Normalization

$|p,\sigma\rangle = H(p)\,|\bar{p},\sigma\rangle$

$P_\mu |p,\sigma\rangle = p_\mu |p,\sigma\rangle \Rightarrow \langle p'\sigma' | p,\sigma\rangle = A_{\sigma'\sigma}\,\delta^3(\underline{p}-\underline{p}')$

(In principle it should be $\delta^4(p-p')$, but p^0 is not an independent variable over the irrep)

Now; what can we say about $A_{\sigma'\sigma}(p)$?

- Let us consider the spin basis first

$$U(R)\,|p,\sigma\rangle = |R(p),\sigma'\rangle\, D_{\sigma'\sigma}(R)$$

$U^\dagger(R)\,U(R) = \mathbb{1} \Rightarrow$

$$A_{\sigma'\sigma}(p)\,\delta^3(p-p') = A_{\tilde{\sigma}'\tilde{\sigma}}(R(p))\,D^*_{\tilde{\sigma}'\sigma'}\,D_{\tilde{\sigma}\sigma}\,\delta^3(R(p-p'))$$

$$\Rightarrow A_{\sigma'\sigma}(p) = A_{\tilde{\sigma}'\tilde{\sigma}}(R(p))\,D^{*}_{\tilde{\sigma}'\sigma'}(R)\,D_{\tilde{\sigma}\sigma}(R)$$

Not sufficient to conclude $A_{\sigma'\sigma} = \delta_{\sigma'\sigma}\, f(|p|)$

More specifically

Better to proceed as follows

Consider $\bar{P} = (M, \vec{0})$ $\quad K = (E_K, \underline{k})$

$$\langle k, \sigma' | \bar{P}, \sigma \rangle = A_{\sigma'\sigma} \, \delta^3(\vec{k})$$

- $A_{\sigma'\sigma}$, because of $\delta^3(\vec{k})$, is just a constant
- Applying a pure rotation like before we get:

$$A_{\sigma'\sigma} = A_{\tilde{\sigma}'\tilde{\sigma}} \, D^*_{\tilde{\sigma}'\bar{\sigma}'}(R) \, D_{\tilde{\sigma}\sigma}(R)$$

$$\Longrightarrow A_{\sigma'\sigma} = \delta_{\sigma'\sigma} A$$

Now insert $U^\dagger(\Lambda) U(\Lambda) = \mathbb{1}$

$$\delta_{\sigma'\sigma} \delta^3(k) A = \langle k, \sigma' | U^\dagger(\Lambda) U(\Lambda) | \bar{P}, \sigma \rangle =$$

$$\equiv \langle p', \tilde{\sigma}' | p \, \tilde{\sigma} \rangle \, D_{\tilde{\sigma}\sigma}(W(\Lambda, \bar{P})) \, D^*_{\tilde{\sigma}'\sigma'}(W(\Lambda, k))$$

where $p = \Lambda \bar{P}$ $\qquad p' = \Lambda k$

Using orthogonality of $D_{\tilde{\sigma}'\sigma}$ we can then write

$$\langle p' \tilde{\sigma}' | p \, \tilde{\sigma} \rangle = D^*_{\tilde{\sigma}\sigma}(W(\Lambda, \bar{P})) \, D_{\tilde{\sigma}'\sigma}(W(\Lambda, k)) \, \delta^3(k) \cdot A$$

but because of $\delta^3(k)$ we can set $k=\bar{p}$ in W

$$\Rightarrow \langle p' \tilde{\sigma}' | p, \tilde{\sigma} \rangle = \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(k) \cdot A$$

$$= \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(\Lambda^{-1}(p'-p)) \, A$$

$$= \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(p'-p) \, \frac{E_p}{M} \cdot A$$

the standard choice is $A = (2\pi)^3 2M$

$$\Rightarrow \langle p', \sigma' | p, \sigma \rangle = (2\pi)^3 2E_p \, \delta^3(p-p') \, \delta_{\sigma'\sigma}$$

