# Lecture 1 : Representations of Poincaré group

$\boxed{\text{Goal}}$ In QFT 1 and 2 we studied the consequences of Poincaré symmetry by

1. Realizing ISO(3,1) via a local field theory
2. Canonical Quantization

We found :

1. Hilbert space ≡ Fock space ⟹ particles
2. ISO(3,1) unitarily realized, with Noether charges as generators
3. Single particle states ≡ irreps

Our approach was based on specific free or near free lagrangians. We also limited ourselves to particles of spin 0, ½, 1.

In the first lectures of this course, I will go back to these basic issues in order to give a more general and more transparent derivation, not relying on specific lagrangians but purely on symmetry (and basic principles).
I will thus add 2 missing chapters

I. General representations of Poincaré ⟺ one particle states

II. LSZ approach to S-matrix

# Representation of ISO(3,1)

$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$ with $\Lambda \in SO(3,1)$

$g \equiv g(\Lambda, a)$ $\qquad g(\Lambda_1, a_1)\, g(\Lambda_2, a_2) = g(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$

$$\Rightarrow \quad g^{-1}(\Lambda, a) = g(\Lambda^{-1}, -\Lambda^{-1} a)$$

$$g(\Lambda, a) = g(\mathbb{1}, a)\, g(\Lambda, 0)$$

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Exponential map & generators

$$g(\mathbb{1}, a) = e^{i P^{\mu} a_{\mu}}$$

$$g(\Lambda, 0) = e^{-\frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma}}$$

in any representation $P, J$ will take different form, but their commutation relations will be fixed by group multiplication law

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$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\rho} J^{\nu\sigma}\right)$$

$$[P^{\mu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}\right)$$

$$[P^{\mu}, P^{\nu}] = 0$$

$J^i \equiv \frac{1}{2} \epsilon^{ijk} J^{jk}$ $\qquad K^i \equiv J^{i0}$

- For a unitary representation, $J$, $P$ are hermitean operators. As $ISO(3,1)$ is non-compact, unitary representations are $\infty$ dimensional.

- It is useful to construct other covariant objects starting from $J, P$. One goal is to write down the Casimir operators.

⊙ $W^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$ Pauli-Lubanski vector

$\epsilon^{0123} = 1$

⊙ $g^\mu \equiv J^{\mu\nu} P_\nu$ not famous

<u>Notice</u>
- $\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma = \epsilon^{\mu\nu\rho\sigma} P_\sigma J_{\nu\rho}$
- $P_\nu J^{\mu\nu} = g^\mu - 3i P^\mu$
- $W$, $g$ are not in general in the generator algebra!

- $P_\mu W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\mu P_\sigma J_{\nu\rho} = 0$
- $g^\mu P_\mu = J^{\mu\nu} P_\mu P_\nu = 0$

- $[W^\mu, P_\lambda] = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}P_\sigma, P_\lambda] =$

$$= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}, P_\lambda]P_\sigma$$

$$= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(i)(\eta_{\rho\lambda}P_\nu - \eta_{\nu\lambda}P_\rho)P_\sigma$$

$$= -i\epsilon^{\mu\nu\sigma}{}_\lambda P_\nu P_\sigma = 0$$

because of this property $W^\mu$ leaves the eigenspaces of $P_\mu$ invariant

- $P^\mu|\psi\rangle = k^\mu|\psi\rangle \Rightarrow P^\mu W^\nu|\psi\rangle = W^\nu|\psi\rangle k^\mu$

- $W^\mu|\psi\rangle = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma|\psi\rangle =$

$$= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}|\psi\rangle k_\sigma$$

$\Rightarrow$ $W^\mu$ acts on eigenspace as a combination of generators: a combination of generators that leaves $k_\sigma$ invariant!

<u>Ex</u> $k = (M, \underline{0})$ center of mass for particle of mass $M$

$$W^\mu \rightarrow \frac{1}{2}\epsilon^{\mu\nu\rho 0}J_{\nu\rho}M = (0, -M\vec{J})$$

indeed $(M, \underline{0})$ is invariant under rotations

- $[g^\mu, P_\nu] = i(P^\mu P_\nu - \delta^\mu_\nu P^2) \neq 0$

- From the above results we quickly conclude that

  - $W^\mu W_\mu$ and $P^\mu P_\mu$ are Casimirs
  - $[g^2, P_\mu] \neq 0 \quad [g \cdot W, P_\nu] \neq 0$

- $W^\mu W_\mu \equiv W^2$, $P^\mu P_\mu \equiv P^2$ are the two Casimirs of $ISO(3,1)$

  - they label irreps (but they are not complete labels)

- What is their physical meaning?

