Canonical formalism in free case

choose $\quad F_a = \partial_\mu A_a^{\ \mu}$

$$\mathcal{L}_{ghost} = -\partial_\mu \bar{\omega}_a \partial^\mu \omega_a$$

$$\left.\begin{aligned} \pi_\omega &= +\dot{\bar{\omega}} \\ \pi_{\bar{\omega}} &= -\dot{\omega} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \{+\dot{\bar{\omega}}(x), \omega(y)\} &=: \delta^3(x-y) \\ \{-\dot{\omega}(x), \bar{\omega}(y)\} &=: \delta^3(x-y) \end{aligned}$$

all other $\{..,..\}$ vanish

- can check that these reproduce time evolution for $H = \pi_\omega \pi_{\bar{\omega}} + \vec{\nabla}\omega \cdot \vec{\nabla}\bar{\omega}$

Now following Weinberg:

$$A_\mu = \int d\Omega_k \left(e^{-ikx} a_\mu(k) + h.c.\right)$$

$$\omega = \int d\Omega_k \left(e^{-ikx} c(k) + e^{ikx} c^*(k)\right)$$

$$\bar{\omega} = \int d\Omega_k \left(e^{-ikx} b(k) + e^{ikx} b^*(k)\right)$$

$$c^*(k) = c^\dagger(k) \qquad b^*(k) = -b^\dagger(k)$$

$$\omega = \int d\Omega_k \left(e^{-ikx} c(k) + e^{ikx} c^\dagger(k)\right)$$

$$\bar{\omega} = \int d\Omega_k \left(e^{-ikx} b(k) - e^{ikx} b^\dagger(k)\right)$$

$$\{c^\dagger(k), b(p)\} = -(2\pi)^3 2\omega_k \, \delta^3(p-k)$$

$$\{c(k), b^\dagger(k)\} = -(2\pi)^3 2\omega_k \, \delta^3(p-k)$$

All other $\{ ,$

- Single particle states [photon + b + c]

$$\varepsilon^\mu a_\mu^\dagger(k)|0\rangle \qquad \langle 0|a_\mu \varepsilon^{\mu *}$$

$$b^\dagger(k)|0\rangle \qquad \langle 0|b(k)$$

$$c^\dagger(k)|0\rangle \qquad \langle 0|c(k)$$

Lets us compute, modulo $(2\pi)^3 2\omega_k \delta^3(p-k)$ the metric

$$\langle Q_\mu Q_\nu^+ \rangle = -\eta_{\mu\nu}$$

$$\langle b\, b^+ \rangle = \langle c\, c^+ \rangle = 0$$

$$\langle b\, c^+ \rangle = \langle c\, b^+ \rangle = -1$$

$$g_{\alpha\beta} = \left(\begin{array}{cccc|cc} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ \hline & & & & 0 & -1 \\ & & & & -1 & 0 \end{array}\right)$$

with 0 in the off-diagonal blocks.

diagonalizing the ghost part

$$g = \mathrm{diag}\,(-,+,+,+,-,+)$$

$\mathrm{Tr}\, g = 2$ consistent with just 2 physical states

- diagonal ghosts $(b^+ + c^+)|0\rangle$
  $(b^+ - c^+)|0\rangle$

$$\langle b+c | b+c \rangle = \langle 0 | (b+c)(b^+ + c^+) | 0 \rangle$$

$$= \langle b c^+ + c b^+ \rangle = -2$$

$$\langle b-c | b-c \rangle = -\langle b c^+ + c b^+ \rangle = 2$$

- ghost-antighost state

$$c^+(k_1)\, b^+(k_2) |0\rangle = c_1^+ b_2^+ |0\rangle$$

$$= |k_1, c; k_2, b\rangle$$

$$\langle k_1, c; k_2, b| = \langle 0| b(k_2)\, c(k_1)$$

$$\langle k_2, c; k_1, b| = \langle 0| b(k_1)\, c(k_2)$$

$$\langle \tilde{k}_2\, c; \tilde{k}_1\, b | k_1, c; k_2, b\rangle =$$

$$= \langle 0| b(\tilde{k}_1)\, c(\tilde{k}_2)\, c^+(k_1)\, b^+(k_2) |0\rangle$$

$$= -\langle 0| b(\tilde{k}_1)\, c^+(\tilde{k}_1)\, c(k_2)\, b^+(k_2) |0\rangle$$

$$= -(2\pi)^3 2\omega_1 \delta^3(\tilde{k}_1 - k_1)(2\pi)^3 2\omega_2 \delta^3(\tilde{k}_2 - k_2)$$

———— o ————

better

$$c_A^+(k)\, b_B^+(p)\,|0\rangle = |k, A, \omega;\, p, B, \bar{\omega}\rangle$$

$$\langle 0|\, b_B(p)\, c_A(k) = \langle k, A, \omega;\, p, B\, \bar{\omega}|$$

$$\langle k', A', \omega;\, p', B'\, \bar{\omega} | k, A, \omega;\, p, B, \bar{\omega}\rangle$$

$$= \langle 0|\, b_{B'}(p')\, c_{A'}(k' |\, c_A^+(k)\, b_B^+(p)\, |\rangle$$

$$= -\, 2\omega_k 2\omega_p (2\pi)^3 \delta^3(p' - k)\,\delta^3(k' - p)$$

crucial !