$$\Lambda^{-1}_p(p') = \frac{-|p\rangle}{m} E' + p'_\perp + \frac{E}{m} p'_L$$

$$= p'_\perp + n_L \frac{E p'_L - p E'}{m}$$

$$\Rightarrow \delta^2(p'_\perp) \, \delta(p'_L - p) \left| \partial_{p'_L} \left(\frac{E p'_L - p E'}{m} \right) \right|^{-1}$$

$$\frac{\partial}{\partial p'_L} \left(\frac{E p'_L - p E'}{m} \right) = \left(E - p \frac{p'_L}{E} \right) \frac{1}{m} = \frac{E - p p'_L}{E m}$$

$$= \frac{m^2}{E m} = \frac{m}{E}$$

$$\Rightarrow \delta^3(\Lambda_p^{-1}(p')) = \frac{E_p}{m} \, \delta^3(p'-p)$$

$\mathrm{II}_A)$ $\bar{P}^\mu = (k, 0, 0, k)$

The Pauli-Lubenski vector becomes

$$W^0 = \frac{1}{2} \epsilon^{0123} J_{12} (-k) \times 2 = -J_3 \cdot k$$

$$W^3 = \frac{1}{2} \epsilon^{3120} J_{12} \cdot k \cdot 2 = -J_3 k$$

$$W^1 = \frac{1}{2} \epsilon^{1203} J_{20} \cdot 2 \cdot (-k) + \frac{1}{2} \epsilon^{1230} J_{23} \cdot 2 \cdot k$$

$$= k (K^2 - J^1)$$

$$W^2 = \frac{1}{2} \epsilon^{2103} J_{10} \cdot 2 \cdot (-k) + \frac{1}{2} \epsilon^{2130} J_{13} \cdot 2 \cdot k$$

$$= k (-K^1 - J^2)$$

$$[W^1, W^2] = -k^2 [K^2 - J^1, K^1 + J^2] =$$

$$= -k^2 (iJ^3 - iJ^3) = 0$$

$$[W^3, W^1] = \varkappa^2 [J^3, K^2 - J^1] =$$

$$= \varkappa^2 (-iK^1 - iJ^2) = i\varkappa W^2$$

$$[W^3, W^2] = -\varkappa^2 [J^3, K^1 + J^2]$$

$$= -\varkappa^2 (iK^2 - iJ^1) = -i\varkappa W^1$$

$$-W^3/\varkappa = J^3 \qquad \begin{cases} [J^3, W^1] = iW^2 \\ [J^3, W^2] = -iW^1 \\ [W^1, W^2] = 0 \end{cases} \qquad \boxed{ISO(2)}$$

$$W_\pm = W^1 \pm iW^2 \qquad [J^3, W^1 \pm iW^2] = iW^2 \pm W^1$$

$$= \pm (W^1 \pm iW^2)$$

$$* \begin{cases} [J^3, W_\pm] = \pm W_\pm \\ [W_\pm, W_\pm] = 0 \end{cases}$$

Moreover one has $W^2 = W^{0\,2} - \vec{W}^2 = -W_+ W_-$

Consistently with $[J_{\mu\nu}, W^2] = 0$ one can check that $(*)$ implies $[W^\pm, W^2] = [J^3, W^2] = 0$

⚠ $|\bar{p}, \sigma\rangle$ must host a unitary representation of $ISO(2)$

$ISO(2)$ is non-compact and non semi-simple: unitary irreps have either $\dim = 1$ or $\dim = \infty$.

From the algebra, given $|\lambda\rangle \Rightarrow J^3|\lambda\rangle = \lambda|\lambda\rangle$, $W^\pm$ act as raising and lowering operators

$$W^+|\lambda+n\rangle = b_n|\lambda+(n+1)\rangle$$

$$W^-|\lambda+n+1\rangle = a_n|\lambda+n\rangle = b_n^*|\lambda+n\rangle$$

$$\langle\lambda+(n+1)|W^+|\lambda+n\rangle = \langle\lambda+n|W^-|\lambda+n+1\rangle^*$$

$$\Rightarrow b_n = a_n^*$$

therefore $W^-W^+|\lambda+n\rangle = |b_n|^2|\lambda+n\rangle$

but given $W^- W^+ = -W^2$ is a Casimir, $|b_n|^2$ must be n independent. There are then only two options