  - $P^2 \rightarrow (\text{Mass})^2$

  - $W^2 \rightarrow$ · consider eigenstate of $P$
    · $W^2$ invariant $\Rightarrow$ compute in c.m.

$$\Rightarrow W^\mu = M(0, \vec{J}) \Rightarrow W^2 = -M^2 \, \vec{J} \cdot \vec{J}$$
$$= -M^2 s(s+1)$$

$W^2 \rightarrow$ spin

- Irreps are labelled by <u>mass</u> and <u>spin</u>.

# Construction of irreps (unitary)

$$U(\Lambda, a) \equiv e^{iP\cdot a}\, e^{-\frac{i}{2}\omega\cdot J}$$

$U(\Lambda, 0) \equiv U(\Lambda)$ in what follows

⊙ check on transformation property of $P^\mu$

$$g(\Lambda^{-1}, 0)\, g(\mathbb{1}, a)\, g(\Lambda, 0) = g(\mathbb{1}, \Lambda^{-1} a)$$

$$\Downarrow$$

$$U^+(\Lambda)\, e^{iP\cdot a}\, U(\Lambda) = e^{iP\cdot(\Lambda^{-1} a)} = e^{i(\Lambda P)\cdot a}$$

$$\Rightarrow \quad U^+(\Lambda)\, P^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu\, P^\nu$$

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$P_0, \dots, P_3$ commute among themselves

$\Rightarrow$ they can be simultaneously diagonalized

$$|\psi\rangle \Rightarrow |p, \sigma\rangle$$

$p^0, \dots p^3$ ; $\sigma$: all other q-numbers (including spin)

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle$$

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$$P^\mu U(\Lambda) |p, \sigma\rangle = U(\Lambda)\, U^+(\Lambda)\, P^\mu\, U(\Lambda)\, |p, \sigma\rangle$$

$$= U(\Lambda)\, \Lambda^\mu{}_\nu\, P^\nu |p, \sigma\rangle = \left(\Lambda^\mu{}_\nu\, p^\nu\right) U(\Lambda) |p, \sigma\rangle$$

therefore $U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$

- all $\Lambda p$ are featured in spectrum of irrep
- constructing irrep $\Longleftrightarrow$ finding $C_{\sigma\sigma'}(\Lambda,p)$

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- Given $|p,\sigma\rangle$ in irrep we can always bring $p^\mu$ to a <u>reference</u> $\bar{p}^\mu$ by suitable $\Lambda$
- The possibilities for $\bar{p}^\mu$ depend on
  - $p^\mu p_\mu \equiv p^2$
  - $\mathrm{sign}(p^0)$ which we shall however assume $\geq 0$

<u>Cases</u>

I. $p^2 > 0$, $p^2 = M^2$ $\Longrightarrow$ $\bar{p}^\mu = (M, \underline{0})$

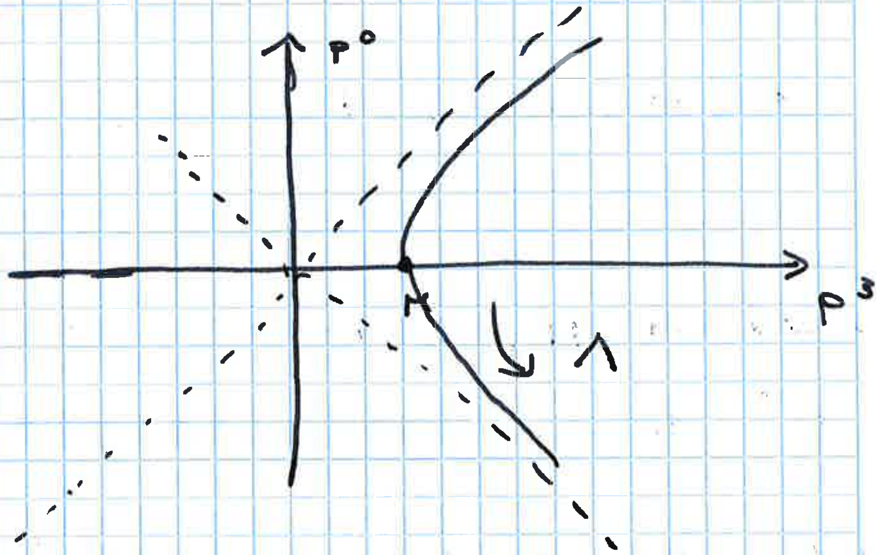
II. $p^2 = 0$ $\hookrightarrow$ 2 sub-cases
$$\begin{cases} \text{II A}: & \bar{p}^\mu = (k,0,0,k) \\ \text{II B}: & \bar{p}^\mu = (0,0,0,0) \end{cases}$$

