Unitarity $\Longleftrightarrow$ $H = H^{+}$

$\langle \psi | \psi \rangle > 0$

- Lie parameters real $\Longleftrightarrow$ F_a = real

- $\left(\bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b \right)^{+} = \omega_b^{+} \frac{\delta F_a}{\delta \alpha_b} \bar{\omega}_a^{+} = - \bar{\omega}_a^{+} \frac{\delta F_a}{\delta \alpha_b} \omega_b^{+}$

$\Longrightarrow$ $\omega_b^{+} = \omega_b$ is ok

$\bar{\omega}_a^{+} = - \bar{\omega}_a$

- compatible with $Q = Q^{+}$ $(\theta = -\theta^{+})$

$e^{iqJ} \hat{\phi} e^{-iqJ} = \phi + iq[J, \hat{\phi}] + \ldots$

$\underbrace{iqJ}_{J^{+} = J} \to i \theta Q$

$(\theta Q)^{+} = Q^{+} \theta^{+} = - \theta^{+} Q^{+}$

$Q = Q^{+} \Rightarrow \theta = -\theta^{+}$

Ex

$$\delta_\theta A_{a\mu} = \theta D_\mu \omega_a = \theta\left[\partial_\mu \omega_a + f_{abc} A_{b\mu} \omega_c\right]$$

$$\delta \bar{\omega}_a = -\theta B_a$$

- $\delta A^\dagger = \delta A \iff \partial_\mu \omega^\dagger \theta^\dagger = -\theta^\dagger \partial_\mu \omega = \theta \partial_\mu \omega$

- $\delta \bar{\omega}^\dagger = -\delta \bar{\omega} \iff \theta^\dagger = -\theta$

- S real $\Rightarrow$ $H = H^{\dagger}$

$$U^{\dagger}(t) = \left(e^{-iHt}\right)^{\dagger} = e^{iHt} = U(t)^{-1}$$

$$\Rightarrow \quad U^{\dagger}(t)\,U(t) = \underline{\mathbb{1}}$$

- state norm preserved by time evolution

- time evolution $|\psi\rangle \longrightarrow U(t)|\psi\rangle \equiv |\psi(t)\rangle$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi|\,U^{\dagger}(t)\,U(t)\,|\psi\rangle = \langle\psi|\psi\rangle$$

$\langle \psi | \psi \rangle > 0$ for $\psi \in \frac{\mathrm{Ker}\, Q}{\mathrm{Im}\, Q}$

- Perturbation theory $\Longleftrightarrow$ enough to prove in free limit

- $F_a = \partial_\mu A_a^\mu \longrightarrow \mathcal{L}_{ghost} = -\partial_\mu \bar{\omega}_a \partial^\mu \omega_a$

$$\boxed{\begin{aligned} \pi_{\bar{\omega}} &= \frac{\partial}{\partial \dot{\bar{\omega}}} \mathcal{L} = -\dot{\omega} \\ \pi_{\omega} &= \frac{\partial}{\partial \dot{\omega}} \mathcal{L} = \dot{\bar{\omega}} \end{aligned}}$$

$$\{\dot{\bar{\omega}}(x), \omega(y)\} = i\,\delta^3(x-y)$$

$$\{-\dot{\omega}(x), \bar{\omega}(y)\} = i\,\delta^3(x-y)$$

$B\,\partial^2\omega$ $\qquad$ $\partial_\mu B\,\partial\omega$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2}B^2 + B\partial_\mu A^\mu - \partial\bar{\omega}\,\partial\omega$$

$\delta_\theta A_\mu = \theta\,\partial_\mu\omega$

$\delta\bar{\omega} = -\theta B$

$\delta\omega = 0$

$$\rightarrow -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \partial\bar{\omega}\,\partial\omega$$

$\xi = 1$ | $$H = -\frac{1}{2}\pi_\mu\pi^\mu - \frac{1}{2}\vec{\nabla}A_\mu\cdot\vec{\nabla}A^\mu + \pi_\omega\pi_{\bar{\omega}} + \vec{\nabla}\omega\cdot\vec{\nabla}\bar{\omega}$$

$$A_\mu = \int d\Omega_k\left(e^{-ikx}a_\mu(k) + h.c.\right)$$

$\underline{k^2 = 0}$

$$\omega = \int d\Omega_k\left(e^{-ikx}c(k) + e^{ikx}c^\dagger(k)\right)$$

$$\bar{\omega} = \int d\Omega_k\left(e^{-ikx}b(k) - e^{ikx}b^\dagger(k)\right)$$

$$[a^\dagger_\mu(k), a_\nu(p)] = -(2\pi)^3 2\omega_k \eta_{\mu\nu} \delta^3(p-k)$$

$$\{c^\dagger(k), b(p)\} = -(2\pi)^3 2\omega_k \delta^3(p-k)$$

$$\{c(k), b^\dagger(p)\} =$$

- Q-action (see Weinberg)