1) $W^\pm \neq 0 \Rightarrow |b_n|^2 = \alpha \neq 0 \ \forall n$

$\Longrightarrow$ representation ∞ dimensional

$\ldots, |\lambda-1\rangle, |\lambda\rangle, |\lambda+1\rangle, |\lambda+2\rangle, \ldots$

$$\boxed{W^2 = -\alpha}$$

2) $W^\pm = 0 \Rightarrow$ representation 1-dim

$|\lambda\rangle$ $\qquad W^\pm |\lambda\rangle = 0$

$J^3 |\lambda\rangle = \lambda |\lambda\rangle$

$$\Rightarrow \boxed{W^2 = 0}$$

Experience shows that only case 2) is realized in nature. Case 1) would correspond to a massless particle endowed with infinitely

many degrees of freedom (polarizations).
Apparently that would not lead to any stark inconsistency (see Schuster-Toro) if not for the fact that all these states must be coupled extremely weakly.

Case 2. $J^3 |\bar{p}, \lambda\rangle = \lambda |\bar{p}, \lambda\rangle \qquad W^\pm |\bar{p}, \lambda\rangle = 0$

$$\Rightarrow \quad W^\mu |\bar{p}, \lambda\rangle = -\lambda \, \bar{p}^\mu |\bar{p}, \lambda\rangle$$

$$W^\mu = -\lambda \bar{p}^\mu = -\lambda (k, 0, 0, k)$$

$$\lambda = \frac{\vec{J} \cdot \vec{\bar{p}}}{|\bar{p}|}$$

irrep fully characterized by helicity λ

$$\boxed{W^2 = P^2 = 0 \qquad W^\mu = -\lambda P^\mu}$$

Constructing the induced representation

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} k \\ k \end{pmatrix} = e^{\eta}\begin{pmatrix} k \\ k \end{pmatrix}$$

$$e^{\eta} k = p \qquad \eta = \ln p/k \equiv \eta_{p/k}$$

There is here no analogue of the spin basis
Convenient to use analogue of helicity basis

$$|p,\lambda\rangle \equiv \underbrace{U(R(\varphi,\theta,-\varphi))\, e^{i\eta_{p/k} K_3}}_{H(p)\,\equiv\, U(\Lambda_p)} |\bar{p},\lambda\rangle$$

corresponds to a state with momentum

$$p^{\mu} = p(1,\vec{n}) \qquad p = e^{\eta} k$$
$$\vec{n} \equiv \vec{n}(\theta,\varphi)$$

and helicity λ

Given $U^\dagger(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$

$$U^\dagger(\Lambda) W^\mu U(\Lambda) = \Lambda^\mu{}_\nu W^\nu$$

the relation $W^\mu = -\lambda P^\mu$, valid for our reference state $|\bar{p}, \lambda\rangle$ is preserved for all states $|p, \lambda\rangle$ and under all changes of frame : <u>helicity</u> <u>is</u> <u>Lorentz</u> <u>invariant</u>

⚠ Two things to conclude

a) λ at this stage $\in \mathbb{R}$.

For arbitrary real λ we have a projective representation of $U(1)_{J_3}$. However we need a consistent projective representation of $SO(3,1)$. The topology of $SO(3,1)$, its $\mathbb{Z}_2$ homotopy group implies

$$e^{iJ_3 \cdot 4\pi} = 1 \implies \lambda = \frac{n}{2}$$

b) Local relativistic QFT also realizes CPT $\equiv \Theta$ as a symmetry

$$\Theta \vec{J} \Theta = -\vec{J} \qquad \Theta \vec{P} \Theta = \vec{P}$$

$$\Rightarrow \Theta \vec{J} \cdot \vec{P} \Theta = -\vec{J} \cdot \vec{P}$$

so that $\lambda \xrightarrow{\Theta} -\lambda$

In order to represent Θ we must supplement $|\vec{p}, \lambda\rangle$ with an additional state $|\vec{p}, -\lambda\rangle$

We thus finally conclude that massless particles always come in two helicity states $\pm\lambda$.

