Haar measure

- given $\Omega = e^{i\pi \cdot T}$ π^a = Lie parameters

define the differential form

$$\Omega^{-1} d\Omega \equiv i T^a \hat{f}^a$$

$$= i T^a f^{ab} d\pi^b$$

- Consider now the transformation of $\hat{f}^a$ under group action
- Left action $\Omega \longrightarrow g\Omega = e^{i\pi_L(g)\cdot T}$

$$\pi^a \longrightarrow \pi_L^a = F_L^a(\pi, g)$$

$$d\pi_L^a = \frac{\partial F_L^a}{\partial \pi^b} d\pi^b$$

$$i T^a \hat{f}^a = \Omega^{-1} d\Omega = (g\Omega)^{-1} d(g\Omega) = i T^a f^{ab}(\pi_L)\, d\pi_L^b$$

$$\boxed{f^{ab}(\pi)\, d\pi^b = f^{ab}(\pi_L)\, d\pi_L^b} \quad L$$

- Right action $\quad \Omega \to \Omega g = e^{i\pi_R(g)\cdot T} \equiv \Omega_R$

$$\pi^a \to \pi_R^a = F_R^a(\pi, g)$$

$$iT^a f^{ab}(\pi_R(g))\, d\pi_R^b = \Omega_R^{-1} d\Omega_R = g^{-1}(\Omega\, d\Omega)\, g$$

$$= i\, g^{-1} T^a g\, f^{ab}(\pi)\, d\pi^b$$

$$= i\, D_A^{ac}(g)\, T^c f^{ab}(\pi)\, d\pi^b$$

↳ adjoint representation of g

$$\Rightarrow \quad R \quad \boxed{\begin{aligned} f^{ab}(\pi_R)\, d\pi_R^b &= D^{ca}(g)\, f^{cb}(\pi)\, d\pi^b \\ &= D^{ac}(g^{-1})\, f^{cb}(\pi)\, d\pi^b \end{aligned}}$$

- We can now define

$$d\mu_\pi = f^{a_1} \wedge \cdots f^{a_N} \cdot \frac{\varepsilon^{a_1 \ldots a_n}}{n!}$$

$N \equiv \dim G$

with $\varepsilon^{a_1 \dots a_n}$ the Levi-Civita tensor on adjoint indices and $f^a = f^{ab} d\pi^b$

- By L, f^a is invariant under left action and so is $d\mu_\pi$
- IR reads $f^a \longrightarrow D^{ab}(g)\, f^b$ under right action. $\Rightarrow$

$$d\mu_\pi \to \det D(g)\, d\mu_\pi$$

under right action. For compact groups $D(g)$ is orthogonal $\Rightarrow$ $d\mu_\pi$ is invariant also under right action

0-dimensional incarnation of Faddeev-Popov

- unitary Lie Group G
- $M_\alpha \equiv$ representation $M \to M_g \equiv D_{\alpha\beta}(g)\, M_\beta$
- $g = e^{i\pi \cdot T}$ $\qquad \pi_A \qquad A = 1 \dots N$ Lie parameters
- Left action $\quad g \to hg = e^{i\pi_h^L \cdot T}$

$$g \to gh = e^{i\pi_h^R \cdot T}$$

- Haar measure $\quad d\mu_\pi = d\mu_{\pi_h^L} = d\mu_{\pi_h^R}$

Consider the integral

$$Z_1 = \int dM\, e^{-S(M)}$$

- $S(M) = S(M_g)$
- $dM_g = dM$ (imagine exists)

- We want to factor out the "group orbits" from dM integral
- Ideally $dM \sim d\mu_\pi \times dI$ $\leftarrow$ invariants
- Concretely we can do as follows

Consider N functions $F_A(M)$ and assume

1) $\forall M \; \exists g \mid F_A(M_g) = 0 \quad \forall A$

2) $\partial_B F_A(M_{g(\pi)}) \equiv \Delta_{AB} =$ invertible

$$\partial_B \equiv \frac{\partial}{\partial \pi_B}$$

I want to claim that

$$\int d\mu_\pi \prod_A \delta\big(F_A(M_{g(\pi)})\big) \det \Delta(M_{g(\pi)})$$

is independent of the choice of $F_A(M)$ (as long as 1) et 2) hold)

$$M_\alpha \xrightarrow{g \in G} D_{\alpha\beta}(g) M_\beta$$

$$g = e^{i\pi \cdot T} \qquad \delta M_\beta = i\pi_A \cdot (T_A M)_\beta$$

Under group action

$$\delta F_A(M) = i\, \partial_B F_A\, \pi_C (T_C)_{B\alpha} M_\alpha$$

$$= \pi_C \left[i\, \partial_\beta F_A (T_C)_{\beta\alpha} M_\alpha \right]$$

$$= \pi_C \Delta_{AC}$$

$$F_A(M(\pi)) = F_A(M) + \pi_C \Delta_{AC} + O(\pi^2)$$

around $\bar{M}$ such that $F_A(\bar{M}) = 0$

$$F_A(M(\pi)) = \pi_C \Delta_{AC}(\bar{M}) + O(\pi^2)$$

Thus

$$\int d\mu_\pi \prod_A \delta\left(F_A\left(M_{g(\pi)}\right)\right) \cdot \left|\det \Delta\left(M_{g(\pi)}\right)\right|$$