$$[Q, a^\mu(p)] = -p^\mu c(p)$$

$$\{Q, b(p)\} = p^\mu a_\mu(p)$$

$$\{Q, c(p)\} = 0$$

$$\delta_\theta A_\mu = \theta \partial_\mu \omega$$

$$\delta \bar{\omega} = -\theta B$$

$$\delta \omega = 0$$

- ground state $|0\rangle$:

$$Q|0\rangle = 0 \qquad \text{BRST}$$

$$J^{\mu\nu}|0\rangle = 0 \qquad \text{Lorentz}$$

$$Q_\mu(k)|0\rangle = 0$$

$$c(k)|0\rangle = b(k)|0\rangle = 0$$

- single particle states

$$\varepsilon^\mu(k)\, Q^+_\mu(k)|0\rangle \equiv |\varepsilon(k)\rangle$$

$$b^+(k)|0\rangle \equiv |b(k)\rangle$$

$$c^+(k)|0\rangle \equiv |c(k)\rangle$$

- $Q|\varepsilon\rangle = \varepsilon^\mu [Q, a^\dagger_\mu]|0\rangle = -\varepsilon^\mu p_\mu c^\dagger(k)|0\rangle$

$\Rightarrow \quad Q|\varepsilon\rangle = 0 \quad \Longleftrightarrow \quad \boxed{p^\mu \varepsilon_\mu = 0}$

- $|c\rangle = c^\dagger(k)|0\rangle = -\dfrac{1}{p\cdot\epsilon} Q|\varepsilon\rangle \quad \leftarrow$ exact

- $|b\rangle$: $\quad Q|b\rangle = \{Q, b^\dagger\}|0\rangle = -p^\mu a^\dagger_\mu(p)|0\rangle$

unphysical

- $p^\mu a^\dagger_\mu |0\rangle \equiv |p\rangle \leftarrow$ exact

- general one particle state

$\varepsilon \cdot p = 0$

$$|\varepsilon\rangle + a\left(|b\rangle + |c\rangle + |\varepsilon'\rangle\right)$$

$$= \underbrace{|\varepsilon\rangle + p^{\mu} a_{\mu}^{+} |0\rangle}_{\text{Gupta Bleuler result}} - \underbrace{p \cdot \varepsilon' |c\rangle}$$

- metric (modulo $(2\pi)^3\, 2\omega_k\, \delta^3(p-k)$)

$$\langle a_\mu\, a_\nu^\dagger \rangle = -\eta_{\mu\nu}$$

$$\langle b\, b^\dagger \rangle = \langle c\, c^\dagger \rangle = 0$$

$$\langle b\, c^\dagger \rangle = \langle c\, b^\dagger \rangle = -1$$

$$g_{\alpha\beta} = \left(\begin{array}{cccc|cc} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ \hline & & & & 0 & -1 \\ & & & & -1 & 0 \end{array}\right) \xrightarrow{\text{diag}} \sim (-,+,+,+,-,+)$$

$$|6_\pm\rangle = |b\rangle \pm |c\rangle$$

$$\langle 6_- | 6_- \rangle = -2$$

$$\langle 6_+ | 6_+ \rangle = 2$$

$$\langle 6_+ | 6_- \rangle = 0$$

2-particle states

- ghost–antighost

$$c^\dagger(k_1)\, b^\dagger(k_2)|0\rangle \equiv |k_1, c; k_2, b\rangle$$

$$\langle 0|\, b(k_2)\, c(k_1) \equiv \langle k_1, c; k_2, b|$$

$$\langle p_2, c; p_1, b | k_1, c; k_2, b\rangle = \langle 0|\, b(p_1) c(p_2) c^\dagger(k_1) b^\dagger(k_2)|0\rangle$$

$$= -\langle 0| \{ b(p_1), c^\dagger(k_1) \} \{ c(p_2), b^\dagger(k_2) \} |0\rangle$$

$$= -(2\pi)^3 2\omega_1 \delta^3(p_1 - k_1)\, (2\pi)^3 2\omega_2 \delta^3(p_2 - k_2)$$

• 2-photons $\quad \varepsilon_1^\mu(k_1)\, a_\mu^\dagger(k_1)\, \varepsilon_2^\nu(k_2)\, a_\nu^\dagger(k_2)|0)$

$$= |\varepsilon_1, k_1 ; \varepsilon_2, k_2 >$$

$$< \tilde{\varepsilon}_1, p_1 ; \tilde{\varepsilon}_2, p_2 | \varepsilon_1 p_1 ; \varepsilon_2, p_2 > = (0| \tilde{\varepsilon}_1^* \cdot a(p_1)\, \tilde{\varepsilon}_2^* \cdot a(p_2)\, \varepsilon_1 \cdot a^\dagger(k_1)\, \varepsilon_2 \cdot a^\dagger(k_2)|0)$$

$$= \left(-\tilde{\varepsilon}_1^* \cdot \varepsilon_1\right)\left(-\tilde{\varepsilon}_2^* \cdot \varepsilon_2\right)(2\pi)^6\, 2\omega_1 2\omega_2\, \delta^3(p_1-k_1)\,\delta^3(p_2-k_2)$$

$$+ \left(-\tilde{\varepsilon}_1^* \cdot \varepsilon_2\right)\left(-\tilde{\varepsilon}_2^* \varepsilon_1\right)(2\pi)^6\, 2\omega_1 2\omega_2\, \delta^3(p_1-k_2)\,\delta^3(p_2-k_1)$$

