Multiparticle states and cluster principle

- particle wave packet $|\varphi\rangle \equiv \int d\Omega_p \, \varphi_\sigma(p) |p,\sigma\rangle$

$$\langle 0| \mathcal{O}_A(x) |\varphi\rangle = \int d\Omega_p \, e^{-ipx} \varphi_\sigma(p) \, \psi_{A,\sigma}(p) = \hat{\psi}_A(x)$$

- if φ_σ smooth function, $\psi_A(x)$ is localiz-
- we can interpret $|\varphi\rangle$ as a state where <u>one</u> particle is localized within a finite region : A

- By the cluster principle
  1. there should exist states featuring other 1-particle blobs provided they are sufficiently separated
  2. the corresponding matrix elements factorize

A A

In practice,

given $|p_r, r, \sigma\rangle$ single particle states

$p_r^2 \equiv M_r^2,\ S,\ Q, \dots \qquad -s, -s+1, \dots s$

1) the Hilbert space must include states

$|(p_1, r_1, \sigma_1); (p_2, r_2, \sigma_2) \dots (p_n, r_n, \sigma_n)\rangle$

$\propto (p_1, r_1, \sigma_1) \otimes (p_2, r_2, \sigma_2) \dots \otimes (p_n, r_n, \sigma_n)$

that is a state that transforms like the tensor product of n-single particle states

2) the corresponding localized states

$$|\Phi\rangle = |\varphi_1, \dots \varphi_r\rangle = \int \prod_{i=1}^{r} \left(d\Omega_i\, \varphi^i_{\sigma_i}(p_i)\right) |(p_1, r_1, \sigma_1) \dots (p_n, r_n, \sigma_n)\rangle$$

satisfy

$$\langle 0| \mathcal{O}_{\alpha_1 A_1}(x_1) \dots \mathcal{O}_{\alpha_n A_n}(x_n) |\Phi\rangle =$$

$$= \sum_{\pi} \prod_i \langle 0 | \mathcal{O}_{\alpha_i A_i}(x_i) | q_{\pi(i)} \rangle$$

$\pi(i)$ permutation of $1, \ldots n$

provided the q_i are sufficiently separated

- We should make the last statement a bit more precise
- In free field theory statement easily seen to be true, by simple application of Wick's theorem
- In interacting theory multiparticle states can only be made sense of as asymptotic states $|\{\theta_1\} \ldots \{\theta_n\}\rangle_{\pm}$ with the corresponding wave packets

There the statement should be

$$\lim_{\min t_i \to \infty} \langle 0 | T\left(\mathcal{O}_{\alpha_1 A_1}(t_1) \dots \mathcal{O}_{\alpha_n A_n}(t_n)\right) | q_1 \dots q_n \rangle_-$$

$$= \sum_{\pi} \prod_i \langle 0 | \mathcal{O}_{\alpha_i A_i}(x_i) | q_{\pi(i)} \rangle$$

$$\lim_{\max t_i \to -\infty} {}_+\langle q_1 \dots q_n | T\left(\varphi_{\alpha_1 A_1}(x_1) \dots \varphi_{\alpha_n A_n}(x_n)\right) | 0 \rangle$$

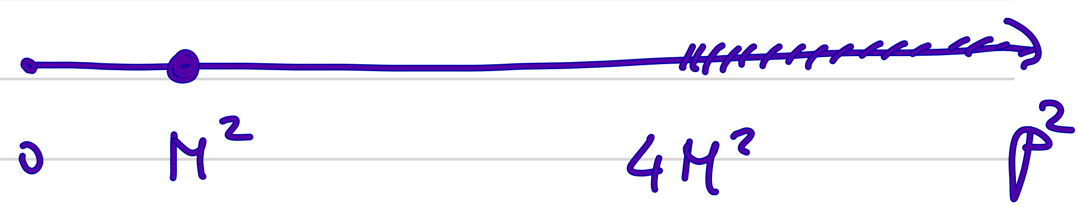
$$= \sum_{\pi} \prod_i \langle q_{\pi(i)} | \mathcal{O}_{\alpha_i A_i}(x_i) | 0 \rangle$$

These, here simply just postulated, properties are the basis of the LSZ reduction formulae, to be derived next time.