- BRST on Fock space

From $i[Q, \phi] = s\phi$

it follows

$$[Q, a^\mu(p)] = -p^\mu c(p)$$

$$\{Q, b(p)\} = \frac{1}{\xi} p^\mu a_\mu(p)$$

$$\{Q, c(p)\} = 0$$

- On single photon states

$$\varepsilon^\mu a_\mu^\dagger |0\rangle \equiv |\psi\rangle$$

$$Q|\psi\rangle = \varepsilon^\mu p_\mu c^\dagger(p)|0\rangle$$

$$\boxed{Q|\psi\rangle = 0 \iff p^\mu \varepsilon_\mu = 0}$$

$$c^{+}(p)|0\rangle = \frac{Q\,\varepsilon^{\mu} a^{+}_{\mu}(p)}{\varepsilon^{\mu}\cdot p_{\mu}}|0\rangle$$

$\Longrightarrow$ Q-exact

$$Q\,b^{+}(p)|0\rangle = \frac{p^{\mu}a_{\mu}^{+}}{\xi}|0\rangle \neq 0$$

$\longrightarrow$ un-physical

Cohomology of 1-particle states

$$\left[\varepsilon_{\mu}\,a^{+\mu}(p) + Q\left(\tilde{\varepsilon}_{\mu}\,a^{+\mu} + \varepsilon\, b^{+}\right)|0\rangle\right]$$

$$= \left(\varepsilon_{\mu} + \frac{\varepsilon}{\xi}p_{\mu}\right)a^{+\mu}|0\rangle + \tilde{\varepsilon}\cdot p\; c^{+}|0\rangle$$

null states

norm $\propto \quad \varepsilon^{\mu *}\varepsilon_{\mu} \qquad (p^{\mu}\varepsilon_{\mu}=0)$

⚠ Looking better into non-positive metric

- normally by $|a\rangle$ and $\langle b|$ we mean

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{pmatrix} \qquad \langle b| = (b_1^*, b_2^*, \dots)$$

$$\langle b|a\rangle = \sum_i b_i^* a_i \quad \Rightarrow \quad \langle a|a\rangle = \sum |a|^2 > 0$$

This picture is clearly incorrect in the presence of negative norms. In general we can define

$$\langle b| \to (b_1^* \dots) \underbrace{\begin{pmatrix} & g & \end{pmatrix}}_{\to \text{ metric}}$$

$$\langle b|a\rangle \equiv b_i^* g_{ij} a_j \qquad g_{ij} = g_{ji}^*$$

- In this case the adjoint of an operator is not simply the transposed conjugated.

$$\langle b | A | a \rangle \equiv \langle a | A^\dagger | b \rangle^*$$

$$\Rightarrow b_i^* (g A)_{ij} a_j = \left(a_i^* (g A^\dagger)_{ij} b_j \right)^*$$

$$= b_j^* (g A^\dagger)^*_{ij} a_i$$

$$(g A)^T = (g A^\dagger)^*$$

$$A^T g^T = g^* A^{\dagger *}$$

$$A^\dagger = g^{-1} (A^T)^* (g^T)^* = g^{-1} (A^T)^* g$$

- Completeness

$$\sum_a |a\rangle\langle a = g$$

$$\langle \gamma | \left(\sum |a\rangle\langle a| \right) |\beta\rangle = \langle \gamma | \beta \rangle$$

- Unitarity : $S^\dagger S = \mathbb{1}$

$$\langle \alpha | S^\dagger S | \beta \rangle = \langle \alpha | \beta \rangle$$

· In the cases of interest g is invertible

$\Rightarrow g^2 > 0 \Rightarrow$ can choose normalization of states such that $g^2 = \mathbb{1}$

$$\Rightarrow \mathbb{1} = \sum_{\alpha,\beta} |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta$$

$$\Rightarrow \langle\gamma|\delta\rangle = \langle\gamma| S^\dagger S |\delta\rangle$$

$$= \sum_{\alpha,\beta} \langle\gamma| S^\dagger |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| S |\delta\rangle$$

$$= \sum_{\alpha\beta} S^*_{\alpha\gamma} S_{\beta\delta} \langle\alpha|\beta\rangle = \langle\gamma|\delta\rangle$$

<u>Ex</u> $\gamma = \delta \quad \langle\delta|\delta\rangle = 1$ (physical)

$$\sum_{\alpha,\beta} S^*_{\alpha\delta} S_{\beta\delta} \langle\alpha|\beta\rangle = 1$$

- $\mathrm{Ker}_Q \Rightarrow$ physical states

  More precisely: the physical Hilbert space is the Q cohomology of $\mathrm{Ker}\, Q$

- Consider the subspace H_0 obtained by picking one representative for each element in the cohomology.