$S \times S \to$	3×3	$\to 9$	$+$
$T \times T \to$	1×1	$\to 1$	$+$
$S \times T \to$	3×1	$\to 3$	$-$
$T \times S$	1×3	$\to 3$	$-$

Working with general Hilbert metric

- $|a\rangle \equiv \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}$ $\quad\quad \langle b| \equiv (b_1^*, b_2^*, \dots)$

- normally

$$\langle b|a\rangle = \sum_i b_i^* a_i \implies \langle a|a\rangle = \sum |a_i|^2 \geq 0$$

- generally $\quad \langle b| \equiv (b_1^*, b_2^*, \dots)\begin{pmatrix} & g & \end{pmatrix}$ → metric

$$\langle b|a\rangle = b_i^* g_{ij} a_j \qquad g_{ij} = g_{ji}^*$$

- adjoint: $\langle b|A|a\rangle = \langle a|A^{+}|b\rangle^{*}$

$$\Updownarrow$$

$$b^{*}_{i}(gA)_{ij}\,a_j = \left(a^{*}_{i}(gA^{+})_{ij}\,b_j\right)^{*} = b_j\,(gA^{+})^{*}_{ij}\,a_i$$

$$\Rightarrow \quad (gA)^{T} = (gA^{+})^{*}$$

$$\boxed{A^{+} = g^{-1}(A^{T})^{*}g}$$

- completeness $\quad \sum |i\rangle\langle i| = \hat{g} \neq \mathbb{1}$ in general

$$\sum \langle i|j\rangle\langle j|k\rangle = \sum g_{ij}\,g_{jk} = (g^{2})_{ik}$$

- g invertible $\Rightarrow g^2 > 0 \Rightarrow$ choose normalization $g^2 = \mathbb{1}$

$$\sum_{i,j} |i\rangle\langle i|j\rangle\langle j| = g^2 = \mathbb{1}$$

- unitarity : $S^\dagger S = \mathbb{1}$

$$\langle\gamma|\delta\rangle = \langle\gamma|S^\dagger S|\delta\rangle = \sum_{\alpha\beta} \langle\gamma|S^\dagger|\alpha\rangle\langle\alpha|\beta\rangle\langle\beta|S|\delta\rangle$$

$$= \sum_{\alpha,\beta} S^*_{\alpha\gamma} S_{\beta\delta} \langle\alpha|\beta\rangle$$

• for physical states $|\gamma\rangle = |\delta\rangle$ $\qquad \langle \delta | \delta \rangle = 1$

$$\Rightarrow \boxed{\sum_{\alpha,\beta} S^{*}_{\alpha\delta} S_{\beta\delta} \langle \alpha | \beta \rangle = 1}$$

$$\mathcal{H}_{phys} = \frac{\operatorname{Ker} Q}{\operatorname{Im} Q}$$

- $\mathcal{H} = H_0 \oplus H_1$ $\qquad H_0 \perp H_1 \qquad H_0 \sim \frac{\operatorname{Ker} Q}{\operatorname{Im} Q}$

- pick diagonal basis for $g_{\alpha\beta}$ in H_0 et H_1

- $g_{\alpha\beta} > 0$ in H_0

- $g_{\alpha\beta} \begin{smallmatrix} <0 \\ >0 \end{smallmatrix}$ in H_1 $\Rightarrow$ $H_1 = H_1^- \oplus H_1^+$

$$\Rightarrow \quad g = \begin{array}{c} \left(\begin{array}{c|c|c} 1 & & \\ \hline & 1 & \\ \hline & & -1 \end{array}\right) \end{array} \begin{array}{l} H_0 \\ H_1^+ \\ H_1^- \end{array}$$

