskian one

$$Z_M = \int \mathcal{D}q \; e^{i\int \frac{\dot{q}^2}{2} - V(q) + \int J q}$$

$$= \int \mathcal{D}q \; e^{iS_M + Jq}$$

$\Rightarrow$ e.o.m are

$$\int \mathcal{D}q \left[i \frac{\delta S}{\delta q} + J \right] e^{iS + Jq} = 0$$

$$\left[i \frac{\delta S}{\delta q}\left(\frac{\delta}{\delta J}\right) + J \right] Z_1[J] = 0$$

From whence

$$\left\langle \frac{\delta S}{\delta \phi_a} \phi_{b_1} \cdots \phi_{b_n} \right\rangle = i\delta_{ab_1} \langle \phi_{b_2} \cdots \phi_{b_n} \rangle$$

$$+ i\delta_{ab_2} \langle \phi_{b_1} \phi_{b_3} \cdots \phi_{b_n} \rangle + \cdots i\delta_{ab_n} \langle \phi_{b_1} \cdots \phi_{b_{n-1}} \rangle$$

- **Ward identities**

Similarly to what we did with equations of motion we can derive the implications of Noether theorem on correlators.

- Quick recap of Noether:

$\phi \to \phi + \Delta\phi$ is a symmetry if

$$\frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \Delta\phi = \partial_\mu K^\mu$$

then I can write

$$\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)\Delta\phi = \partial_\mu K^\mu - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\Delta\phi\right)$$

$$\equiv -\partial_\mu J^\mu$$

from whence $\partial_\mu J^\mu = 0$ on e.o.m.

Consider now the change of integration variables $\phi_a = \phi'_a + \varepsilon(\Delta\phi)_a$ where $\varepsilon(x)$ is an infinitesimal function with compact support. That way ϕ and ϕ' satisfy the same boundary conditions. We have

$$Z[J] = \int D\phi \; e^{-S + J\cdot\phi} =$$

$$= \int D(\phi' + \varepsilon\Delta\phi) \; e^{-S[\phi' + \varepsilon\Delta\phi] + J(\phi' + \varepsilon\Delta\phi)}$$

expanding in ε we have

from measure

$$= \int D\phi \left(1 + \int dx \, \varepsilon \frac{\delta\Delta\phi(x)}{\delta\phi(x)}\right) \cdot$$

$$\left(1 - \int \frac{\delta S}{\delta\phi}(\varepsilon\Delta\phi) + \int J\varepsilon\Delta\phi\right) e^{-S + J\phi} + O(\varepsilon^2)$$

Notice

$$\int \frac{\delta S}{\delta \phi}(\varepsilon \Delta \phi) = \int \frac{\partial \mathcal{L}}{\partial \phi}(\varepsilon \Delta \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\varepsilon \Delta \phi)$$

$$\overset{hp}{=} \int \varepsilon \partial_\mu K^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \, \partial_\mu \varepsilon$$

$$= \int \partial_\mu \varepsilon J^\mu$$

where in the last step we integrated by parts

hence

$$0 = \int D\phi \left[\int \varepsilon \left[\frac{\delta (\Delta\phi)_a}{\delta \phi_a} + \partial_\mu J^\mu + J_a (\Delta \phi)_a \right] \right] \times e^{-S + J\cdot\phi}$$

Notice $\frac{\delta (\Delta\phi)_a (x)}{\delta \phi_a (x)} = \delta^{(4)}(0) F(\phi)$

- If $\mathcal{F}(\phi) \neq 0$ the measure of integration is not invariant
- The resulting effect is UV divergent:

$$\delta^{(4)}(0) = \lim_{x \to y} \delta^{(4)}(x-y) = \int \frac{d^4 k}{(2\pi)^4}$$

However:

1) in most cases of interest $\mathcal{F}(\phi) = 0$
   Ex for Poincaré, internal symmetries

2) if dimensional regularization available (that is $S^{(d)}$ is invariant) then $\delta^{(d)}(0) = 0$

$$\int \frac{d^d k}{(2\pi)^d} (k^2)^r = 0 \qquad \forall\, r \text{ including } r=0$$

equivalently one can add local UV divergent counterterms to S

3) There exist situations where the classical

action is invariant but the UV regulated path integral is not. In that case we have an anomalous symmetry. See later on.

- Focusing on $J=0$ we have

$$\left[\partial_\mu J^\mu\left(\frac{\delta}{\delta J}\right) + J_a (\Delta\phi)_a\right] Z_1[J] = 0$$

which on n-point function gives the Ward identity

$$\partial_\mu \langle J^\mu(y)\, \phi_1(x_1) \dots \phi_n(x_n) \rangle$$

$$+\, \delta(x_1 - y) \langle (\Delta\phi)_1 (x_1) \dots \phi_n(x_n) \rangle +$$

$$+ \dots \delta(x_n - y) \langle \phi_1 \dots (\Delta\phi)_n (x_n) \rangle = 0$$

The Ward id. is a <u>local</u> operator relation implementing Noether theorem in QFT the analogue in real time is obtained by $\partial^\mu \to -i\partial^\mu$

- When integrating the Ward id. in d^dy we have two possibilities

I) <u>Linearly realized symmetry</u>:

this arises when $\langle \partial^\mu(y)\, \phi_1(x_1) \cdots \phi_n(x_n)\rangle$ decays fast enough at $y \to \infty$ that we can drop the boundary term. We thus obtain

$$\langle (\Delta\phi)_1(x_1)\, \phi_2(x_2) \cdots \phi_n(x_n)\rangle + \cdots \langle \phi_1 \cdots (\Delta\phi)_n\rangle = 0$$

$$\Leftrightarrow \quad \delta\langle \phi_1 \cdots \phi_n\rangle = 0 \qquad (*)$$

expressing the invariance of correlators

II) <u>Non-linearly realized symmetry</u>:

This arises when $\langle J^\mu \cdots \rangle \propto \frac{1}{y^3}$

$\Rightarrow \partial_\mu \langle \partial^\mu \cdots \rangle \sim \frac{1}{y^4}$

in which case $\int \partial_\mu \langle \partial^\mu \cdots \rangle \, d^4y$ gives a finite boundary term so that (*) does not hold anymore. Formally this corresponds to the vacuum state not being invariant. Indeed by assuming the symmetry transformation is generated by a conserved charge Q, i.e. $[Q, \phi_a] = (\Delta\phi)_a$ and that $Q|0\rangle = 0$, eq. (*) simply follows. When (*) is violated we can thus also say that the symmetry is spontaneously broken: the dynamics is symmetric as evidenced by the Ward identity, but the vacuum is not invariant.

The asymptotic power law decay $\langle J^\mu X \rangle \propto \theta^\mu \frac{1}{y^2}$ corresponds to the presence of a massless scalar: the Goldstone Boson.