Towards Gauge Theories

The remarkable properties of $m=0$, $J=\pm 1$

We here want to study the "wave functions" for massless spin 1 states. The little group $ISO(2)$ plays a crucial role.

Remember the generators of $ISO(2)$

$$J^3, \quad W^1 = K^1 + J^2, \quad W^2 = K^2 - J^1$$

(Cavalier on overall signs..)

$$J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad W^1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad W^2 = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

an element of little group can be written as

$$W(\underline{\alpha}, \varphi) \equiv e^{i \underline{\alpha} \cdot \underline{W}} e^{-i \varphi J^3} \qquad \underline{\alpha} = (\alpha_1, \alpha_2)$$

In particular

$$W(\alpha) \equiv W(\alpha, 0) \equiv e^{i\underline{\alpha}\cdot\underline{W}} = \left(e^{i\alpha_1 W_1}\right)\left(e^{i\alpha_2 W_2}\right)$$

$$= \begin{pmatrix} 1+\gamma & \alpha_1 & \alpha_2 & -\gamma \\ \alpha_1 & 1 & 0 & -\alpha_1 \\ \alpha_2 & 0 & 1 & -\alpha_2 \\ \gamma & \alpha_1 & \alpha_2 & 1-\gamma \end{pmatrix} \qquad \gamma = \frac{\alpha_1^2 + \alpha_2^2}{2}$$

One can check that $W^\mu{}_\nu \bar{P}^\nu = \bar{P}^\mu$

with $\bar{P}^\mu = (\kappa, 0, 0, \kappa)$

We shall need these expressions shortly.

Let us now consider the structure of the interpolating matrix elements $\langle 0 | \mathcal{O}_1 | p, \sigma \rangle$

We shall consider two cases

i. $\mathcal{O}_A \Longrightarrow F_{\mu\nu} = (1,0) \oplus (0,1)$

ii. $\mathcal{O}_1 \longrightarrow A_\mu = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$\underline{i\,|}\qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & -B_3 & B_2 \\ & & 0 & -B_1 \\ & & & 0 \end{pmatrix}$$

$$\langle 0| F_{\mu\nu}(x)|p,\sigma\rangle = \langle 0| F_{\mu\nu}(0)|p,\sigma\rangle e^{-ipx} \equiv f_{\mu\nu}(p,\sigma) e^{-ipx}$$

- what constraints does symmetry place on $f_{\mu\nu}(p,\sigma)$? We saw in the massive case that these constraint corresponded to "relativistic wave equations".
- Use $\begin{cases} J_3 |\bar{p}, \pm 1\rangle = \pm 1 |\bar{p}, \pm\rangle \\ W_{1,2} |\bar{p}, \pm 1\rangle = 0 \end{cases}$

- $[J_3, E_3] = 0 \implies$

$$0 = \langle 0|[J_3, E_3]|\bar{p},\sigma\rangle = -\sigma \langle 0| E_3 |\bar{p},\sigma\rangle$$

similarly for $B_3 = 0 \quad f_{03} = f_{12} = 0$

operator

- $$[\hat{W}_2, F^{\mu\nu}] = (W_2 F)^{\mu\nu} + (F W_2^T)^{\mu\nu}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -B_1 - E_2 \\ & 0 & E_1 - B_2 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

Now $\hat{W}_{1,2} |\bar{p}, \sigma) = 0 \qquad \hat{W}_{1,2} |0) = 0$

$$\Rightarrow \quad 0 = \langle 0 | [\hat{W}_2, F^{\mu\nu}] | \bar{p}, \sigma)$$

$$\Rightarrow \quad B_1 + E_2 = B_2 - E_1 = 0$$

$$\Rightarrow \quad E_i = \epsilon_{ij} B_j \qquad i = 1, 2$$

- How can then the constraints

$E_3 = B_3 = 0 \qquad E_i = \epsilon_{ij} B_j$

be written?

$$
* \begin{cases} \bar{P}^{\mu} f_{\mu\nu} = 0 \\ \epsilon^{\mu\nu\rho\sigma} \bar{P}_{\nu} f_{\rho\sigma} = 0 \end{cases}
$$