III. $p^2 = -M^2 < 0$ $\Longrightarrow$ $\bar{p}^\mu = (0,0,0,M)$

(6)

Notice however that case III implies the existence of states with $p^0 < 0$

$p = \Lambda \bar{p}$




III. will then be discarded as un-physical

▲ Little group ≡ subgroup $G_L$ of $SO(3,1)$ that leaves $\bar{p}^\mu$ invariant

■ $\forall \Lambda_L \in G_L$, $U(\Lambda_L)|\bar{p},\sigma\rangle = \sum_{\sigma'} C_{\sigma\sigma'} |\bar{p},\sigma'\rangle$

$\{|\bar{p},\sigma\rangle\}$ furnish a representation of $G_L$

| | | $G_L$ |
|---|---|---|
| I. | $\bar{p}^\mu = (M,0,0,0)$ | $SO(3)$ |
| $II_A$. | $\bar{p}^\mu = (K,0,0,K)$ | $ISO(2)$ |
| $II_B$. | $\bar{p}^\mu = (0,0,0,0)$ | $SO(3,1)$ |
| III. | $\bar{p}^\mu = (0,0,0,M)$ | $SO(2,1)$ |

Apart from degenerate case $II_B$, $G_L$ is generated by $W^\mu(\bar{p}) = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho}\bar{p}_\sigma$

▲ Basic idea (<u>induced representation</u>)

- construct irrep of $G_L$ over $|\bar{p},\sigma\rangle$
- construct all other state by boosting $U(\Lambda)|\bar{p},\sigma\rangle$

■ Discarding III, let us study I, $II_A$, $II_B$

$\mathrm{II}_B$) Simplest case: $|0,\sigma\rangle \equiv$ unitary irrep of $SO(3,1)$

As $SO(3,1)$ is non compact the only finite dimensional unitary irrep is the one dim trivial one $U(G_L) = \mathbb{1}$. (Otherwise there would exist infinitely ~~different~~ many states at $p^\mu = 0$.)

therefore $|0,\sigma\rangle \to |0\rangle \equiv$ Poincaré invariant vacuum state

I) $P^2 = M^2 > 0$

$$\bar{p} = (M, \underline{0}) \Rightarrow G_L \text{ generated by } J^i:$$

$$G_L = SO(3)$$

Discarding IV let us study in detail I – III

$\mathrm{II}_B$) Simplest case : $|0,\sigma\rangle \equiv$ unitary irrep of $SO(3,1)$

As $SO(3,1)$ the only option for a unitary finite dimensional irrep is the one dimensional trivial

representation $|0,\sigma\rangle \longrightarrow |0\rangle \equiv$ Poincaré inv vacuum

I) $p^2 = M^2 > 0$

$\bar{p} = (M, \underline{0}) \Rightarrow J^i$ live $\bar{p}$ invariant $\Rightarrow$ L.G $\equiv SO(3)$ generated by $J^i$

$|\bar{p},\sigma\rangle$ is a representation of $SO(3)$

- must be irreducible for induced rep to be so
- $|\bar{p},\sigma\rangle$ characterized by half integers $s$ (neglecting other q-numbers) $\dim\{|\bar{p},\sigma\rangle\} = 2s+1$

can choose $|\bar{p},\sigma\rangle$ eigenstates of $J_3$

$$J_3|\bar{p},\sigma\rangle = \sigma|\bar{p},\sigma\rangle \qquad \sigma = -s, -s+1, \ldots, +s$$

interpretation $\begin{cases} \bar{p} = (M, \underline{0}) \\ J^2 = s(s+1) \\ J_3 = -s, \ldots +s \end{cases}$

particle of $\begin{pmatrix} \text{mass} & M \\ \text{spin} & s \end{pmatrix}$ in center of mass

- $U(\Lambda)|\bar{p},\sigma\rangle = C_{\sigma\sigma'}|\Lambda\bar{p},\sigma'\rangle$

Now, $\forall\, p$ such that $p \in \Lambda\bar{p}$ choose a specific $\Lambda_p$ such that $p = \Lambda_p\bar{p}$

define $\quad |p,\sigma\rangle = H(p)|\bar{p},\sigma\rangle$

$$H(p) = U(\Lambda_p)$$

The choice $\Lambda_p\,(H(p))$ is not unique.
Different choices correspond to different basis vectors.