- $\mathcal{H} = H_0 \oplus H_1$ $\quad H_1 \perp H_0$

  pick "orthogonal" basis in H_0, H_1.
  - the basis of H_0 has positive norm
  - the basis of H_1 will contain necessarily negative norm states $H_1 = H_1^- \oplus H_1^+$

The metric in this basis will have ± 1 on diagonal.

$$H_0 \to \{|\psi_0^a\rangle\}$$
$$H_1^+ \to \{|\psi_+^i\rangle\}$$
$$H_1^- \to \{|\psi_-^\alpha\rangle$$

Consider now $|\psi_0\rangle \in H_0 \quad \langle\psi_0|\psi_0\rangle = 1$

$$1 = \langle\psi_0|\psi_0\rangle = \langle\psi_0|\, S^\dagger S\, |\psi_0\rangle =$$

$$= \sum_a |S_{a0}|^2 + \sum_{i_+} |S_{i_+0}|^2 - \sum_{\alpha_-} |S_{\alpha_-0}|^2$$

(*)

As $[S, Q] = 0$ $\quad S|\psi_0\rangle = s_{a0}|\psi_0^a\rangle$

$+ Q|q\rangle$

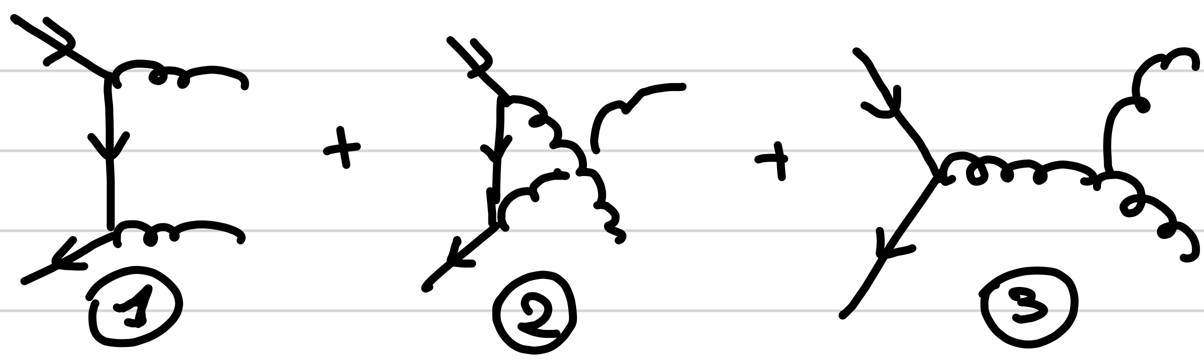
$\Rightarrow \langle\psi_0|S^\dagger S|\psi_0\rangle = \sum_a |S_{a0}|^2$

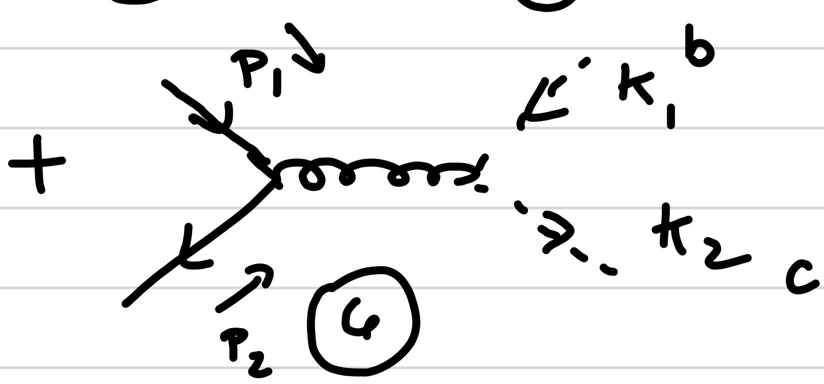
given $\langle\psi_0^a|Q|q\rangle = 0 \quad \forall a$

$\Rightarrow$ In eq (*) we must have

$$\sum_{i_+} |S_{i_+0}|^2 - \sum_{\alpha-} |S_{\alpha-0}|^2 = 0$$

Unitarity in $q\bar{q} \to gg$





• ④ $i(-ig)\, T^a_{\alpha\beta}\, \bar{v}(p_2)\gamma^\mu u(p_1) \cdot$

$$\cdot(-i)\left(\eta_{\mu\nu} - \underbrace{(1-\xi)\frac{(p_1+p_2)^\mu (p_1+p_2)^\nu}{(p_1+p_2)^2}}_{\text{drops}}\right)\frac{k_2^\mu}{(k_1+k_2)^2} f^{abc} g$$

$$= -ig^2\, \bar{v}(p_2)\gamma^\mu u(p_1)\, \frac{k_2^\mu}{s}\, T^a_{\alpha\beta} f^{abc}$$