Ex photon $\lambda = \pm 1$

graviton $\lambda = \pm 2$

(massless gravitino) $\lambda = \pm 3/2$

A bit of axiomatism

Locality $\checkmark$, Causality $\checkmark$, cluster property $\checkmark$: Fields

0. $\exists$ vacuum state ($ISO(3,1)$ invariant)

1. $\exists$ local excitations of vacuum $\Longleftrightarrow$ local experiment

   A

2. excitations can be exported and replicated elsewhere

   A A

3. If "sufficiently" separated the different experiments do not influence one another

$$\Downarrow$$

A. $\exists$ local observables $\equiv$ field operators $\mathcal{O}_\alpha(x)$

B. $e^{iPa}\,\mathcal{O}(x)\,e^{-iPa} = \mathcal{O}(x+a)$

C. $\mathcal{O}(x)_\alpha |0\rangle$ generate local excitations

D. $\langle 0| \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)\, \mathcal{O}_{\beta_1}(y_1+a) \cdots \mathcal{O}_{\beta_m}(y_m+a)|0\rangle$

$$\xrightarrow{|a|\to\infty} \langle 0| \mathcal{O}_{\alpha_1}(x_1) \cdots \mathcal{O}_{\alpha_n}(x_n)|0\rangle\langle 0| \mathcal{O}_{\beta_1}(y_1+a) \cdots \mathcal{O}_{\beta_m}(y_m+a)|0\rangle$$

E. $[\mathcal{O}_\alpha(x_\alpha), \mathcal{O}_\beta(x_\beta)] = 0$ if $x_\alpha \times x_\beta$

Field operators will be organized into irreducible representations of $SO(3,1)$

$$\mathcal{O}_{\alpha,A}(x) \equiv [j_-, j_+] \qquad A = 1 \ldots (2j_-+1)(2j_++1)$$

$$U^\dagger(\Lambda)\,\mathcal{O}_{\alpha,A}(x)\,U(\Lambda) = D_{AB}(\Lambda)\,\mathcal{O}_{\alpha,B}(\Lambda^{-1}x)$$

in particular $U^\dagger(\Lambda)\,\mathcal{O}_{\alpha,A}(0)\,U(\Lambda) = D_{AB}(\Lambda)\,\mathcal{O}_{\alpha,B}(0)$

$$\Rightarrow \mathcal{O}_\alpha(0) = \bigoplus_{\substack{\text{irreps of}\\ SO(3)}} \tilde{\mathcal{O}}_{\alpha,J} \qquad \bar{\mathcal{O}}_{\alpha,J} \equiv \tilde{\mathcal{O}}_{\alpha,J,J_3}$$

$$J_3 = -J, -J+1, \ldots J$$

Let us in what follows concentrate on a

given $\mathcal{O}_\alpha$ and drop α:

$$\begin{cases} \mathcal{O} = \bigoplus_J \tilde{\mathcal{O}}_J & \tilde{\mathcal{O}}_J \Longrightarrow \tilde{\mathcal{O}}_{J,J_3} \\ \\ \mathcal{O}_{\alpha A} \to \mathcal{O}_A \end{cases}$$

⚠ Fields interpolate for 1-particle states: relativistic wave functions

$$\langle 0 | \mathcal{O}_A(x) | \underbrace{p, \sigma}_{\substack{p^2 = M^2 \\ J = s \quad \sigma = -s, -s+1, \dots, s}} \rangle \equiv \psi_{A,\sigma}(p, x)$$

$$= \langle 0 | \mathcal{O}_A(0) | p, \sigma \rangle e^{-ipx} =$$

$$= \langle 0 | H(p) H^+(p) \mathcal{O}_A(0) H(p) | \bar{p}, \sigma \rangle e^{-ipx}$$

$$= D_{AB}(\Lambda_p) \langle 0 | \mathcal{O}_A(0) | \bar{p}, \sigma \rangle e^{-ipx}$$

$$\psi_{A\sigma}(p) \equiv D_{AB}(\lambda_p) c_{B,\sigma}$$

These two eqs are the generalization of what we had found in QFT I-II:

(2) is wave eq. in C.M.