⚠ Polology: Källén-Lehmann, LSZ, ...

- The necessity of multiparticle states implies that the spectrum of P^2 possesses a continuum

$\exists$ in a theory with just one particle species ($\equiv$ just one type of particle can appear in asymptotic states). Given the mass M of the particle, the spectrum of P^2 has clearly the form



- with a continuum starting at $4M^2$
- even if the particle is a scalar, in the continuum we shall find states with $\vec{J} \neq 0$, resulting from orbital $\vec{L}$.

- The correlators of a QFT, starting with 2-point ones, encode information about its spectrum.

(LSZ)

Lehmann-Symanzik-Zimmermann reduction formula

- Concept: the 1-particle singularities of N-point correlators are associated to S-matrix

Focussing on N identical scalars the claim is

$$G(p_1 \cdots p_n; k_1, \dots k_m) \equiv \prod_i^n \int d^4x_i \, e^{ip_i x_i} \prod_j^m d^4y_j \, e^{-ik_j y_j} \times$$

$$\times \langle 0 | T \left(\mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \, \mathcal{O}(y_1) \cdots \mathcal{O}(y_m) \right) | 0 \rangle$$

$$\sim \left(\prod_i^n \frac{\sqrt{Z}\, i}{p_i^2 - m^2 + i\epsilon} \right) \left(\prod_j^m \frac{\sqrt{Z}\, i}{k_j^2 - m^2 + i\epsilon} \right) \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle$$

where $\sim$ holds for $p_i^0 \to E_{p_i} = \sqrt{\mathbf{p}_i^2 + m^2}$

$k_i^0 \to E_{k_i} = \sqrt{\mathbf{k}_i^2 + m^2}$

and where

$$\sqrt{Z} \equiv \langle 0 | \mathcal{O}(0) | p \rangle$$

Proof

Singularities in Fourier transforms of distributions arise from large $|x|$ region.
At finite volume no singularities.
In the present case the singularities arise in the region $x_i \to +\infty$

$$y_j \to -\infty$$

(in an obvious notation)
For these coordinates the correlator is

$$= T\langle 0 | T\left(O(x_1)\ldots O(x_n)\right) T\left(O(y_1)\ldots O(y_m)\right) | 0 \rangle$$

- We can insert complete set of states between the T's. However singularities can only come from states that have the right number of particles

Consider inserting

$$\prod_i d\Omega_{q_i} \prod_j dw_j \; |q_1 \cdots q_n\rangle_- \underbrace{{}_-\langle q_1 \cdots q_n | w_1 \cdots w_m\rangle_+}_{\equiv \langle q_1 \cdots q_n | S | w_1 \cdots w_m\rangle} {}_+\langle w_1 \cdots w_m|$$

- Now from cluster hypothesis

$$\lim_{x_i \to \infty} \langle 0| T(\mathcal{O}(x_1) \cdots \mathcal{O}(x_n)) | q_1 \cdots q_n\rangle_-$$

$$= \prod_i \langle 0 | \mathcal{O}(x_i) | q_i\rangle + \text{perms}$$

$$= \prod_i \sqrt{Z}\, e^{-i q_i x_i} + \text{perms}$$

and similarly for the in states

$$\lim_{y_j \to -\infty} \langle w_1 \cdots w_m | T(\mathcal{O}(y_1) \cdots \mathcal{O}(y_m)) | 0\rangle$$

$$= \prod_j \sqrt{Z}\, e^{i w_j y_j}$$

Of course these relations are valid only as long as the x's (and the y's) are widely separated among themselves. However in the future (past) this is basically true everywhere apart from a set of measure zero where some x_i's are close.