The action of $U(\Lambda, a)$ on $\{|p,\sigma\rangle\}$ is now fully determined (and calculable)

Translations
$$e^{iP a}|p,\sigma\rangle = e^{iP\cdot a} H(p)\,|\bar{p},\sigma\rangle$$
$$= H(p)\,H(p)^{+}\,e^{iP\cdot a} H(p)\,|\bar{p},\sigma\rangle$$
$$= H(p)\; e^{i(\Lambda_p P\cdot a)}\,|\bar{p},\sigma\rangle$$
$$= H(p)\; e^{i(\Lambda \bar{p}\cdot)a} \;=\; e^{ipa}\,|p,\sigma\rangle$$

Lorentz
$$U(\Lambda)\,|p,\sigma\rangle =$$
$$H(\Lambda p)\,\underbrace{H^{+}(\Lambda p)\,U(\Lambda)\,H(p)}\,|\bar{p},\sigma\rangle$$
$$= U\left(\Lambda_{\Lambda p}^{-1}\,\Lambda\,\Lambda_p\right).$$

$\Lambda^{-1}_{\Lambda p} \Lambda \Lambda_p$ clearly belongs to $G_L$ ($= SO(3)$)

indeed $\bar{p} \xrightarrow{\Lambda_p} \Lambda_{\bar{p}} = p \xrightarrow{\Lambda} \Lambda p \xrightarrow{\Lambda^{-1}_{\Lambda p}} \bar{p}$

$$\Rightarrow \Lambda^{-1}_{\Lambda p} \Lambda \Lambda_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & W_{ij} & \\ 0 & & & \end{pmatrix} \qquad \begin{array}{l} W^T W = \mathbb{1} \\ W \in SO(3) \end{array}$$

in general $W \neq \mathbb{1}$

$W = W(\Lambda, p) \equiv$ rotation that non-trivially depends on $\Lambda$ & $p$

the explicit form of $W$ depends on choice of $\Lambda_p$.

$W \equiv$ Wigner rotation $\in G_L$

We thus have

$$U(\Lambda)|p,\sigma\rangle = H(\Lambda_p)\, U(W)\, |\bar{p},\sigma\rangle$$

$$= H(\Lambda_p)\, D_{\sigma'\sigma}(W)\, |\bar{p},\sigma'\rangle$$

$$= D_{\sigma'\sigma}(W)\,|p,\sigma'\rangle$$

To represent $U(\Lambda)$ we only need to represent $W$ in the corresponding spin $s$ irrep $D_{\sigma'\sigma}$

We are now basically done. To make $\underbrace{\text{things}}^{\text{concrete}}$ the only missing step is to choose a specific $\Lambda_p$

Among infinitely options two choices are particularly convenient (and popular)

A) The spin basis (ex Weinberg)

$$\Lambda_p^{\text{spin}} = e^{i\vec{\eta}(p)\cdot\vec{K}} \qquad \text{where } \vec{\eta} = \tanh^{-1}\left|\frac{p}{p^0}\right| \frac{\vec{p}}{p}$$

$$p \equiv |\vec{p}|$$

Wigner matrix conveniently worked out explicitly for pure rotations and pure boosts

**Notation**: various parametrizations of rotations

$$R(\underline{\theta}) = e^{-i\,\vec{\theta}\cdot\vec{J}}$$

$$R(\varphi,\theta,\psi) = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{-i\psi J_3}$$

Euler angles

$$R(\underline{n}) = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{i\varphi J_3}$$

where $\underline{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$

Notice
$$R(\underline{n}) = e^{-i\theta\, e^{-i\varphi J_3} J_2\, e^{i\varphi J_3}}$$
$$= e^{-i\theta(\cos\varphi\, J_2 - \sin\varphi\, J_1)}$$

$$\odot\ R(\underline{\theta}) = e^{-i\vec{\theta}\cdot\vec{\hat{J}}} \qquad U(R_\theta) = e^{-i\vec{\theta}\cdot\vec{J}}$$

$$U(R)\,K^i\,U^\dagger(R) = R^{ji}\,K^j = (R^{-1})^{ij}K^j$$

$$\Rightarrow U(R)\,\eta^i K^i\,U^\dagger(R) = (R\eta)^i K^i$$

$$U(R)\,e^{i\vec{\eta}\cdot\vec{K}}\,|\bar{p},\sigma\rangle = U e^{i\eta K} U^\dagger\, U|p,\sigma\rangle$$

$$= e^{i(R\eta)\cdot K}\,U(R)|p,\sigma\rangle$$

$$= H(Rp)\,D_{\sigma'\sigma}(R)\,|p\,\sigma'\rangle$$

$$U(R)|p,\sigma\rangle = |Rp,\sigma'\rangle\,D_{\sigma'\sigma}(R)$$

orbital part      spin part

Same result of non-relativistic particles with spin.

