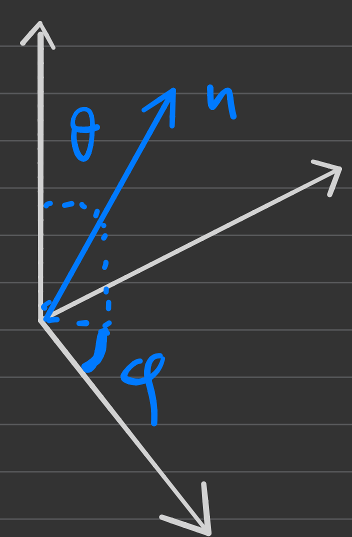
Parametrizing rotations

1) $R(\underline{\theta}) \equiv e^{-i\vec{\theta}\cdot\vec{J}}$ $\qquad \vec{\theta} \equiv (\theta_1, \theta_2, \theta_3)$

2) $R(\varphi, \theta, \psi) = e^{-i\varphi J_3} e^{-i\theta J_2} e^{-i\psi J_3}$ $\qquad \varphi, \theta, \psi \equiv$ Euler angles

2') 

$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$R(\vec{n}) \equiv R(\varphi, \theta, -\varphi) = e^{-i\varphi J_3} e^{-i\theta J_2} e^{i\varphi J_3}$$

$$\mathcal{R}(\vec{n})^{i3} = n^i \quad \text{or} \quad \boxed{\mathcal{R}\hat{z} = \vec{n}}$$

$$R^{ij} z^j = R^{i3} \qquad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \hat{z}_i = \delta_{i3}$$

B) The helicity basis

Jacob & Wick

Annals Phys 7 (1959) 404

$$\vec{n} \equiv \frac{\vec{p}}{|\vec{p}|}\,, \qquad \vec{p} = \mathcal{R}(\vec{n}) \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$$

$$\begin{pmatrix} p^0 \\ 0 \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} \sqrt{m^2+p^2} \\ 0 \\ 0 \\ p \end{pmatrix} = e^{i\eta K_3} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p \equiv |\vec{p}| = \frac{\operatorname{tgh} \eta}{\sqrt{1-\operatorname{tgh}^2\eta}}\, m$$

$$\eta = \operatorname{tgh}^{-1}\left(\frac{p}{p_0}\right)$$

$$\boxed{\Lambda^h_{\vec{p}} \equiv R(\vec{n})\, e^{i\eta K_3}}$$

$$(\vec{J}\cdot\vec{n})\,U(R) = U(R)\,U(R)^{-1}(J\cdot n)\,U(R) = U(R)\,R_{ij}\,J^{j}\,n^{i}$$

$$\Rightarrow (\vec{J}\cdot\vec{n})\,\Lambda^{n}_{\vec{p}} = \Lambda^{n}_{\vec{p}}\,J_3$$

- $|p,\sigma\rangle_n \equiv U(\Lambda^{n}_{\vec{p}})\,|\bar{p},\sigma\rangle$

$$J\cdot n\,|p,\sigma\rangle = (J\cdot n)\,U(\Lambda^{n}_{\vec{p}})\,|\bar{p},\sigma\rangle = U(\Lambda^{n}_{p})\,J_3\,|\bar{p}\,\sigma\rangle$$

$$\boxed{J\cdot n\,|p,\sigma\rangle = \sigma\,|p,\sigma\rangle}$$

$$R(\vec{n})\, e^{i\eta K_3} = e^{i\eta R(\vec{n}) K_3 R^{+}(\vec{n})} \cdot R(\vec{n})$$

$$= e^{i\eta \mathcal{R}_{i3} K_i} R(\vec{n}) = e^{i\vec{\eta}\cdot\vec{K}} R(\vec{n})$$

$$|p,\sigma\rangle_n \equiv U\left(R(\vec{n})e^{i\eta K_3}\right)|\bar{p},\sigma\rangle = U\left(e^{i\vec{\eta}\cdot\vec{K}}\right)U(R(\vec{n}))\,|\bar{p},\sigma\rangle$$