Because of the above the leading contribution to the Fourier transform factorizes in products of terms of the forms

1) future $\Rightarrow$ $\displaystyle\int_{t_i > t_+} dx_i^4 \, e^{i(p_i - q_i)x_i} \, d\Omega_{q_i} \times \sqrt{Z}$

$$\sim \int_{t_i > t_+}^{\infty} dt \, e^{i(p_i^0 - E_{q_i})t} (2\pi)^3 \delta^3(p_i - q_i) \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \times \sqrt{Z}$$

$$\simeq \frac{1}{2E_{p_i}} \frac{\sqrt{z}\, i e^{i(p_i^0 - E_{p_i})t}}{p_i^0 - E_{p_i} + i\epsilon} \approx \frac{i\sqrt{z}}{p_i^2 - m^2 + i\epsilon}$$

2) rest $\displaystyle\int_{-\infty}^{t_-} e^{-i(k_j - w_j)y_j}\, d\Omega_{w_j} \times \sqrt{z}$

$$\simeq \frac{1}{2E_{k_j}} \frac{i\sqrt{z}}{k_j^0 - E_{k_j} + i\epsilon} \sim \frac{i\sqrt{z}}{k_j^2 - m}$$

Putting all the factors together we thus obtain the claimed result

Notice

$$\langle p_1 \dots p_n | k_1 \dots k_m \rangle_+ = \langle p_1 \dots p_n | S | k_1 \dots k_m$$

⑪ This result is readily generalized to operators $O_{\alpha A}(x)$ in arbitrary irrep of Lorentz and to the corresponding spin ≠ 0 irreps they interpolate for

LSZ for arbitrary spin

- consider now $|p,\sigma\rangle$ $\quad p^2 = M^2$, $\quad \sigma = -s \dots s$

- and an interpolating operator $\mathcal{O}_A^{j_-j_+}$

$$\langle 0| \mathcal{O}_A^{j_-j_+} |p,\sigma\rangle = \psi_{A\sigma}^{j_-j_+s}(p)$$

- The spinless case directly generalizes to the spinning case: for each external leg we shall have factors

- outgoing $\quad \dfrac{\langle 0| \mathcal{O}_A^{j_-j_+} |p,\sigma\rangle}{p^2-M^2} = \dfrac{\sqrt{Z}\, \psi_{A\sigma}^{j_-j_+s}(p)}{p^2-M^2}$

- incoming $\quad \dfrac{\langle p,\sigma| \mathcal{O}_A^{j_-j_+\dagger} |0\rangle}{p^2-M^2} = \dfrac{\sqrt{Z}\left(\psi_{A\sigma}^{j_-j_+s}\right)^*(p)}{p^2-M^2}$

indicating $(j_-, j_+) = J$ $\quad (j_+, j_-) = J_P$

$\equiv$ parity conjugated

We showed before that

$$\psi^{+J_P}_{A\sigma'} \psi^{J}_{A\sigma} = \delta_{\sigma\sigma'}$$

So that by using $\psi^{+J_P}_{A\sigma}(p)\, \mathcal{O}^{J}_{A}(x) = \mathcal{O}_{\sigma}(x)$

$$\psi^{J_P}_{A\sigma}(p)\left(\mathcal{O}^{J}_{A}\right)^{+}(x) = \mathcal{O}^{J\,+}_{\sigma}(x)$$

as interpolating fields we practically reduce to the scalar case.

$$\langle \prod_i (p_i, \sigma_i, s_i) \mid \prod_j (q_j, \bar{\sigma}_j, \bar{s}_j) \rangle_{+} =$$

$$\xrightarrow{\text{on-shell}}\ \prod_i \frac{(p_i^2 - M_i^2)}{\sqrt{Z_i}} \psi^{+J_P^i}_{A_i \sigma_i} \prod_j \frac{(q_j^2 - M_j^2)}{\sqrt{Z_j}} \psi^{J_P^j}_{B_j \bar{\sigma}_j}$$

$$\times \langle T\left(\prod_i \mathcal{O}^{J_i}_{A_i}(p_i) \prod_j \mathcal{O}^{+}_{B_j}(-q_j)\right) \rangle$$