- $|\psi_0\rangle \in H_0 \qquad \langle\psi_0|\psi_0\rangle = 1$

$$1 = \langle\psi_0|\psi_0\rangle = \langle\psi_0|S^\dagger S|\psi_0\rangle =$$

$$= \sum_{a} |S_{a0}|^2 + \sum_{i_+} |S_{i_+0}|^2 - \sum_{i_-} |S_{i_-0}|^2$$

$H_0 \qquad\qquad H_1^+ \qquad\qquad H_1^-$

$$\big|S|\psi_0\rangle\big|^2 = \big|S_{a0}|a\rangle + Q|\varphi\rangle\big|^2$$

$\circ \; [S, Q] = 0 \implies S|\psi_0\rangle = S_{a0}|a\rangle + Q|q\rangle$

$\implies$ given $\langle a|Q|q\rangle = 0 \;\; \forall a$

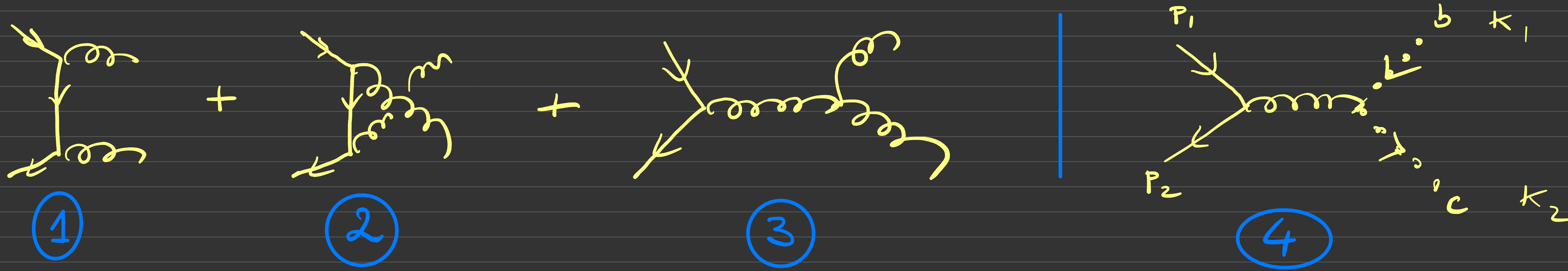
$$\boxed{\langle\psi_0|S^\dagger S|\psi_0\rangle = \sum_a |S_{a0}|^2}$$

$$\Downarrow$$

$$\boxed{\sum_{i_+} |S_{i_+0}|^2 - \sum_{i_-} |S_{i_-0}|^2 = 0}$$

let us check this explicitly $\Longrightarrow$

Unitarity in $q\bar{q} \to gg$

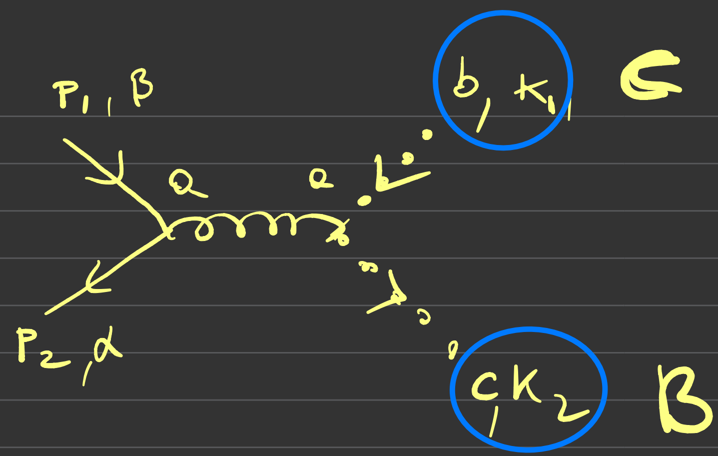


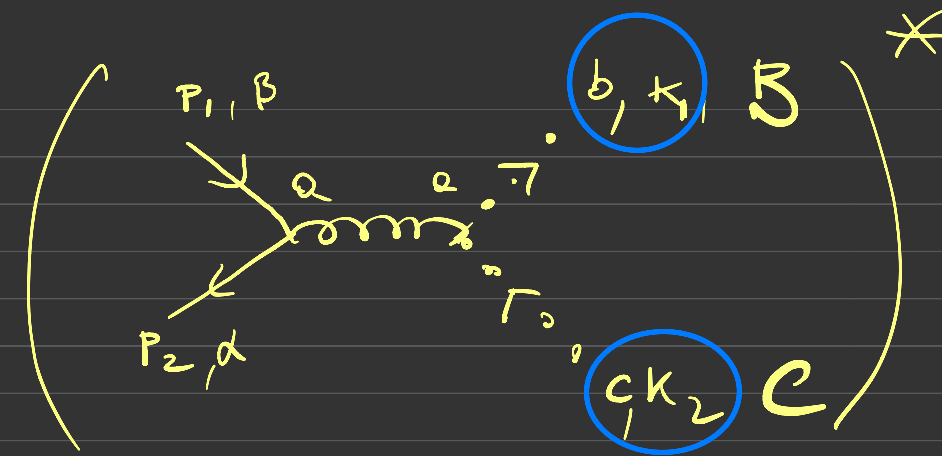
$$④ \quad g^2 \, \bar{v}(p_2)\gamma^\mu u(p_1) \frac{k_2^\mu}{s} T^a_{\alpha\beta} \cdot f^{abc}$$

contribute to $\langle in | S^\dagger S | in \rangle$

$$\Longrightarrow \quad = -g^4 \, \bar{v}(p_2)\gamma^\mu u(p_1) \, \bar{u}(p_1)\gamma^\nu v(p_2) \frac{k_2^\mu k_1^\nu}{s^2} \cdot$$