- pure boost $U(\Lambda) = e^{i\vec{\eta}_\Lambda \cdot \vec{K}}$

$$e^{i\vec{\eta}_\Lambda \cdot \vec{K}} |p,\sigma\rangle = e^{i\vec{\eta}_\Lambda \cdot \vec{K}} e^{i\eta(p)\cdot K} |\bar{p},\sigma\rangle$$

$$= e^{i\eta(\Lambda p)\cdot K} \underbrace{e^{-i\eta(\Lambda p) K} e^{i\eta_\Lambda \cdot K} e^{i\eta(p)\cdot K}}_{W} |\bar{p},\sigma\rangle$$

- explicit result is nice exercise

- infinitesimal $\Lambda$ $(\eta_\Lambda \ll 1)$

$$W = \mathbb{1} + i \frac{(\vec{\eta}_\Lambda \wedge \vec{p})}{p^0 + M} \cdot \vec{J}$$

3) the helicity basis (Jacob-Wick)

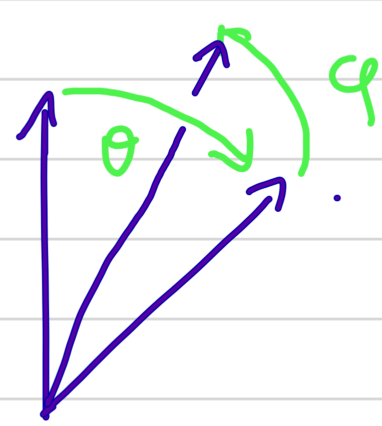
$$\Lambda_{\underline{p}}^{\text{helicity}} = R(\varphi, \theta, -\varphi)\, e^{i\eta K_3}$$

$$\eta = \operatorname{tgh}^{-1} p/p_0$$

$$R(\varphi, \theta, -\varphi) \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} p$$

$$= \vec{p}$$

$$R(\varphi, \theta, -\varphi) = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{i\varphi J_3}$$




$$\Lambda_{\underline{p}} = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{i\eta K_3}\, e^{i\varphi J_3}$$

$U(\Lambda_{\underline{p}}) \,|\, \underline{p}, s_3\rangle$ obviously corresponds to a state of helicity $= s_3$ & momentum $p^\mu$

indeed $U(\chi_p)|\bar{p}, s_3\rangle = \underbrace{e^{i\varphi s_3}}_{\text{trivial phase}} \underbrace{e^{-i\varphi J_3^3} e^{-i\theta J_2}}_{\text{rotate to } \vec{n}(\varphi,\theta)} \underbrace{e^{i\eta k_3}}_{\text{boost} \parallel z} |\bar{p}, \sigma_3\rangle$

Now $U(\chi_p^{hel}) \equiv H^{helicity}(p)$

Normally helicity $\frac{\vec{S}\cdot\vec{p}}{p}$ is indicated by $\lambda$

$\Rightarrow |p,\lambda\rangle \equiv H^{helicity}(p)|\bar{p}, s_3 = \lambda\rangle$

Using $R(\vec{\eta}\cdot\vec{K})R^T = (R\eta)\cdot\vec{K}$

$\Lambda_p^{helicity} = e^{i R(\eta \cdot K_3) R^+} R = e^{i\vec{\eta}(p)\cdot K} R(\varphi, \theta, -\varphi)$

$\Rightarrow \Lambda_p^{helicity} = \Lambda_p^{spin} R(\varphi, \theta, -\varphi)$

⊙ States in spin & helicity bases are related via a rotation in c.m.

## Wigner rotation in helicity basis

- pure rotation; convenient to parametrize with Euler angles

$$R = R(\alpha, \beta, \gamma)$$

$$R(\alpha,\beta,\gamma)\Lambda_p^{hel} = R(\alpha,\beta,\gamma)R(\varphi,\theta,-\varphi)e^{i\eta K_3}$$

$$= R(\varphi',\theta',\psi')e^{i\eta K_3} = \underbrace{R(\varphi',\theta',-\varphi')e^{i\eta K_3}}_{H(p')}\underbrace{e^{-i(\varphi'+\psi')J_3}}_{W}$$

$$W = e^{-i(\varphi'+\psi')J_3} \implies$$ helicity is preserved under rotations

phase factor $e^{-i(\varphi'+\psi')\lambda}$ can be worked out explicitly, for instance using the spin 1/2 representation.

**Pure boost**. It is useful to related Wigner rotation in spin and helicity basis.

$$W_h(\Lambda, p) = H_h(p')^+ \, U(\Lambda) \, H_h(p) \qquad p' = \Lambda p$$

$$= e^{-i\eta' K_3} R^+(\underline{n}') \, U(\Lambda) \, R(\underline{n}) \, e^{i\eta K_3}$$

$$= R^+(n') \, e^{-i\vec{\eta}'\cdot\vec{K}} \, U(\Lambda) \, e^{i\vec{\eta}\cdot\vec{K}} \, R(n)$$

$$= R^+(\underline{n}') \, W_s \, R(\underline{n})$$

. To compute $W_h$ for an infinitesimal pure boost we can use the use the result for $W_s$ and then apply the relations.