$$= U\left(e^{i\vec{\eta}\cdot\vec{K}}\right)|\bar{p},\sigma'\rangle\, D^{(s)}_{\sigma'\sigma}(R(n))$$

$$= |\vec{p},\sigma'\rangle_s\, D^{(s)}_{\sigma'\sigma}(R(n))$$

Wigner Rotation in helicity basis

- pure rotation $R \equiv R(\alpha, \beta, \gamma)$

$$\underline{R(\alpha,\beta,\gamma)\,\Lambda^h_{p}} = R(\alpha,\beta,\gamma)\,R(\varphi,\theta,-\varphi)\,e^{i\eta K_3}$$

$$= R(\varphi',\theta',\psi')\,e^{i\eta K_3}$$

$$= e^{-i\varphi' J_3}\,e^{-i\theta' J_2}\,e^{-i\psi' J_3}\,e^{i\eta K_3}$$

$$= \underbrace{e^{-i\varphi' J_3}\,e^{-i\theta' J_2}\,e^{i\varphi' J_3}}\,e^{i\eta K_3}\cdot e^{-i(\varphi'+\psi')J_3}$$

$$= R(n')\;e^{i\eta K_3}\;e^{-i(\varphi'+\psi')J_3}$$

$$= \Lambda^{h}_{\vec{p}'} \; e^{-i(\varphi'+\psi')J_3}$$

$W \equiv$ Wigner rotation

$$U(R(\alpha,\beta,\gamma))\,|\vec{p},\sigma\rangle_h = U(R)\,U(\Lambda^{h}_{\vec{p}})\,|\bar{p},\sigma\rangle$$

$\vec{p}\,' \equiv R\vec{p}$

$$= U(\Lambda_{R\vec{p}}) \quad e^{-i(\varphi'+\psi')\sigma} \quad |\bar{p},\sigma\rangle$$

$$= e^{-i(\varphi'+\psi')\sigma} \quad |Rp,\sigma\rangle$$

$$e^{i\alpha \vec{J}\cdot\vec{n}} = R(\vec{n})\, e^{i\alpha J_3}\, R^{-1}(n)$$

$$|\vec{p},\sigma\rangle_n = \underbrace{U(R(\vec{n}))\, e^{i\eta K_3}}\, |\bar{p},\sigma\rangle$$

$$\underbrace{e^{i\alpha \vec{J}\cdot\vec{n}}}\, |\vec{p},\sigma\rangle_n = \underbrace{U(R(\vec{n}))\, e^{i\alpha J_3}}\, \underbrace{U(R^{-1}(\vec{n}))\, U(R(n))}\, e^{i\eta K_3}\, |\bar{p},\sigma\rangle$$

$$= U(R(\vec{n}))\, e^{i\alpha J_3}\, e^{i\eta K_3}\, |\bar{p},\sigma\rangle$$

$$= U(R(\vec{n}))\, e^{i\eta K_3}\, |\bar{p},\sigma\rangle\, e^{i\alpha\sigma}$$

$$= |\vec{p},\sigma\rangle\, e^{i\alpha\sigma}$$

- pure boost $\quad \underline{H_h(p) \equiv U(\Lambda^h_{\vec{p}})}$ $\quad \Lambda^h_{\vec{p}} = R(n)\, e^{i\eta K_3}$

$$W_h(\Lambda, p) \equiv H_h(\Lambda p)^\dagger\, U(\Lambda)\, H_h(p) \qquad p' = \Lambda p$$

$$= e^{-i\eta' K_3}\, U^\dagger(R(n'))\, U(\Lambda)\, \underbrace{U(R(n))\, e^{i\eta K_3}}$$

$$= U^\dagger(R(n'))\, \underbrace{e^{-i\vec{\eta}'\cdot\vec{K}}\, U(\Lambda)\, e^{i\vec{\eta}\cdot\vec{K}}}\, U(R(n))$$

$$\underline{U^\dagger(R(n'))}\; \underline{W_s(\Lambda, p)}\; \underline{U(R(\vec{n}))}$$