$$T^a_{\alpha\beta} T^{a'}_{\beta\alpha} f^{abc} f^{a'cb} \qquad d\phi^{(2)}$$





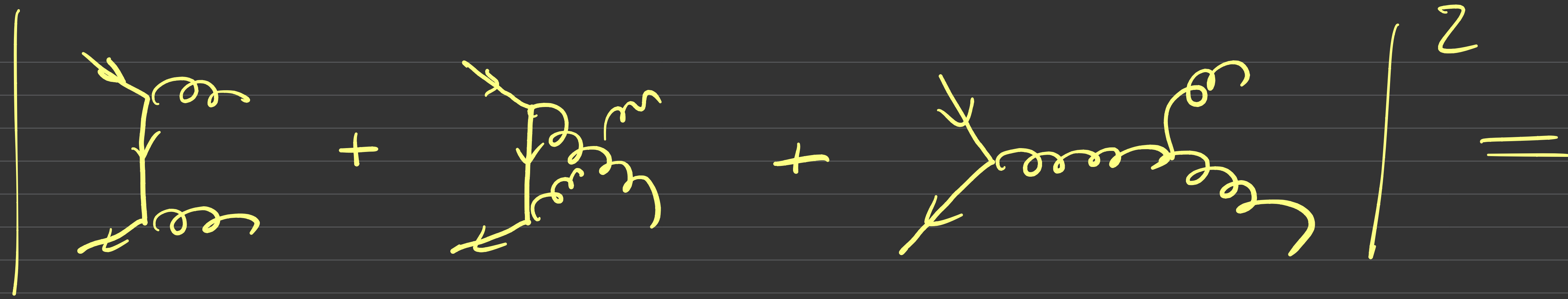
$\langle B(k_1) | C(k_1) \rangle$

$\langle C(k_2) | B(k_2) \rangle$

- $k_2 = p_1 + p_2 - k_1$
- $f^{abc} f^{a'bc} = -f^{abc} f^{a'bc}$

$\mathrm{Tr}\, T^a T^{a'} = t_q \, \delta^{aa'}$ → Dynkin index

$$\Rightarrow \boxed{-g^4 \left| \bar{v}(p_2) \not{k}_1 u(p_1) \right|^2 t_a f^{abc} f^{abc}}$$



$$= \left| S(q\bar{q} \to \text{transverse}) \right|^2 + \left| S(q\bar{q} \to \text{longitudinal}) \right|^2$$

$\Downarrow$ physical

$\Downarrow$ unphysical $\Downarrow$

$$= g^4 \left| \bar{v} \not{k}_1 u \right|^2 t_q \frac{f^{abc} f^{abc}}{s^2}$$

$$\langle q\bar{q}|\, S^{\dagger} S\, |q\bar{q}\rangle =$$

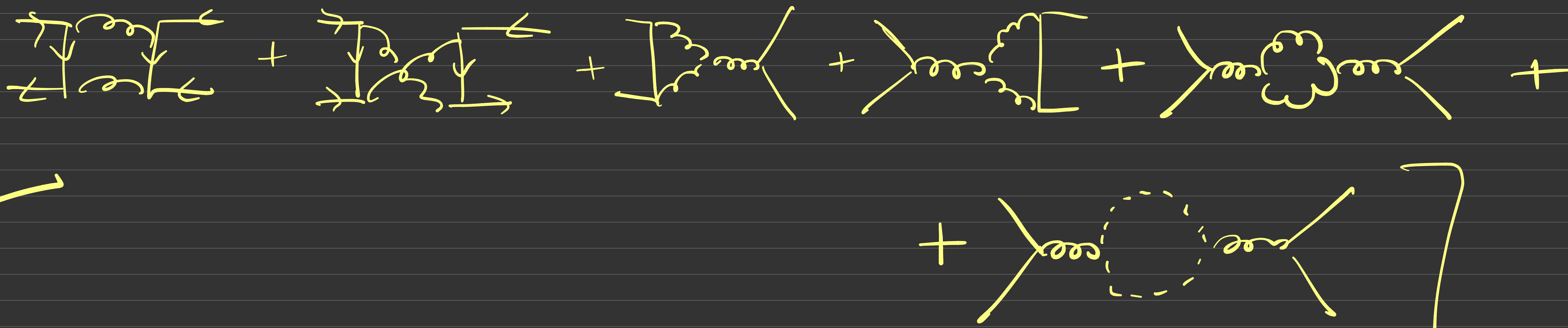
$$\sum_{g_T} \left|\langle g_T g_T | S | q\bar{q}\rangle\right|^2 + \sum_{g_L} \left|\langle g_L g_L | S | q\bar{q}\rangle\right|^2$$

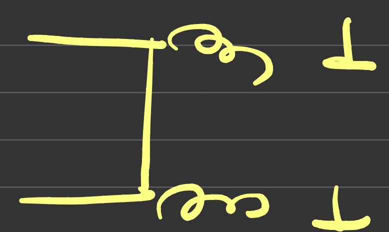
$$- \sum_{ghost} \left|\langle c b | S | q\bar{q}\rangle\right|^2$$

$$= \sum_{g_T} \left|\langle g_T g_T | S | q\bar{q}\rangle\right|^2$$

$$\sum_{i} \varepsilon^{*}_{\mu} \varepsilon_{\nu} = -\eta_{\mu\nu}$$

$$-\eta_{\mu\nu}$$

$\mathrm{Im}\Big[$  $\Big]$

$= \Big|$  $\Big|^2$

⚠ Slavnov-Taylor ids

- integrated BRST Ward id:

$$\langle \delta_\theta (O_1 \dots O_n) \rangle = \theta \langle \delta (O_1 \dots O_n) \rangle = 0$$