(1) is the general solution obtained by boosting the C.M. one

Indeed (2) can be written as a covariant constraint on $\psi_{A\sigma}(p,x)$

def $$\tilde{\Pi}_{AB}(p) \equiv D_{AC}(\lambda_p)\, \Pi^{\perp}_{CD}\, D^{-1}_{cB}(\lambda_p)$$

obviously $$\tilde{\Pi}_{AB}(p)\psi_{B,\sigma}(p,x) = 0$$

moreover $\tilde{\Pi}(p)$ properly transforms under boosts. Indeed

$$\tilde{\Pi}(\Lambda p) = D(\lambda_{\Lambda p})\; \Pi^{\perp} D^{-1}(\lambda_{\Lambda p})$$

$$D(\lambda_{\Lambda p}) = D(\Lambda) D(\lambda_p) D^{-1}(W(\Lambda, p))$$

Now, by the Wigner-Eckart theorem,
$\langle 0 | O_A(0) | \bar{p}, \sigma \rangle$ can only be $\neq 0$
if spin s irrep is contained in O_A

$\langle A | \equiv \langle 0 | O_A(0)$ $\qquad \langle A | = \oplus \langle J |$

- $\langle 0 | O_A(0) | \bar{p}, \sigma \rangle \equiv C_{A\sigma}$

$\langle 0 | U(R) O_A(0) U(R^{-1}) U(R) | \bar{p}, \sigma \rangle = \langle 0 | O_A(0) | \bar{p}, \sigma \rangle$

$\Downarrow$

$$D^{+}_{AB}(R) D_{\sigma'\sigma}(R) C_{B\sigma'} = C_{A\sigma}$$

- Now C is a $\dim O \times (2s+1)$ matrix with $\dim O \geq (2s+1)$. Indeed

$\dim O = \sum_{s_i} (2s_i + 1)$ $\qquad O = \oplus \tilde{O}_i$

- Dividing the indices A into blocks each corresponding to each S_i

We have by Schur's lemma that $C_{A\sigma}$ can be $\neq 0$ only in correspondence with blocks with $s_i = s$.
In fact we could choose the A-index basis so as to have $C \propto \mathbb{1}$ on those blocks.

For instance if s irreps contained twice in $C_{A\sigma}$ we have

$$C_{A\sigma} = \begin{array}{cc} \overbrace{\hphantom{c_1 \cdot \mathbb{1}_1}}^{2s+1} & \\ \left(\begin{array}{c} c_1 \cdot \mathbb{1}_1 \\ \hline c_2 \mathbb{1}_2 \\ \hline 0 \end{array}\right) & \begin{array}{c} s_1 = s \\ s_2 = s \\ \\ \end{array} \end{array}$$

In the normal practice however the A-basis is such that the non-vanishing $C_{A\sigma}$ blocks are <u>not</u> $\propto \mathbb{1}$

<u>Ex</u> consider $\langle 0 | A_\mu(0) | \vec{p}, \sigma \rangle_s$ $\quad s=1$, $\sigma = -1, 0, +1$

- $\langle 0 | A_0 | \vec{p}, \sigma \rangle = 0$
  $\downarrow$ spin $= 0$

- $\langle 0 | A_i | \vec{p}, \sigma \rangle = \sqrt{Z}\, \varepsilon_{i\sigma}$

by using algebra $[J_i, J_j] = i\varepsilon_{ijk} J_k \quad [J_i, A_j] = i\varepsilon_{ijk} A_k$

plus

$$J_3 |\sigma\rangle = \sigma |\sigma\rangle \qquad J_\pm |0\rangle = \sqrt{2} |\pm\rangle$$

$$J_+ |+\rangle = J_- |-\rangle = 0 \qquad J_\pm |\mp\rangle = \sqrt{2} |0\rangle$$

and normalizing $\varepsilon_{30} = 1$ we have

$$\varepsilon_{i-} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} \qquad \varepsilon_{i0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \varepsilon_{i+} = \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