Last time

$$\boxed{W_s(\Lambda, p) \simeq \mathbb{1} + \frac{i(\vec{\eta}_\Lambda \wedge \vec{p}) \cdot \vec{J}}{p^0 + m} + \dots}$$

$$\Lambda = \mathbb{1} + \omega \dots \;\searrow\; \eta_\Lambda$$

- In helicity basis $(p^0 \gg m)$ $\quad \vec{n} = \frac{\vec{p}}{|\vec{p}|}$

$$\boxed{W_h = \mathbb{1} + \frac{i(\vec{\eta}_\Lambda \wedge \vec{n})_3 J_3}{1 + n_3} + O\left(\frac{m}{p_0}\right)}$$

$$U(\Lambda)\,|p,\sigma\rangle_h = U(\Lambda)\,H^h(p)\,|\bar{p},\sigma\rangle$$

$$= H^h(\Lambda p)\,U(\omega_h(\Lambda,p))\,|\bar{p},\sigma\rangle$$

$$= H^h(\Lambda p)\,|\bar{p},\sigma'\rangle\,D_{\sigma'\sigma}(\omega_h(\Lambda,p))$$

$$= |\Lambda p,\sigma'\rangle_h \left(\delta_{\sigma'\sigma} + i\,\frac{(n_\Lambda \times n)_3}{1+n_3}\,J^3_{\sigma'\sigma} + \dots\right)$$

$$= |\Lambda p,\sigma\rangle_h \left(1 + i\,\frac{(\vec{n}_\Lambda \wedge \vec{n})_3}{1+n_3}\cdot\sigma + \dots\right)$$

⚠ In spin basis, action of pure rotation R, described by $W = R$, like in non-relativistic QM:

$$U(R)|\vec{p},\sigma\rangle = |\underbrace{R\vec{p}}_{L},\sigma'\rangle \underbrace{D(R)}_{S}{}_{\sigma'\sigma}$$

⚠ In helicity basis spin is referred to $\vec{p}$ in lab frame: $W \neq R$

$\mathrm{II}_A)$ $\quad \bar{P}^\mu = (K, 0, 0, K)$

$$W^0 = \frac{1}{2}\varepsilon^{0\mu\nu\rho} J_{\mu\nu} P_\rho = \frac{1}{2}\left(\varepsilon^{0123} J_{12} P_3 + \underline{\varepsilon^{0213}} J_{21} P_3\right)$$

$$= -J_3 \cdot K$$

$$W^3 = -J_3 K \quad = \quad \frac{1}{2}\varepsilon^{3120} J_{12} K \cdot 2$$

$$W^1 = K\left(K^2 - J^1\right) \qquad [J_3, J_1] = iJ_2 \quad [J_3, J_2] = -iJ_1$$

$$W^2 = K\left(-K^1 - J^2\right)$$

$$[W^1, W^2] = \text{exercise} = -K^2(iJ^3 - iJ^3) = 0$$

$$[W^3, W^1] = -iK\,W^2$$

$$[W^3, W^2] = +iK\,W^1$$

$$\boxed{\begin{aligned}[J^3, W^1] &= i\,W^2 \\ [J^3, W^2] &= -i\,W^1\end{aligned}}$$

$$\frac{W^3}{K} = J^3$$

$$\boxed{\begin{aligned}[J^3, W^1] &= i\,W^2 \\ [J^3, W^2] &= -i\,W^1 \\ [W^1, W^2] &= 0\end{aligned}}$$

ISO(2)

Convenient $\quad W^{\pm} \equiv W^1 \pm i W^2 \qquad W^- = (W^+)^\dagger$

$$\boxed{\begin{aligned} [J^3, W^{\pm}] &= \pm W^{\pm} \\ [W^{\pm}, W^{\pm}] &= 0 \end{aligned}}$$