$\Rightarrow$ contribution to $\langle in| S^\dagger S |in\rangle \Longrightarrow$

$$- g^4 \bar{v}(p_2)\gamma^\mu u(p_1)\, \bar{u}(p_1)\gamma^\nu v(p_2)\, \frac{K_2^\mu K_1^\nu}{s^2}$$

$$\cdot\, T^a_{\alpha\beta} T^{a'}_{\beta\alpha} f^{abc} f^{a'cb} \cdot d\phi$$

- $K_2 = P_1 + P_2 - K_1$

- $f^{abc} f^{a'cb} = - f^{abc} f^{a'bc}$

$$\Downarrow$$

$$- g^4 \left| \left(\bar{v}(p_2)\, \not{K}_1\, u(p_2) \right) \right|^2 \underbrace{t_q}_{\text{Dynkin}} f^{asb} f^{asc}$$

$$\times\, d\phi$$

- We can now compare this to ~~the~~ gluon amplitude. Following Peskin (Section 16.1) we can choose a complete basis for the polarization of each gluon as follows : · $\underline{\text{transverse}}$ $\quad \varepsilon^{T\mu}_{i} \quad i = 1, 2$

· $\underline{\text{forward}}$ $\quad \varepsilon^{\mu}_{+} = \dfrac{k^\mu}{\sqrt{2}|k|}$

· backward $\quad \varepsilon^{\mu}_{-} = \dfrac{\bar{k}^\mu}{\sqrt{2}|k|}$

$$k^\mu = (|k|, \vec{k}) \qquad \bar{k}^\mu = (|k|, -\vec{k})$$

$$\varepsilon^T_i \cdot \varepsilon^{T*}_j = -\delta_{ij} \qquad \varepsilon_\pm \cdot \varepsilon^{T*}_i = 0$$

$$(\varepsilon_+)^2 = (\varepsilon_-)^2 = 0 \qquad \varepsilon_+ \cdot \varepsilon_- = 1$$

With the completeness relation

$$\sum_i \varepsilon^T_{i\mu} \varepsilon^{T*}_{i\nu} - \varepsilon_{-\mu} \cdot \varepsilon^*_{+\nu} - \varepsilon_{+\mu} \cdot \varepsilon^*_{-\nu} = -\eta_{\mu\nu}$$

We can thus define states

$$|k, i\rangle = \varepsilon^T_i \cdot a^\dagger(k)|0\rangle = \varepsilon^T_{i\mu} a^{\dagger\mu} |0\rangle$$

$$|k, \pm\rangle = \varepsilon_\pm \cdot a^\dagger(k)|0\rangle$$

$$\langle k, i | k', j\rangle = 2\omega_k (2\pi)^3 \delta^3(k - k')$$

$$\langle k, \pm | k', \pm \rangle = 0 \qquad \langle k, i | k', \pm \rangle = 0$$

$$\langle k, \mp | k', \pm \rangle = -2\omega_k (2\pi)^3 \delta^3(k - k')$$

- Consider now the longitudinal 2-gluon states: their q-numbers resemble those of the ghosts

$$| k, A, + ; p, B, - \rangle$$

$$\langle p', B', - ; k', A', + | k, A, + ; p, B, - \rangle$$

$$= + \, 2\omega_k \, 2\omega_p (2\pi)^3 \delta_{A'B} \, \delta_{B'A} \, \delta^3(p' - k) \, \delta^3(p - k')$$

$\Downarrow$

positive !

- We now need to compute the amplitude

$$\mathcal{M}(p_1, p_2 \longrightarrow k_1 \varepsilon_-, k_2 \varepsilon_+)$$

This is done in Peskin, eq. 16.21

the result is

$$i\mathcal{M} = ig\,\bar{v}(p_2)\cancel{k}_1 u(p_1)\, T^a_{\alpha\beta}\, \frac{-ig}{s} f^{abc} \left|\frac{k_1}{k_2}\right|$$

So that putting q-numbers properly

$$\mathcal{M}(p_1,\alpha;p_2,\beta|k_1,\varepsilon_-,b;k_2,\varepsilon_+,c)\,\mathcal{M}^*(p_1,\alpha;p_2,\beta|k_1,\varepsilon_+,b;k_2,\varepsilon_-,c)$$

$$= g^4 (\bar{v}\cancel{k}_1 u)(\bar{u}\cancel{k}_2 v)\,\mathrm{Tr}(T^a T^{a'})\frac{f^{abc}}{s^2} f^{a'cb} \left|\frac{k_1}{k_2}\right| \left|\frac{k_2}{k_1}\right|$$

$$= g^4 (-) |\bar{v}\cancel{k}_1 u|^2 d_q (-) \frac{f^{abc} f^{abc}}{s^2}$$

$$= g^4 |\bar{v}\cancel{k}_1 u|\, d_q \frac{f^{abc} f^{abc}}{s^2}$$

which is precisely equal and opposite to the ghost piece

Writing $\mathcal{M}(\dots; k_1, \varepsilon_1, a; k_2, \varepsilon_2, b) \equiv \mathcal{M}^{ab}_{\mu\nu}(k_1,k_2)\,\varepsilon_1^{\mu *}\varepsilon_2^{\nu *}$