For a boost $\Lambda \simeq \begin{pmatrix} 1 & \vec{\beta} \\ \beta & \eta \end{pmatrix}$ one finds (Exercise)

$$W_h = \mathbb{1} + \frac{i(\vec{\beta}\times\vec{n})_3}{1+n_3} J_3 + O\left(\frac{m}{p^0}\right)$$

$\Rightarrow$ helicity is boost invariant in ultra-relativistic limit $m/p^0 \to 0$

⚠ In the spin basis for an ordinary rotation $R$ one has $W = R$, precisely like in ordinary non-relativistic QM.

Stated otherwise, in the spin basis $\vec{J} = \vec{L} + \vec{S}$ where $[L, S] = 0$.

- The spin basis is thus well suited to study the non-relativistic limit

- The spin q-numbers in the spin basis are referred to a fixed frame in the CM.
- In the helicity basis the spin q-numbers are referred to $\vec{p}$ in the lab frame.
  The helicity basis is thus well suited to study the ultrarelativistic limit, in which CM is related to lab by $\to \infty$ boost.
- but in helicity basis cannot write $J = L + S$

## Normalization

$$|p,\sigma\rangle = H(p)\,|\bar{p},\sigma\rangle$$

$P_\mu\,|p,\sigma\rangle = p_\mu\,|p,\sigma\rangle \Rightarrow \langle p'\sigma'\,|\,p,\sigma\rangle = A_{\sigma'\sigma}\,\delta^3(\underline{p}-\underline{p}')$

(In principle it should be $\delta^4(p-p')$, but $p^0$ is not an independent variable over the irrep)

Now; what can we say about $A_{\sigma'\sigma}(p)$?

- Let us consider the spin basis first

$$U(R)\,|p,\sigma\rangle = |R(p),\sigma'\rangle\, D_{\sigma'\sigma}(R)$$

$U^\dagger(R)\,U(R) = \mathbb{1} \Rightarrow$

$$A_{\sigma'\sigma}(p)\,\delta^3(p-p') = A_{\tilde{\sigma}'\tilde{\sigma}}(R(p))\,D^*_{\tilde{\sigma}'\sigma'}\,D_{\tilde{\sigma}\sigma}\,\delta^3(R(p-p'))$$

$$\Rightarrow A_{\sigma'\sigma}(p) = A_{\tilde{\sigma}'\tilde{\sigma}}(R(p))\,D^*_{\tilde{\sigma}'\sigma'}(R)\,D_{\tilde{\sigma}\sigma}(R)$$

Not sufficient to conclude $A_{\sigma'\sigma} = \delta_{\sigma'\sigma}\,f(|p|)$

More specifically

Better to proceed as follows

Consider $\quad \bar{P} = (M, \vec{0}) \quad K = (E_k, \underline{k})$

$$\langle k, \sigma' | \bar{P}, \sigma \rangle = A_{\sigma'\sigma} \, \delta^3(\vec{k})$$

- $A_{\sigma'\sigma}$, because of $\delta^3(\vec{k})$, is just a constant
- Applying a pure rotation like before we get:

$$A_{\sigma'\sigma} = A_{\tilde{\sigma}'\tilde{\sigma}} \, D^*_{\tilde{\sigma}'\sigma'}(R) \, D_{\tilde{\sigma}\sigma}(R)$$

$$\Longrightarrow A_{\sigma'\sigma} = \delta_{\sigma'\sigma} A$$

Now insert $U^\dagger(\Lambda) U(\Lambda) = \mathbb{1}$

$$\delta_{\sigma'\sigma} \delta^3(k) A = \langle k, \sigma' | U^\dagger(\Lambda) U(\Lambda) | \bar{P}, \sigma \rangle =$$

$$\equiv \langle p', \tilde{\sigma}' | p \, \tilde{\sigma} \rangle \, D_{\tilde{\sigma}\sigma}(W(\Lambda, \bar{P})) \, D^*_{\tilde{\sigma}'\sigma'}(W(\Lambda, k))$$

where $\quad p = \Lambda \bar{P} \quad\quad p' = \Lambda k$

Using orthogonality of $D_{\tilde{\sigma}'\sigma}$ we can then write

$$\langle p' \tilde{\sigma}' | p \, \tilde{\sigma} \rangle = D^*_{\tilde{\sigma}\sigma}(W(\Lambda, \bar{P})) \, D_{\tilde{\sigma}'\sigma}(W(\Lambda, k)) \, \delta^3(k) \cdot A$$