- $W^2 = W^\mu W_\mu \equiv$ Casimir

$$= \cancel{(W^0)^2} - (W_1^2 + W_2^2 + \cancel{W_3^2}) = -W_1^2 - W_2^2 = -W^+ W^-$$
$$= -W^- W^+$$

$$\underline{W}^2 \equiv -\underline{W^+ W^-} = -c\,\mathbb{1}$$

⚠ $|\bar{p}, \sigma\rangle$ must host a unitary repr. of $ISO(2)$

- $ISO(2)$ is non compact and non semi-simple

$\Longrightarrow$ unitary irreps either dim $= 1$ or dim $= \infty$

- $[J_3, W^\pm] = \pm W^\pm \Longrightarrow W^\pm$ raising and lowering J_3

$$J_3 |\lambda\rangle = \lambda |\lambda\rangle \longrightarrow \begin{array}{l} J_3 W^+ |\lambda\rangle = (\lambda+1) W^+ |\lambda\rangle \\ J_3 W^- |\lambda\rangle = (\lambda-1) W^- |\lambda\rangle \end{array}$$

$$W^- W^+ = W^+ W^- = (W^+)^\dagger W^+ = c \cdot \mathbb{1} \qquad c \geq 0$$

$$\langle\psi| A^\dagger A |\psi\rangle \geq 0$$

$$\langle\psi| A^\dagger |n\rangle\langle n| A|\psi\rangle = \sum_n |\langle n| A|\psi\rangle|^2 \geq 0$$

- $c > 0$

$$(W^+)^\dagger W^+ = c \cdot \mathbb{1} \qquad c > 0$$

$$\left(\frac{W^+}{\sqrt{c}}\right)^\dagger \left(\frac{W^+}{\sqrt{c}}\right) = \mathbb{1} = V^\dagger V$$

$$\frac{W^+}{\sqrt{c}} \equiv V = \text{unitary}$$

Toro, Schuster

$$\Longrightarrow \ldots, \lambda-2, \lambda-1, \lambda, \lambda+1, \lambda+2, \ldots$$

all eigenvalues !!

$\boxed{\text{irrep is } \infty \text{ dimensional !}}$

$\triangle$ $c = 0$ $\qquad (W^+)^\dagger W^+ = 0 \Rightarrow W^+ = W^- = 0$

$J_3 |\lambda\rangle = \lambda |\lambda\rangle \qquad W^\pm |\lambda\rangle = 0$

$\Rightarrow$ 1-dimensional irrep

⚠ $c=0$

$$\underbrace{J^3|\bar{p},\lambda\rangle = \lambda|\bar{p},\lambda\rangle \qquad W^{\pm}|\bar{p},\lambda\rangle = 0}$$

$$\Downarrow$$

$$W^{\mu}|\bar{p},\lambda\rangle = -\lambda\,\bar{p}^{\mu}|\bar{p},\lambda\rangle$$

- $\underline{W^{\mu}} = -\lambda\,\underline{\bar{p}^{\mu}} = -\lambda(k,0,0,k)$
- $\lambda = \dfrac{\vec{J}\cdot\vec{\bar{p}}}{|\bar{p}|}$

loSel

⚠ Casimirs: $P^2 = W^2 = 0$ $\qquad \underline{W^{\mu}} = -\lambda\,\underline{P^{\mu}}$

$\lambda \equiv$ Lorentz inv

Induced representation

$$\begin{pmatrix} K \\ 0 \\ 0 \\ K \end{pmatrix} \longrightarrow \begin{pmatrix} K \\ K \end{pmatrix} \xrightarrow{\text{boost}} \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} K \\ K \end{pmatrix}$$

$$= e^{\eta} \begin{pmatrix} K \\ K \end{pmatrix} \equiv \begin{pmatrix} P \\ P \end{pmatrix}$$

$$\eta_P \equiv \ln \frac{P}{K}$$

$$|\vec{P}, \lambda\rangle = \underline{U(R(\vec{n}))\, e^{i\eta_P \cdot K_3}} \; |\bar{P}, \lambda\rangle$$

$$\vec{P} = \vec{n} \cdot P \qquad P^0 \equiv P$$

$$\downarrow \quad U(\Lambda_p)\,|\bar{p},\lambda\rangle$$

⚠ given

$$U^\dagger(\Lambda)\,P^\mu\,U(\Lambda) = \Lambda^\mu{}_\nu\,P^\nu$$

$$U^\dagger(\Lambda)\,W^\mu\,U(\Lambda) = \Lambda^\mu{}_\nu\,W^\nu$$

$$\boxed{W^\mu = -\lambda P^\mu \qquad \text{true for all } p^\mu \quad (\text{of course } p^\mu p_\mu = 0)}$$