Maxwell eqs $\Longleftrightarrow$ group theoretic constraints on $\langle 0 | F_{\mu\nu} | \bar{P}, \sigma \rangle$

(*) also allow us to write

$$
f_{\mu\nu}(\bar{P}, \sigma) = \sqrt{Z} \left(\bar{P}_{\mu} e_{\nu}(\bar{P}, \sigma) - \bar{P}_{\nu} e_{\mu}(\bar{P}, \sigma) \right)
$$

with

$$
e_{\nu}(\bar{P}, \pm 1) = \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}
$$

ii) $\langle 0 | A_\mu(x) | \bar{p}, \sigma \rangle = \langle 0 | A_\mu(0) | \bar{p}, \sigma \rangle e^{-i\bar{p}x}$

$$= Q_\mu(\bar{p}, \sigma)\, e^{-i\bar{p}\cdot x}$$

- $[J_3, A_{0,3}] = 0 \Rightarrow Q_{0,3} = 0$

- $[\hat{W}_i, A^\mu] = k^\mu A_i - \delta_i^\mu k^\nu A_\nu \qquad (*)$

$$\left.\begin{aligned} \Rightarrow [W_2, A_0] &= A_2 \\ [W_1, A_0] &= A_1 \end{aligned}\right\} \quad 0 = \langle 0 | [W_i, A_0] | \bar{p}, \sigma \rangle \;!! \quad \Rightarrow Q_{2,1} = 0$$

- Poincaré + unitarity of irrep $\Rightarrow Q_\mu(\bar{p}, \sigma) = 0$

- Alternatively, in order to have $Q_\mu \neq 0$ we either need
  - A_μ is <u>not</u> a covariant tensor (in particular $(*)$ not true)

manifest

- give up unitarity ≡ represent Poincaré on Hilbert space with non-positive metric

⚠ Consider the first option

if $\pi^\mu A_i$ term where absent, then $Q_{0,3}=0$ $Q_{1,2}\neq 0$ would be consistent. The elimination of this terms correspond to complementing the standard transformation $U^\dagger(\lambda) A^\mu U(\lambda) = \Lambda^\mu{}_\nu A^\nu$ with a term $\equiv \partial^\mu \Omega$

$$U^\dagger(\lambda) A^\mu(x) U(\lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) + \partial^\mu \Omega$$

i.e. if A^μ transformed like a vector up to a <u>gauge</u> shift.

In such a situation Lorentz symmetry can only be recovered if the change of gauge has no physical consequence

<u>Ex</u> if $\mathcal{L} \supset A^\mu J_\mu$

with $\partial^\mu J_\mu = 0$

⚠ In more detail

$$\left(a_{0,3} = 0 \Rightarrow a_0 \, , \, \vec{\nabla} \cdot \vec{a} = 0 \equiv \text{Coulomb gauge} \right)$$

$$|p,\sigma\rangle = H(p) \, | \bar{p}, \sigma \rangle \qquad H(p) = R(\varphi, \theta, -\varphi) \, e^{i \eta_p / \bar{p} \, K_3}$$

$$U(\Lambda) |p,\sigma\rangle = H(\Lambda p) \, W(p,\Lambda) \, |\bar{p}, \sigma\rangle$$

$$= H(\Lambda p)\,|\bar{p},\sigma\rangle\, e^{-i\varphi(p,\Lambda)\sigma}$$

$$= |\Lambda p,\sigma\rangle\, e^{-i\varphi(p,\Lambda)\sigma}$$

- define $e_\mu(p,\sigma) = H_\mu{}^\nu(p)\, e_\nu(\bar{p},\sigma)$

$$e(\bar{p},\pm) = \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

we have $\begin{cases} e_0(p,\sigma) = 0 \\ \vec{p}\cdot\vec{e}(p,\sigma) = 0 \end{cases} \quad \forall p,\sigma$

- <u>Assume</u> $\langle 0|\, A_\mu(0)\,|p,\sigma\rangle = \sqrt{Z}\, e_\mu(p,\sigma)$

- <u>Question</u> : under the above assumption how does A_μ transform under Lorentz?