- The constraints imposed on $C_{A\sigma}$ by Wigner Eckart when covariantly expressed are just the relativistic wave eq. for the corresponding spin
- Let us see this

$$O_A = [j_+, j_-] \qquad |\bar{p}, \sigma\rangle \longmapsto s$$

$$\langle 0| O_A(0) |\bar{p}, \sigma\rangle = C^{(j_+,j_-)\,s}_{A\sigma}$$

Now the indices A host a direct sum of irreps of $SO(3)$.

$\Rightarrow$ In the space A we can write:

$$\mathbb{1}_{AA'} = \bigoplus_r \Pi^{s_r, r}_{AA'}$$

where $\Pi^{s_r,r}$ are projectors on the r-th irrep

By Wigner-Eckart

- $$\Pi^{s_r, r}_{AA'} C^{(j_+ j_-)\,s}_{A'\sigma} = 0 \qquad \text{for } s_r \neq s$$

- if $s_r = s$ appears more than once a basis can be chose such that only one of them satisfies $\Pi^{s_r,r}_{AA'} C^{(j_+ j_-)\,s}_{A'\sigma} \neq C^{(j_+ j_-)s}_{A'\sigma}$ $\quad s_r = s$

Apart from such case, all other maps satisfy $\Pi_{AA'}^{s_r, r}\, C_{A'\sigma}^{(j_+ j_-)\, s} = 0$

This set of constraint corresponds to the relativistic wave eq. when expressed in an arbitrary reference frame and defining $\Pi_{A'A}^{s_r, r}(p) = \left[D(\Lambda_p)\, \Pi^{s_r, r}\, D^{-1}(\Lambda_p)\right]_{A'A}$

we have $\Pi_{A'A}^{s_r, r}(p)\, \psi_{A'\sigma}^{(j_+ j_-) s}(p) = 0$

and of course $\Pi_{A'A}^{s_r, r}$ transforms covariantly under Lorentz

$$D(\Lambda)\, \Pi^{s_r, r}(p)\, D^{-1}(\Lambda) =$$

$$= D(\Lambda) D(\Lambda_p)\, \Pi^{s_r, r} \left(D(\Lambda) D(\Lambda_p)\right)^{-1}$$

$$= D(\Lambda_{\Lambda p}) \left[D(\Lambda_{\Lambda p})^{-1} D(\Lambda) D(\Lambda_p)\right] \Pi^{s_r, r} \left[D(\Lambda_{\Lambda p})^{-1} D(\Lambda) D(\Lambda_p)\right]^{-1} D(\Lambda_{\Lambda p})^{-1}$$

$$= D(\Lambda_{\Lambda p})\left[W(\Lambda, p)\,\Pi^{s_r, r}\,W^{-1}(\Lambda, p)\right] D^{-1}(\Lambda_{\Lambda p})$$

$$= D(\Lambda_{\Lambda p})\,\Pi^{s_r, r}\,D^{-1}(\Lambda_{\Lambda p}) = \Pi^{s_r, r}(\Lambda p)$$

where we use the SO(3) invariance of the projectors

- Regardless of Parity being preserved we can define its action on Lorentz multiplets : $(j_-, j_+) \xmapsto{\mathcal{P}} (j_+, j_-)$

$$\mathcal{O}_A^{(j_-, j_+)} \xrightarrow{\mathcal{P}} \eta_{A\bar{B}}\, \mathcal{O}_{\bar{B}}^{j_+ j_-} \qquad \eta\eta^\dagger = \mathbb{1}$$

where in the simplest case of $(\frac{1}{2}, 0) \to (0, \frac{1}{2})$, in the convention chosen in QFT 1-2, we simply have $\eta = \mathbb{1}_{2\times 2}$: $\psi_L \xrightarrow{\mathcal{P}} \psi_R$