and the ghost amplitude $S^{ab}(k_1, k_2)$

and the projector

$$-\eta_{\mu\nu} = \sum_i \varepsilon_i^{T\mu} \varepsilon_i^{T\nu *} - \varepsilon_+^{\mu} \varepsilon_-^{\nu} - \varepsilon_-^{\mu} \varepsilon_+^{\nu}$$

$$= P_T^{\mu\nu}(k) + P_L^{\mu\nu}(k)$$

We can thus write $\left(|\psi\rangle = |p, \alpha; p_2 \beta\rangle \right)$

$$\langle\psi| T^+ T |\psi\rangle = \langle\psi|\psi\rangle \times$$

$$\times \int d\phi \left\{ \frac{1}{2} \mathcal{M}^{ab}_{\mu\nu}(k_1, k_2) \mathcal{M}^{ab*}_{\mu'\nu'} \eta^{\mu\mu'} \eta^{\nu\nu'} - S^{ab}(k_1, k_2) S^{ba}(k_2, k_1)^* \right\}$$

$$\boxed{= \int d\phi \left\{ \frac{1}{2} \mathcal{M}^{ab}_{\mu\nu} \mathcal{M}^{ab*}_{\mu'\nu'} P_T^{\mu\mu'}(k_1) P_T^{\nu\nu'}(k_2) \right\}}$$

The longitudinal and ghost part giving

$$\int d\phi \left\{ \mathcal{M}^{ab}_{\mu\nu}(k_1, k_2) \mathcal{M}^{ab*}_{\mu'\nu'} \varepsilon_+^{\mu}(k_1) \varepsilon_-^{\mu'}(k_1) \varepsilon_-^{\nu}(k_2) \varepsilon_+^{\nu'}(k_2) - S^{ab}(k_1, k_2) S^{ba}(k_2, k_1)^* \right\}$$

$$= \int d\phi \left\{ |S^{as}(k_1,k_2)|^2 - |S^{as}(k_1,k_2)|^2 \right\}$$

$$= 0$$

- We have thus explicitly verified that, given $P_0 \equiv$ projector on physical states, we can define the reduced S-matrix $S_{phys} \equiv P_0 S P_0$

satisfying

$$P_0 = P_0 S^\dagger S P_0 = P_0 S^\dagger P_0 S P_0$$

$$= S^\dagger_{phys} S_{phys}$$

the reduced S-matrix acts unitarily on physical states.

Deriving nilpotency using matrix form of A_μ and ω, $\bar{\omega}$: $A_\mu = A_\mu^a T^a$
$\omega = \omega^a T^a$
$\bar{\omega} = \bar{\omega}^a T^a$

$$\delta M = i\theta \omega M$$

$$\delta \omega = i[\theta\omega, \omega]\frac{1}{2}$$
$$= i\theta\, \omega^2$$

$$\begin{cases} s\omega = i\omega^2 \\ sM = i\omega M \end{cases}$$

$$s^2\omega = i\left((s\omega)\omega - \omega s\omega\right)$$
$$= i^2\left(\omega^2\omega - \omega\omega^2\right) = 0$$

$$s(\omega M) = i\omega^2 M - i\omega^2 M = 0$$

$$\delta A = D\omega$$

$$\frac{\delta F_a}{\delta \alpha_b}\omega_b = sF_a$$

$$-\frac{1}{2\xi} F_a^2 + \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b}\omega_b$$

$$D = \partial_\mu - i A_\mu \qquad \delta_\theta A = \theta D_\mu \omega$$

$$D_\mu \omega = \partial_\mu \omega - i [A_\mu, \omega]$$

$$= \partial_\mu \omega - i A_\mu \omega + i \omega A_\mu$$

$$\delta_\theta D\omega = \theta \Big[\partial_\mu i \omega^2 - i D_\mu \omega\, \omega - i \omega D_\mu \omega$$

$$+ A_\mu \omega^2 - \omega^2 A_\mu \Big]$$

$$= \theta \Big[i \partial_\mu \omega\, \omega + i \omega \partial_\mu \omega - i \partial_\mu \omega\, \omega - i \omega \partial_\mu \omega$$

$$- [A_\mu, \omega] \omega - \omega [A_\mu, \omega]$$

$$+ A_\mu \omega^2 - \omega^2 A_\mu \Big]$$

$$= \theta \Big[- [A_\mu, \omega^2] + [A_\mu, \omega^2] \Big] = 0$$

BRS in toy example

$$\int dM\, d\bar{\omega}\, d\omega\, e^{-S(M) - \frac{1}{2\xi}(F^a(M))^2 + \bar{\omega}_a \frac{\delta F^a}{\delta \pi_b}\omega_b}$$