$$U^\dagger(\Lambda)\, A_\mu(0)\, U(\Lambda) = \,?$$

$$\cdot \; \langle 0 | U(\Lambda)^{\dagger} A_{\mu}(0) U(\Lambda) | p, \sigma \rangle =$$

$$= \langle 0 | A_{\mu}(0) U(\Lambda) | p, \sigma \rangle = (*)$$

$$U(\Lambda) H(p) | \bar{p}, \sigma \rangle = H(\Lambda p) W(p, \Lambda) | \bar{p}, \sigma \rangle$$

$$= H(\Lambda p) | \bar{p}, \sigma \rangle e^{-i \sigma \varphi(p, \Lambda)}$$

$$= | \Lambda p, \sigma \rangle e^{-i \sigma \varphi(p, \Lambda)}$$

$$(*) = e_{\mu}(\Lambda p, \sigma) e^{-i \sigma \varphi(p, \Lambda)}$$

Now $e_{\mu}(\Lambda p, \sigma) = H(\Lambda p)_{\mu}{}^{\nu} e_{\nu}(\bar{p}, \sigma)$

$$= \left(\Lambda H(p) W^{-1}(p, \Lambda) \right)_{\mu}{}^{\nu} e_{\nu}(\bar{p}, \sigma)$$

$$W(p, \Lambda) = e^{-i \varphi J_3} e^{i \alpha_i W_i} \qquad \varphi, \alpha_i \text{ of } (\Lambda, p)$$

$$= \left(\Lambda H(p) e^{-i \alpha_i W_i} \right)_{\mu}{}^{\nu} e_{\nu}(\bar{p}, \sigma) e^{i \sigma \varphi(p, \Lambda)}$$

So

$$\langle 0| U^\dagger(x)\, A_\mu(0)\, U(x) |p,\sigma\rangle = \left[\left(\Lambda H(p)\right) \left(e^{-i\underline{\alpha}.\underline{W}}\, e(\bar{p},\sigma)\right)\right]_\mu$$

using previous result for $e^{i\underline{\alpha}\,\underline{W}}$

$$\left(e^{-i\alpha W}\right)\begin{pmatrix} 0 \\ e_1 \\ e_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 e_1 + \alpha_2 e_2 \\ e_1 \\ e_2 \\ -\alpha_1 e_1 - \alpha_2 e_2 \end{pmatrix}$$

$$= e_\mu(\bar{p},\sigma) + \frac{\bar{p}_\mu}{K}(\alpha_i e_i)$$

So that

$$\langle 0| U^\dagger(x)\, A_\mu(0)\, U(x) |p,\sigma\rangle = \Lambda_\mu{}^\nu e_\nu(p,\sigma) + \Lambda_\mu{}^\nu \frac{p_\nu}{K}(\alpha_i e_i(\bar{p},\sigma))$$

$$= \Lambda_\mu{}^\nu \left(e_\nu(p,\sigma) + p_\nu \tilde{\Omega} \right)$$

Can check as exercise this corresponds to

$$U^\dagger(\Lambda) A_\mu(x) U(\Lambda) = \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x) + \partial_\mu \Omega(\Lambda^{-1}x)$$

$$\langle 0| U(\Lambda)^\dagger A_\mu(x) U(\Lambda) |p,\sigma\rangle =$$

$$= \langle 0| U(\Lambda)^\dagger e^{iPx} A_\mu(0) e^{-iPx} U(\Lambda) |p,\sigma\rangle$$

$$= \langle 0| U(\Lambda)^\dagger A_\mu(0) U(\Lambda) e^{-i(\Lambda P)x} |p\sigma\rangle$$

$$= \Lambda_\mu{}^\nu \left(e_\nu(p,\sigma) + p_\nu \tilde{\Omega} \right) e^{-i(\Lambda p)x}$$

$$= \Lambda_\mu{}^\nu \left(e_\nu(p,\sigma) e^{-ip(\Lambda^{-1}x)} \right) +$$

$$+ i\partial_\mu \left(\tilde{\Omega} e^{-ip(\Lambda^{-1}x)} \right)$$