- As $\mathcal{P}$ commutes with rotations, $\eta_{A\bar{B}}$ is $SO(3)$ invariant

- If $\mathcal{P}$ were exact we would have

$$\langle 0 | \mathcal{O}_A^{j_- j_+} | \vec{p}, \sigma \rangle = \eta_{A\bar{B}} \langle 0 \mathcal{O}_{\bar{B}}^{j_+ j_-} | \vec{p}, \sigma \rangle \quad (*)$$

$$(**) \qquad C^L_{A\sigma} = \eta_{A\bar{B}}\, C^R_{\bar{B}\sigma} \qquad C^L = \eta\, C^R$$

$$C^R = \eta^\dagger C^L$$

Where we short cut $c^{i+j+} \to c^L$

$c^{i+j-} \to c^R$

- Broken P simply implies that we cannot group the operators of our theory into pairs ~~satisfying~~ satisfying $(*)$!
- $(**)$ is however a suitable relation for the basis of relativistic wave functions

We can thus consistently define

$$\psi^L_{A\sigma}(p) = D_L(\Lambda_p)_{AB}\, C^L_{B\sigma}$$

$$\psi^R_{\bar{B}\sigma}(p) = D_R(\Lambda_p)_{\bar{B}\bar{A}}\, \eta^+_{\bar{A}C}\, C^L_{C\sigma}$$

$$\Rightarrow \psi^{+R}(p)_{\bar{B}\sigma} = c^{L+}\eta\, D_L^{-1}(\Lambda_p)$$

$$\Rightarrow \left(\psi^{R+}\psi^L\right)_{\sigma\sigma'} = \left(c^{L+}\eta\, c^L\right)_{\sigma\sigma'} \equiv \delta_{\sigma\sigma'}$$

where in the last step we picked a suitable normalization

The abstract derivation of the relativistic wave equation for a massive particle with arbitrary spin can be illustrated in the simplest cases.

• S=0

$$\langle 0|\phi|\bar{p}\rangle = C \qquad \bar{p}^2 = 0$$

$$\langle 0|\phi|p\rangle \equiv \psi = C \qquad (p^2 - M^2)\psi = 0$$

• S=1

$$\langle 0|A_\mu|\bar{p},\sigma\rangle \equiv \bar{a}_{\mu\sigma}$$

$$\bar{a}_{0\sigma} = 0 \qquad \bar{a}_{i\sigma} = \varepsilon_{i\sigma}$$

CM $\quad A_0 = 0 \xRightarrow{\text{covariant}} \partial^\mu A_\mu = 0$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu$$

$$(\Box + M^2) A_\mu = 0$$
$$\partial^\mu A_\mu = 0$$

$$\boxed{4 - 1 = 3 \text{ d.o.f.}}$$

• **S=2** $\quad \langle 0 | h_{\mu\nu} | \bar{p}, \sigma \rangle = \bar{h}_{\mu\nu,\sigma}$

$\sigma = -2, \dots 2$

$\sigma = ij$

$\sigma = ii = 0$

$$h_{00} = h_{0i} = h_{ii} = 0$$

$$\Rightarrow \quad \partial^\mu h_{\mu\nu} = 0 \qquad h^\mu{}_\mu = 0$$

$$-4 \qquad\qquad -1 \quad = -5 \text{ d.o.f.}$$

$$\text{total} = 10 - 5 = 5 \text{ d.o.f.}$$

II Fierz-Pauli action for spin 2

$$\mathcal{L} = \frac{1}{2}\left\{ h_{\mu\nu} \Box h^{\mu\nu} - h^\mu{}_\mu \Box h^\nu{}_\nu + 2 h_{\mu\nu} \partial^\mu \partial^\nu h^\rho{}_\rho - 2 h_{\mu\nu} \partial^\mu \partial^\rho h^\nu{}_\rho \right\}$$

$$+ \frac{m^2}{2}\left(h^\mu{}_\mu h^\nu{}_\nu - h^{\mu\nu} h_{\mu\nu} \right)$$