$$\int dM\, dB\, d\bar{\omega}\, d\omega\, e^{-S(M) + \frac{\xi}{2}(B^a)^2 + F^a B^a + \bar{\omega}_a \frac{\delta F^a}{\delta \pi_b}\omega_b}$$

$$\frac{\xi}{2}(B^a)^2 + F^a B^a + \bar{\omega}_a \frac{\delta F^a(M)}{\delta \pi_b}\omega_b$$

$$\delta_\theta B^a = 0$$

$$\delta_\theta \omega = i\theta\omega^2$$

$$\delta M = i\theta\omega M$$

$$\delta\bar{\omega} = -\theta B \qquad\qquad B \equiv B^a T^a$$

$$\delta_\theta F^a = \frac{\partial F^a}{\partial M_\beta}(i\theta\omega M)_\beta = \theta \underbrace{\left[\frac{\partial F^a}{\partial M_\beta} i T^b_{\beta\alpha} M_\alpha\right]}\omega_b$$

$$= \theta \frac{\delta F^a}{\delta \pi_b}\omega_b$$

$$\delta_\theta \left(\frac{\delta F^a}{\delta \pi_b} \omega_b \right) = \delta_\theta \frac{\partial F^a}{\partial M_\beta} i(\omega M)_\beta$$

$$= \theta \frac{\partial^2 F^a}{\partial M_\beta \partial M_\alpha} (i\omega M)_\alpha (i\omega M)_\beta + \frac{\partial F^a}{\delta M_\beta} i \underbrace{\delta_\theta(\omega M)}_{=0}$$

$$(\omega M)_\alpha (\omega M)_\beta = -(\omega M_\beta)(\omega M_\alpha)$$

$$\Rightarrow = 0 \quad M_\alpha M_\beta = T^a_{\alpha\gamma} T^b_{\beta\delta} M_\gamma M_\delta \; \omega_a \omega_b$$

$$= - T^a_{\alpha\gamma} T^b_{\beta\delta} M_\gamma M_\delta \; \omega_b \omega_a$$

$$= - M_\beta M_\alpha$$

$$\frac{3}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F_a}{\delta \pi_b} \omega_b$$

$$= -s \left[\bar{\omega}_a \left(\frac{3}{2} B_a + F_a \right) \right]$$

$$-s(\bar{\omega}_a h_a) = B_a h_a + \bar{\omega}_a \frac{\delta h_a}{\delta \pi_b} \omega_b$$

$$\frac{\delta F^a}{\delta \pi^b} = \Delta^{ab}(M)$$

$$\int DM \ \operatorname{Det}\Delta \ e^{-S(M) - \frac{1}{2\xi}(F^a)^2}$$

$$F \to F + h \qquad h \equiv \delta F$$

$$\Delta \to \Delta + \frac{\delta h}{\delta \pi}$$

$$\det \Delta \Rightarrow \operatorname{Det}\Delta\left(1 + \operatorname{Tr} \Delta^{-1} \frac{\delta h}{\delta \pi}\right)$$

$$\int DM \det \Delta \, e^{-S - \frac{1}{2\xi}F^2}\left(1 - \frac{F\delta h}{\xi} + \operatorname{Tr}\Delta^{-1}\frac{\delta h}{\delta \pi}\right)$$

$$= \int DM \det \Delta \, e^{-S - \frac{1}{2\xi}F^2}$$

$$\int DM \det \Delta \, e^{-S - \frac{1}{2\xi}F^2}\left(\frac{Fh}{\xi} + \operatorname{Tr}\Delta^{-1}\frac{\delta h}{\delta \pi}\right) = 0$$

Using instead BRST we have

$$\int DM D\bar{\omega} D\omega \ e^{-S + \frac{\xi}{2}B^2 + BF + \bar{\omega}\frac{\delta F}{\delta \pi}\omega}\left(B_a h_a + \bar{\omega}\frac{\delta h}{\delta \pi}\omega\right)$$

$$\int D\bar{\omega} D\omega \; e^{\bar{\omega}\Delta\omega} \; (\bar{\omega} H \omega)$$

$$= \operatorname{Det}\Delta \; \operatorname{Tr} \Delta^{-1} H$$

obvious if Δ, H diagonal

$$\Rightarrow \int DH \, D\bar{\omega} \, D\omega \; e^{-S+\cdots} \left(\frac{Fh}{3} + \operatorname{Tr}\left(\Delta^{-1} \frac{\delta h}{\delta H}\right) \right)$$

Slavnov-Taylor ids.

$$\frac{\xi}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F^a}{\delta \alpha_b} \omega_b$$

$$= -s\left[\frac{\xi}{2} \bar{\omega}_a B_a + \bar{\omega}_a F_a \right] \equiv s Y$$

$$\int s\left(\bar{\omega}_a(x) B_b(y)\right) e^{iS} = 0$$

$$\xi B^a + F^a = 0$$

$$\langle (\xi B^a + F^a)(x)\, B^b(y) \rangle = i \delta_{ab}\, \delta(x-y)$$

$$F^a(x)\, B^b(y) = i \delta_{ab} (x-y)$$

$$\langle (\xi B^a + F^a) F^b \rangle = 0$$

$$\langle F^a F^b \rangle = -\xi \langle B^a F^b \rangle = -i\xi \delta_{ab}\, \delta(x-y)$$