<u>proof</u>

$$\Downarrow \quad W^\mu\,|p,\lambda\rangle = U(\Lambda_p)\,U(\Lambda_p)^\dagger\,W^\mu\,U(\Lambda_p)\,|\bar{p},\lambda\rangle$$

$$\boxed{p^\mu = \Lambda_p{}^\mu{}_\nu \bar{p}^\nu}$$

$$= U(\Lambda_p)\, \Lambda_p{}^\mu{}_\nu W^\nu |\bar{p}, \lambda\rangle$$

$$= U(\Lambda_p)\, \Lambda_p{}^\mu{}_\nu \left(-\lambda \bar{p}^\nu\right) |\bar{p}, \lambda\rangle$$

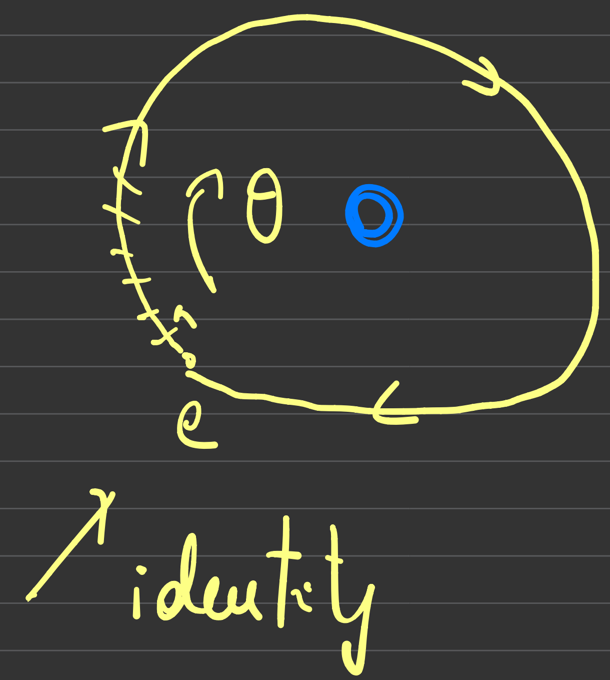
$$= -\lambda p^\mu \quad \underbrace{U(\Lambda_p) |\bar{p}, \lambda\rangle}$$

$$-\lambda p^\mu \quad |p, \lambda\rangle$$

$\lambda \equiv$ Lorentz invariant

Quantization of λ

G



$e^{i\theta X}$

ϑ_f

$g(\theta)$ $\quad g(0) = e \quad g(2\pi) = e$

$$U(g(\theta)) = e^{i\theta X} = e^{i2\pi X} = e^{i\varphi} U(e) = e^{i\varphi} \cdot \mathbb{1}$$

$$e^{i4\pi \cdot J_3} = \mathbb{1}$$

$e^{i4\pi\lambda}$

$4\pi\lambda = 2\pi n$

$\lambda = n/2$

CPT & antiparticles

Θ

$$\Theta \vec{J} \Theta = -\vec{J} \qquad \Theta \vec{P} \Theta = \vec{P}$$

$$\Theta\, \vec{J}\cdot\vec{P}\, \Theta = -\vec{J}\cdot\vec{P}$$

$$\lambda \longrightarrow -\lambda$$

Ex

- Weyl $\lambda = \pm 1/2$
- photon $\lambda = \pm 1$ $\Longleftarrow$ $ISO(3,1) + CPT$
- graviton $\lambda = \pm 2$
- gravitino $\lambda = \pm 3/2$ massive $3/2$: $3/2,\ 1/2,\ -1/2,\ -3/2$