but because of $\delta^3(k)$ we can set $k = \bar{p}$ in $W$

$$\Rightarrow \langle p' \tilde{\sigma}' | p, \tilde{\sigma} \rangle = \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(k) \cdot A$$

$$= \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(\Lambda^{-1}(p'-p)) \, A$$

$$= \delta_{\tilde{\sigma}'\tilde{\sigma}} \, \delta^3(p'-p) \, \frac{E_p}{M} \cdot A$$

the standard choice is $A = (2\pi)^3 2M$

$$\Rightarrow \langle p', \sigma' | p, \sigma \rangle = (2\pi)^3 2E_p \, \delta^3(p-p') \, \delta_{\sigma'\sigma}$$

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$$\Lambda^{-1}_p(p') = -\frac{|p|}{m} E' + p'_\perp + \frac{E}{m} p'_L$$

$$= p'_\perp + n_L \frac{E p'_L - p E'}{m}$$

$$\Rightarrow \delta^2(p'_\perp) \, \delta(p'_L - p) \left| \partial_{p'_L} \left( \frac{E p'_L - p E'}{m} \right) \right|^{-1}$$

$$\frac{\partial}{\partial p'_L} \left( \frac{E p'_L - p E'}{m} \right) = \left( E - p \frac{p'_L}{E} \right) \frac{1}{m} = \frac{E - p p'_L}{E m}$$

$$= \frac{m^2}{E m} = \frac{m}{E}$$

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$$\Rightarrow \delta^3(\Lambda_p^{-1}(p')) = \frac{E_p}{m} \delta^3(p' - p)$$

$\mathrm{II}_A)$ $\bar{P}^\mu = (k, 0, 0, k)$

The Pauli-Lubenski vector becomes

$$W^0 = \frac{1}{2}\epsilon^{0123} J_{12} (-k) \times 2 = -J_3 \cdot k$$

$$W^3 = \frac{1}{2}\epsilon^{3120} J_{12} \cdot k \cdot 2 = J_3 k$$

$$W^1 = \frac{1}{2}\epsilon^{1203} J_{20} \cdot 2 \cdot (-k) + \frac{1}{2}\epsilon^{1230} J_{23} \cdot 2 \cdot k$$

$$= k\left(K^2 - J^1\right)$$

$$W^2 = \frac{1}{2}\epsilon^{2103} J_{10} \cdot 2 \cdot (-k) + \frac{1}{2}\epsilon^{2130} J_{13} \cdot 2 \cdot k$$

$$= k\left(-K^1 - J^2\right)$$

$$[W^1, W^2] = -k^2 [K^2 - J^1, K^1 + J^2] =$$

$$= -k^2 \left(iJ^3 - iJ^3\right) = 0$$

$$[W^3, W^1] = \lambda^2 [J^3, K^2 - J^1] =$$
$$= \lambda^2 (-iK^1 - iJ^2) = iKW^2$$

$$[W^3, W^2] = -\lambda^2 [J^3, K^1 + J^2]$$
$$= -\lambda^2 (iK^2 - iJ^1) = -i\lambda W^1$$

$$W^3/K = J^3 \qquad \begin{cases} [J^3, W^1] = iW^2 \\ [J^3, W^2] = -iW^1 \\ [W^1, W^2] = 0 \end{cases} \qquad \boxed{ISO(2)}$$

$$W_\pm = W^1 \pm iW^2 \qquad [J^3, W_1 \pm iW^2] = iW^2 \pm W^1$$
$$= \pm (W^1 \pm iW^2)$$

$$\ast \begin{cases} [J^3, W_\pm] = \pm W_\pm \\ [W_\pm, W_\pm] = 0 \end{cases}$$

Moreover one has $W^2 = W^{0\,2} - \vec{W}^2 = -W_+ W_-$

Consistently with $[J_{\mu\nu}, W^2] = 0$ one can check that (*) implies $[W^\pm, W^2] = [J^3, W^2] = 0$

⚠ $|\bar{p}, \sigma\rangle$ must host a unitary representation of $ISO(2)$

$ISO(2)$ is non-compact and non semi-simple: unitary irreps have either dim $= 1$ or dim $= \infty$.