One can similarly derive an identity for the n-point function

$$\langle F^{a_1}(x_1) \cdots F^{a_n}(x_n) \rangle = (-i\xi)^{n/2} \cdot \frac{1}{2^{n/2}(n/2)!}$$

$$\sum_{\pi} \delta_{a_{\pi(1)} a_{\pi(2)}} \cdots \delta_{a_{\pi(n-1)} a_{\pi(n)}} \, \delta(x_{\pi(1)} - x_{\pi(2)}) \cdots \delta(x_{\pi(n-1)} - x_{\pi(n)})$$

Idea : integrate out B_a

$$\int DA \, D\bar{\omega} \, D\omega \; e^{iS - \int \frac{i}{2\xi} F_a^2 + i \bar{\omega} \frac{\delta F}{\delta \alpha} \omega}$$

$F_a \to F_a + J_a$ $\hookrightarrow$ field independent

does not affect the path integral

$$\Rightarrow \frac{\delta}{\delta J_{a_1}(x_1)} \cdots \frac{\delta}{\delta J_{a_n}(x_n)} \int D\phi \; e^{i\tilde{S} - \frac{i}{2\xi} \int (F_a + J_a)^2} \Bigg|_{J_a = 0}$$

$$-\frac{i}{2\xi} F_a^2 - i \frac{FJ}{\xi} - i \frac{J^2}{2\xi}$$

$$= e^{iW(-iJ/\xi)} \; e^{-i/2\xi \int J^2}$$

$$\frac{\delta}{\delta(-iJ/\xi)} \frac{\delta}{\delta(-iJ/\xi)} iW = \langle F \, F \rangle_c = \frac{\delta}{\delta(-i\frac{J}{\xi})} \frac{\delta}{\delta(-i\frac{J}{\xi})} \int \frac{i}{2\xi} J^2$$

$$\Rightarrow \quad \langle F_a(x) F_b(y) \rangle_c = \frac{3}{i} \delta_{ab} \delta(x-y) = -i\, 3\, \delta_{ab} \delta(x-y)$$

while the connected $n>2$ correlators vanish

- This is an example of a powerful Ward identity that can be derived from BRST

- Similarly one finds that

$$\langle \underbrace{O_1 \dots O_m}_{\text{gauge inv}} F_{a_1} \dots F_{a_n} \rangle_c = 0$$

the full correlator reduces to

$$\langle O_1 \dots O_m F_{a_1} \dots F_{a_n} \rangle = \underbrace{\langle O_1 \dots O_m \rangle}_{\text{physical}} \underbrace{\langle F_{a_1} \dots F_{a_n} \rangle}_{\text{trivial}}$$

Renormalization of gauge theories

(unbroken case for simplicity)

- Let us rescale $A_\mu \to g A_\mu$; or equivalently $T_a \to g T_a$ $\quad f_{abc} \to g f_{abc}$ in all our eqs. (including BRST).

- Ordinarily symmetries act <u>linearly</u> on the fields. In that case, provided the regulator respects the symmetry, the structure of the counterterms can be directly inferred by using the symmetry. Indeed symmetry in this case is also useful when some coupling breaks it explicitly: such couplings can be formally promoted to fields transforming under the symmetry in such a way as to formally restore it. The resulting **selection rules** still provide a powerful constraint

Ex

$$\int D\phi \; e^{-S[\phi,\lambda] + J\cdot\phi}$$

λ : couplings

assume $\quad S[\phi_g, \lambda_g] = S[\phi, \lambda]$

$$J_g \phi_g = J\phi$$

$$\Rightarrow Z[\lambda, J] = Z[\lambda_g, J_g]$$

$$Z_0[J] + \lambda_a \cdot Z_a^{(1)}[J] + \frac{\lambda_a \lambda_b}{2} Z_{ab}^{(2)} \ldots$$

suitably transforms under G

$\equiv$ selection rules

- The same structure appears when considering UV divs: the divergent part of diagrams must respect on its own the selection rules, given the symmetry transformation does not depend on the regulator and thus there cannot be compensation between the divergent and the finite parts.
- The situation is instead more involved in the case of non-linear symmetry, and more specifically when terms of quadratic and higher order in the fields appear in their transformations. This is easily understood by the need to renormalize composite operators: the symmetry opera=

tion now intrinsecally depends on the regulator. BRST in unfortunately in this class as $\delta_\theta \psi$, $\delta_\theta A$, $\delta_\theta \omega$ are all quadratic in the elementary fields. Still it can be proven that both correlators and the BRS transformation can be renormalized order by order in perturbation theory. The result is as follows. Consider bare fields and couplings