- $\langle \delta (\bar{\omega}_a(x) B_b(y)) \rangle = - \langle B_a(x) B_b(y) \rangle = 0$

- $\left\langle \frac{\delta S}{\delta \phi}(y) \, O(x) \right\rangle = i \left\langle \frac{\delta O(x)}{\delta \phi(y)} \right\rangle$

- $\frac{\delta S}{\delta B_a} = \xi B_a + F_a$

$$S \supset \frac{\xi}{2} B^2 + BF$$

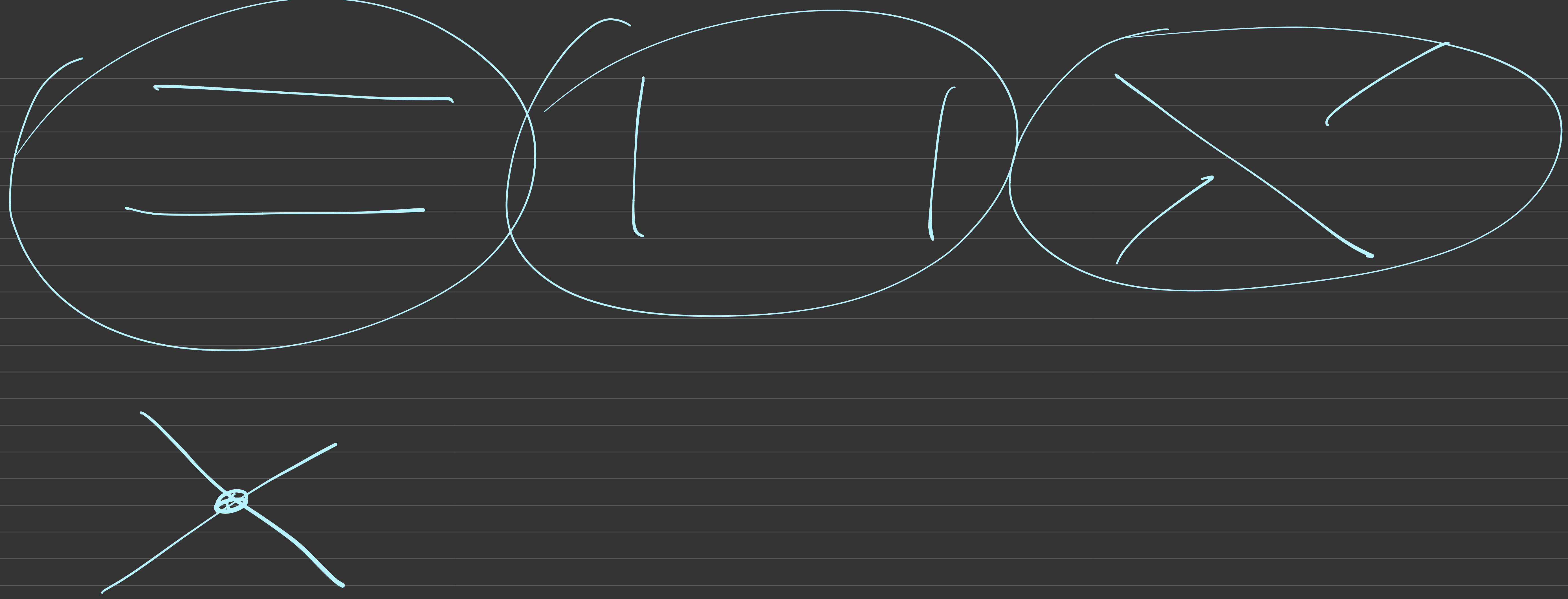
$$\left\langle (\xi B_a + F_a)(y)\, B_b(x) \right\rangle = i \left\langle \frac{\delta B_b(x)}{\delta B_a(y)} \right\rangle = i\, \delta_{ab}\, \delta(x-y)$$

$$\left\langle (\xi B_a + F_a)(y)\, F_b(x) \right\rangle = i \frac{\delta F_b}{\delta B_a} = 0$$

$$\boxed{\left\langle F_a(y)\, B_b(x) \right\rangle = \frac{-1}{\xi} \left\langle F_a(y)\, F_b(x) \right\rangle = i\, \delta_{ab}\, \delta(x-y)}$$

⚠ But there is more (see notes)

- $\langle F_a \rangle = 0$
- $\langle F_a(y)\, F_b(x) \rangle_c = -i \xi\, \delta_{ab}\, \delta(x-y)$
- $\langle F_{a_1} F_{a_2} \cdots F_{a_n} \rangle_c = 0$ for $n > 2$



- $\langle \underbrace{O_1 \cdots O_m}_{\text{gauge inv}} \; F_{a_1} \cdots F_{a_n} \rangle_G = 0$

ii) $\langle O_1 \cdots O_m \; F_{a_1} \cdots F_{a_n} \rangle = \langle O_1 \cdots O_m \rangle \langle F_{a_1} \cdots F_{a_n} \rangle$