From the algebra, given $|\lambda\rangle \Rightarrow J^3|\lambda\rangle = \lambda|\lambda\rangle$, $W^\pm$ act as raising and lowering operators

$$W^+|\lambda+n\rangle = b_n|\lambda+(n+1)\rangle$$

$$W^-|\lambda+n+1\rangle = a_n|\lambda+n\rangle = b_n^*|\lambda+n\rangle$$

$$\langle\lambda+(n+1)|W^+|\lambda+n\rangle = \langle\lambda+n|W^-|\lambda+n+1\rangle^*$$

$$\Rightarrow b_n = a_n^*$$

therefore $W^-W^+|\lambda+n\rangle = |b_n|^2|\lambda+n\rangle$

but given $W^- W^+ = -W^2$ is a Casimir, $|b_n|^2$ must be $n$ independent. There are then only two options

1) $W^\pm \neq 0 \Rightarrow |b_n|^2 = \alpha \neq 0 \ \forall n$

$\Longrightarrow$ representation $\infty$ dimensional

$\ldots, |\lambda-1\rangle, |\lambda\rangle, |\lambda+1\rangle, |\lambda+2\rangle, \ldots$

$$\boxed{W^2 = -\alpha}$$

2) $W^\pm = 0 \Rightarrow$ representation 1-dim

$|\lambda\rangle \qquad W^\pm |\lambda\rangle = 0$

$J^3 |\lambda\rangle = \lambda |\lambda\rangle$

$$\Rightarrow \boxed{W^2 = 0}$$

Experience shows that only case 2) is realized in nature. Case 1) would correspond to a massless particle endowed with infinitely

many degrees of freedom (polarizations).
Apparently that would not lead to any stark inconsistency (see Schuster-Toro) if not for the fact that all these states must be coupled extremely weakly.

<u>Case 2</u>. $J^3|\bar{p},\lambda\rangle = \lambda|\bar{p},\lambda\rangle \qquad W^{\pm}|\bar{p},\lambda\rangle = 0$

$$\Rightarrow W^{\mu}|\bar{p},\lambda\rangle = -\lambda\,\bar{p}^{\mu}|\bar{p},\lambda\rangle$$

$$W^{\mu} = -\lambda\,\bar{p}^{\mu} = -\lambda(k,0,0,k)$$

$$\lambda = \frac{\vec{J}\cdot\vec{\bar{p}}}{|\bar{p}|}$$

irrep fully characterized by helicity $\lambda$

$$\boxed{W^2 = P^2 = 0 \qquad W^{\mu} = \lambda P^{\mu}}$$

Constructing the induced representation

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} \kappa \\ \kappa \end{pmatrix} = e^{\eta}\begin{pmatrix} \kappa \\ \kappa \end{pmatrix}$$

$$e^{\eta}\kappa = p \qquad \eta = \ln p/\kappa \equiv \eta_{p/\kappa}$$

There is here no analogue of the spin basis

Convenient to use analogue of helicity basis

$$|p,\lambda\rangle \equiv \underbrace{U(R(\varphi,\theta,-\varphi))\, e^{i\eta_{p/\kappa}K_3}}_{H(p)\,\equiv\, U(\Lambda_p)} |\bar{p},\lambda\rangle$$

corresponds to a state with momentum

$$p^{\mu} = p(1,\vec{n}) \qquad p = e^{\eta}\kappa$$
$$\vec{n} \equiv \vec{n}(\theta,\varphi)$$

and helicity $\lambda$

Given $U^\dagger(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$

$U^\dagger(\Lambda) W^\mu U(\Lambda) = \Lambda^\mu{}_\nu W^\nu$

the relation $W^\mu = -\lambda P^\mu$, valid for our reference state $|\bar{p}, \lambda\rangle$ is preserved for all states $|p, \lambda\rangle$ and under all changes of frame : <u>helicity</u> <u>is</u> <u>Lorentz</u> <u>invariant</u>

⚠ Two things to conclude

a) $\lambda$ at this stage $\in \mathbb{R}$.

For arbitrary real $\lambda$ we have a projective representation of $U(1)_{J_3}$. However we need a consistent projective representation of $SO(3,1)$. The topology of $SO(3,1)$, its $\mathbb{Z}_2$ homotopy group implies

$$e^{i J_3 \cdot 4\pi} = 1 \implies \lambda = \frac{n}{2}$$

b) Local relativistic QFT also realizes CPT $\equiv \Theta$ as a symmetry

$$\Theta \vec{J}\, \Theta = -\vec{J} \qquad \Theta \vec{P}\, \Theta = \vec{P}$$

$$\Rightarrow \Theta\, \vec{J}\cdot\vec{P}\, \Theta = -\vec{J}\cdot\vec{P}$$

so that $\lambda \xrightarrow{\Theta} -\lambda$

In order to represent $\Theta$ we must supplement $|\vec{p},\lambda\rangle$ with an additional state $|\vec{p},-\lambda\rangle$

We thus finally conclude that massless particles always come in to helicity states $\pm\lambda$.

Ex photon $\lambda = \pm 1$

graviton $\lambda = \pm 2$

(massless gravitino) $\lambda = \pm 3/2$