$A_{0\mu}$, $\psi_{0\alpha}$, B_{0a}, $\bar{\omega}_{0a}$ ω_{0a}

g_0, ξ_0, $\underbrace{M_0}$, matter field masses

$$\Rightarrow D_\mu = \partial_\mu - i g^0 A^a_{0\mu} T^a$$

$$D_\mu \omega_{0a} = \partial_\mu \omega_{0a} + g_0 f_{abc} A^b_{0\mu} \omega_{0c}$$

etc.

and lagrangian

$$S = -\frac{1}{4}(G^0_{\mu\nu})^2 + \bar{\psi}_0 (i\not{D} - M_0)\psi_0 + |D_\mu \phi_0|^2$$

$$+ \ \ldots$$

$$+ \ \frac{\xi_0 B_{0a}^2}{2} + B_0 F_a(A^0) + \bar{\omega}_0 \frac{\delta F(A_0)}{\delta \alpha} \omega_0$$

with $F^a = \partial^\mu A^a_{0\mu}$

one can define renormalized quantities through

$$A^0_{a\mu} = \sqrt{Z_A}\, A_{a\mu} \qquad \psi^0 = \sqrt{Z_\psi}\, \psi$$

$$\phi^0 = \sqrt{Z_\phi}\, \phi \qquad \bar{\omega}_0 = \sqrt{\tilde{Z}}\, \omega$$

$$\omega_0 = \sqrt{\tilde{Z}}\, \omega \qquad g_0 = \frac{X}{\tilde{Z}\, Z_A^{1/2}}\, g$$

$$\xi_0 = \xi Z_A \qquad B_{0a} = \sqrt{Z_B}\, B_a$$

such that $\sqrt{Z_B} = \frac{1}{\sqrt{Z_A}}$

1. The correlators of $A, \psi, \phi, \ldots$ are finite functions of $g, \xi, M, \ldots$

2. given the unrenormalized BRST

$$\delta_{\theta_0} \psi_0 = i\,\theta_0 \left(g_0 T^a \omega_0^a \psi_0 \right)$$

$$\delta_{\theta_0} \psi = i\,\theta_0 \frac{X}{(Z_A \tilde{Z})^{1/2}} g \quad T^a \omega_a \psi$$

$$= i\,\theta_0 \left(\frac{1}{Z_A \tilde{Z}}\right)^{1/2} X \, g \, T^a \omega_a \psi$$

$$= i\,\theta \, \underbrace{X \cdot g \, T \cdot \omega \, \psi}_{\text{finite}}$$

$$\theta = \theta_0 \left(\frac{1}{Z_A \tilde{Z}}\right)^{1/2}$$

Similarly

$$\delta_{\theta_0} A_{0\mu}^a = \theta_0 \left(\partial_\mu \omega_0^a + g_0 f^{abc} A_{0\mu}^b \omega_0^c \right)$$

$$\Rightarrow \delta_{\theta_0} A_\mu^a = \theta_0 \frac{1}{(Z_A \tilde{Z})^{1/2}} \left[\tilde{Z} \partial_\mu \omega^a + g X f^{abc} A_\mu^b \omega^c \right]$$

$$\delta_\theta \omega_a = -\frac{1}{2} g X f^{abc} \omega^b \omega^c$$

$$\delta_{\theta_0} \bar{\omega}_{0a} = -\theta_0 B_{0a}$$

$$= \theta_0 \frac{1}{\xi_0} \partial^\mu A_{\mu 0}^a =$$

$$\Rightarrow \delta_\theta \bar{\omega}_a = \theta \cdot \frac{1}{\xi} \partial^\mu A_\mu^a$$

$$B_{0a} = \frac{1}{\xi} \frac{1}{\sqrt{Z_A}} \partial^\mu A_\mu^a = \frac{1}{\sqrt{Z_A}} B_a$$

- We won't prove 1. and 2. A proof of sort is found in Chapter 12 of Collins. We shall limit ourselves to a few relations. In particular the renormalizations of ξ, B and A are constrained by the BRS Ward identities we previously derived:

$$\langle F_0^a(x)\, F_0^b(y)\rangle = \partial^\mu \partial^\nu \langle A_{0\mu}^a(x)\, A_{0\nu}^b(y)\rangle$$

$$= i\, \xi_0\, \delta_{ab}\, \delta(x-y)$$

$\Rightarrow$ once Z_A is picked to make the transverse part of $\langle A_\mu^a A_\nu^b\rangle$ finite, by the above eq. finiteness of the longitudinal part fixes

$$\frac{\xi_0}{Z_A} \equiv \xi = \text{finite}$$

- similarly $\langle \xi_0 B_0^a \ldots \rangle = \langle -F_0^a \ldots \rangle$

$$\Rightarrow \quad Z_A \sqrt{Z_B} = \sqrt{Z_A} \quad \Rightarrow \quad Z_B = \frac{1}{Z_A}$$