◬ Renormalization of gauge theories

- Bare Action

$$S = -\frac{1}{4}\left(G^{a}_{\mu\nu}\right)^2 + \bar{\psi}_0\left(i\not{D}_0 - m_0\right)\psi + \left|D^{\mu}_0\phi_0\right|^2$$

$$+ \frac{\xi_0}{2}\left(B^{a}_0\right)^2 + B^{a}_0 F^{a}(A^0) + \bar{\omega}_0 \frac{\delta F(A^0)}{\delta\alpha}\omega_0$$

$$F(A^0) = \partial^{\mu} A^{a}_{0\mu} \qquad D_{\mu 0} = \partial_{\mu} - i g_0 T_a A^{a}_{0\mu}$$

- $A^0_{\mu} = \sqrt{Z_A}\, A_{\mu} \qquad \psi^0 = \sqrt{Z_\psi}\,\psi \qquad \phi^0 = \sqrt{Z_\phi}\,\phi$

$$\bar{\omega}_0 = \sqrt{\tilde{Z}}\;\bar{\omega} \qquad \omega_0 = \sqrt{\tilde{Z}}\;\omega \qquad B_0 = \sqrt{Z_B}\, B$$

$$\xi_0 = \xi Z_A \qquad \sqrt{Z_B} = \frac{1}{\sqrt{Z_A}} \qquad g_0 = \frac{X}{\tilde{Z}\sqrt{Z_A}}\, g$$

☆ Results: (~ Collins)

1. Correlators of $A, \psi, \phi, \ldots$
   are finite functions of $g, \xi, M, \ldots$

2. $\delta_{\theta_0} \psi_0 = i\theta_0 (g_0 T^a \omega_0^a \psi_0)$

$$\delta_{\theta_0} \psi = i \underbrace{\left(\theta_0 / \sqrt{Z_A \tilde{Z}}\right)}_{\equiv \theta} \underbrace{X g T^a \omega_a \psi}_{\text{finite}}$$

$$\bullet \quad \delta_{\theta_0} A_{0\,\mu}^{\,a} = \theta_0 \left(\partial_\mu \omega_0^{\,a} + g_0 f^{abc} A_{0\,\mu}^{\,a} \omega_0^{\,c} \right)$$

$$\delta_\theta A_\mu^{\,a} = \theta \left(\tilde{Z} \partial_\mu \omega^a + X g f^{abc} A_\mu^{\,b} \omega^c \right)$$

$$\bullet \quad \delta_\theta \omega_a = - \theta \frac{1}{2} X g f^{abc} \omega^b \omega^c$$

$$\delta_\theta \bar{\omega}_a = \theta \cdot \frac{1}{\xi} \partial^\mu A_\mu^{\,a}$$

- Ex use of bare Slavnov–Taylor id.

$A_0 = \sqrt{Z_A}\, A$

$$\langle F_0^a(x)\, F_0^b(y)\rangle \equiv \partial_x^\mu \partial_y^\nu \langle A_\mu^a(x)\, A_\nu^b(y)\rangle Z_A$$

$$\equiv Z_A \langle A_\mu^a(x)\, A_\nu^b(y)\rangle$$

$$= i\xi_0\, \delta_{ab}\, \delta(x-y)$$

$$\text{finite} \Leftarrow \partial^\mu \partial^\nu \langle A_\mu^a\, A_\nu^b\rangle = i\,\frac{\xi_0}{Z_A}\, \delta_{ab}\, \delta(x-y)$$

$$\Rightarrow \frac{\xi_0}{Z_A} \equiv \xi = \text{finite}$$

• eqs of motion: $\xi_0 B_0^a = -F_0^a$

$$\frac{\xi_0 \sqrt{Z_B}}{\sqrt{Z_A}} B^a = -\sqrt{Z_A}\, \partial^\mu A_\mu^a$$

$$\frac{\xi_0 \sqrt{Z_B}}{\sqrt{Z_A}} = \xi \sqrt{Z_A Z_B} = \text{finite}$$

$$\Longrightarrow \quad Z_B = \frac{1}{Z_A}$$

Composite operators

- $[\phi(x,t), \dot{\phi}(y,t)] = i\delta^3(x-y)$
- $\langle 0|\, \phi(x)\phi(y) |0\rangle = \frac{1}{(x-y)^2}$

quantum fields $\equiv$ operator valued distributions

- $\phi^2(x), \phi^3(x), \dots \equiv$ ill defined
- UV problem $\Longrightarrow$ tackled through renormalization

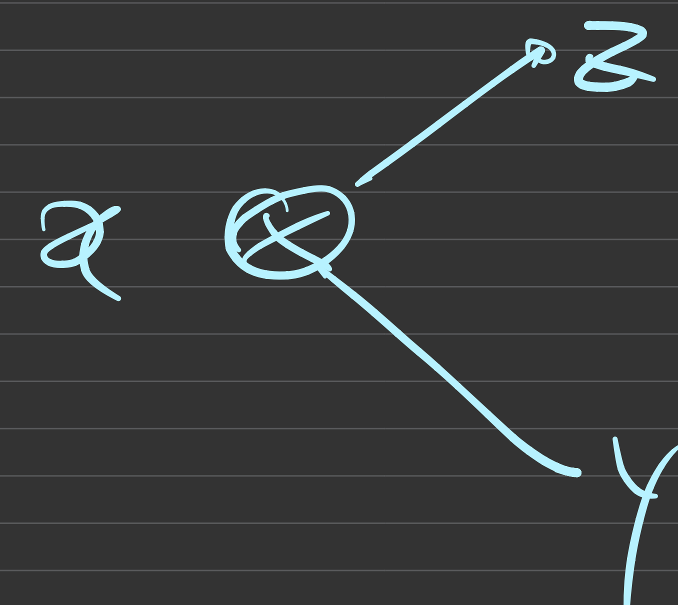
ϕ^2 in free field theory

$$\mathcal{O} \sim \phi^2$$

$$\langle \mathcal{O}(x)\, \phi(z)\phi(y) \rangle = \langle \phi(x)\phi(z)\rangle$$

$$\times \langle \phi(x)\phi(y)\rangle$$

$=$



$$\langle \phi(x)^2 \, \phi(z) \, \phi(y) \rangle = G(x-z)\, G(x-y)$$

2

$$\langle \phi(x)^2 \, \phi(z) \, \phi(y) \rangle$$

$$G(x-x) = G(0) = \int d^4p \, \frac{1}{p^2+m^2}$$

$$\sim \Lambda^2$$

$$\mathcal{O} = \underbrace{\phi(x)^2}_{\text{inf}} - \underbrace{\langle \phi(x)^2 \rangle}_{\text{inf}} = \text{finite}$$

$$\sim \quad \phi^2 \longrightarrow \; :\phi^2:$$

$$Z[c] = \int \mathcal{D}\phi \; e^{-S[\phi] + \sum_i c_i \mathcal{O}_i}$$

All operators : $\phi, \phi^2, \phi^3, \ldots$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{\delta}{\delta c_1(x_1)} \cdots \frac{\delta}{\delta c_n(x_n)} Z[c] \Big|_{c=0}$$

☆ Idea : treat c_i like any other coupling

- Renormalization : $S - \sum_i c_i \mathcal{O}_i + S_{CT}[c_i, \lambda_a, \phi]$

$$\equiv S_0 - \sum_i c_i^0 \mathcal{O}_i^0$$

$$\Rightarrow Z[c, g] = \text{finite}$$

- Simpler problem: single insertion of composites

$$\sum_i c_i \underbrace{\mathcal{O}_i}_{\text{all}} \Rightarrow J\cdot\phi + \sum_i c_i \underbrace{\mathcal{O}_i}_{\text{composite}}$$

$$\langle \mathcal{O}_i \, \phi_1 \cdots \phi_n \rangle = \frac{\delta}{\delta c_i}\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_n} Z[J,c]\Big|_{c,J=0}$$

$\Rightarrow$ work at linear order in c_i's

$$\Rightarrow \mathcal{L}_{CT} = \sum_i c_i (\cdots)$$

$$\Longrightarrow \quad S_0[\Phi_0] + J_0\Phi_0 + \sum_i \underbrace{c_i\, Z^{-1}_{ij}}_{\equiv\, c_j^0} O_j^0$$

- Z^{-1}_{ij} = function of genuine couplings $(m, g, \lambda \ldots)$

- $c_j^0\ ,\ O_j^0 \longrightarrow \mu$-indep

- $c_i\ ,\ Z^{-1}_{ij} O_j^0 \longrightarrow \mu$-dep

- $O_i^R(\mu) \equiv Z^{-1}_{ij}\, O_j^0$ $\qquad O_j^0 = Z_{ji}\, O_i^R$

- $$\mu \frac{d}{d\mu} O_i^R = \underbrace{\left(\mu \frac{d}{d\mu} Z_{ij}^{-1}\right) Z_{jk}}\, O_k^R$$

$$= -\gamma_{ik} \, O_k^R$$

$$\downarrow$$

anomalous dimension matrix

Power counting

- H_p $\mathcal{S}$ renormalizable $\not\Leftrightarrow$ $[\lambda_a] \geq 0$

- $Z_{ij}^{-1} \equiv$ polynomial series in $\lambda_a \Rightarrow [Z_{ij}^{-1}] \geq 0$

- $\underset{4-\Delta_i}{c_i}\ \underset{\Delta_i}{\mathcal{O}_i} \longrightarrow c_i Z_{ij}^{-1} \underset{\Delta_j \leq \Delta_i}{\mathcal{O}_j}$

Ex ϕ^2 in $\lambda\phi^4$ $\qquad \phi \leftrightarrow -\phi$

$$\mathcal{O}_i = \mathbb{1}\,,\ \frac{\phi^2}{2}\,,\ \frac{\phi^4}{4!}\,,\ \Box\phi^2\,,\ \ldots$$

$$J^\mu_{EM} \equiv \bar{\psi}_0 \gamma^\mu \psi_0$$

$$= Z_\psi\, \bar{\psi} \gamma^\mu \psi$$