$$F_A\left(M_{g(\pi)}\right) = 0 \quad \text{at} \quad \pi = \bar{\pi}$$

$$g(\bar{\pi}) \equiv \bar{g} \qquad M_{g(\bar{\pi})} \equiv \bar{M}$$

$$g(\pi) = g(\tilde{\pi})\,\bar{g}$$

$$d\mu_\pi = d\mu_{\tilde{\pi}}$$

$$= \int d\mu_{\tilde{\pi}} \prod_A \delta\left(F_A\left(\bar{M}_{g(\tilde{\pi})}\right)\right) \left|\Delta\left(\bar{M}_{g(\tilde{\pi})}\right)\right|$$

$$\equiv \int \left(1 + O(\tilde{\pi})\right) \prod_A d\tilde{\pi}_A\, \delta\left(\Delta_{AB}\tilde{\pi}_B + O(\tilde{\pi}^2)\right) \cdot \left|\det \Delta_{AB}\right|$$

$$= 1 = \text{indep of } F$$

Now

$$\int dM\ e^{-S(M)} = \int dM\, d\mu_\pi\, e^{-S(M)} \delta(F(M_{g(\pi)})) / \Delta(M_{g(\pi)})|$$

$$= \int dM_{g(\pi)}\, d\mu_\pi\, e^{-S(M_{g(\pi)})} \delta(F(M_{g(\pi)})) |\Delta(M_{g(\pi)})|$$

$$= \underbrace{\int d\mu_\pi}\ \underbrace{\int dM'\, e^{-S(M')} \delta(F(M')) |\Delta(M')|}$$

$$= V_G \times I$$

"I" represents the integral over the invariants

- We can further manipulate this integral by noticing $F_A(M) \to F_A(M) - B_A$ does not affect the result

$$I = \int N\, dB\, e^{-\frac{B^2}{2\xi}}\, dM\, e^{-S(M)} \delta(F-B) |\Delta(M)|$$

$$\overset{\text{F-indep}}{=} N \int dM\, e^{-S(M) - \frac{1}{2\xi}(F(M))^2} \left|\Delta(M)\right|$$

⚠ <u>Faddeev–Popov</u>: the real one (!)

Consider the <u>naive</u> path integral of a gauge theory (simple G)

$$\int D\psi\, DA\, e^{i\int\left\{-\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} + \mathcal{L}_M\right\}} = \int D\psi\, DA\, e^{iS[A,\psi]}$$

All gauge invariant

<u>Ex</u> $\prod_{x,a,\mu} dA^a_\mu(x) \equiv dA$

$$A^a_\mu \to A^a_\mu + \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c$$

$$dA \to dA \prod_{x,a\mu} \left(1 + f^{aac}\delta(0)\alpha^c\right)$$

$$= dA$$

- The integral $D\psi DA$ sums over all group orbits ψ_g, A_g yelding an overall factor V_G. The problem is that $V_G = \infty$ in the case of a local symmetry $\Rightarrow$ the path integral is not defined.

This difficulty is one and the same we encountered in QFT2 when trying to quantize Maxwell theory:

$F_{\mu\nu}F^{\mu\nu}$ does not provide a healthy quadratic term for all variables A_μ^a.

$\Rightarrow$ Gaussian integral not well defined

$\Rightarrow$ To define a proper path integral we can employ a generalization of the factorization trick we saw previously.

This will be quickly recognized to correspond to <u>fixing the gauge</u>.

- Introduce $F^a(A(x))$ N functions satisfying
  - $\forall A \; \exists g \quad F^a(A_g) = 0 \quad \forall x$,
  - $\dfrac{\delta}{\delta \alpha^b} F^a(A_{g(\alpha)}) \equiv \Delta^{ab}$

    non. singular

$$1 = \int d\mu_\alpha \prod_{x,a} \delta\left(F^a(A_{g(\alpha)})\right) \left| \operatorname{Det} \Delta(A_{g(\alpha)}) \right|$$

- <u>Ex 1</u> Lorentz gauge $\quad F^a = \partial^\mu A^a_\mu(x)$

$$\frac{\partial}{\partial \alpha^b(y)} \partial^\mu \left(A_{g(\alpha)}\right)^a_\mu =$$

$$= \frac{\partial}{\partial \alpha^b(y)} \left(\partial^2 \alpha^a(x) + f^{acd} \partial^\mu (A^c_\mu \alpha^d) \right)$$

$$\delta^{ab}\partial^2 \delta^d(x-y) + f^{acb}\partial^\mu\left(A^c_\mu(x)\,\delta^d(x-y)\right)$$

$$= \partial^\mu (D_\mu)^{ab}\,\delta^d(x-y)$$

$$(D_\mu)^{ab} = \delta^{ab}\partial_\mu - f^{abc}A^c_\mu$$

$$= \delta^{ab}\partial_\mu + f^{acb}A^c_\mu$$

• Ex 2 Axial gauge $F^a = A^a_3(x)$

$$\Rightarrow \frac{\partial}{\partial\alpha_b} F^a(A_j) = \partial_3\delta^{ab}\delta(x-y) + f^{acb}A^c_3$$

$$\overset{A^b_3=0}{=} \partial_3\delta^{ab}\delta(x-y)$$

$$\bar{c}(x)\,\partial^2_x\,\delta(x-y)\,c(y) = \bar{c}\,\partial^2 c$$

We can now perform <u>formally</u> the same manipulations of the toy example

$$\int D\psi DA\, e^{iS[A,\psi]} \cdot \int d\mu_\alpha\, \delta(F(A_\alpha)) \left(\det \Delta(A_\alpha)\right)$$

$$= \int D\psi_\alpha DA_\alpha\, e^{iS[A_\alpha,\psi_\alpha]} \int d\mu_\alpha\, \delta(F(A_\alpha)) \det \Delta(A_\alpha)$$

$$= \underbrace{\int d\mu_\alpha}_{\infty} \underbrace{\int D\psi DA\, e^{iS[A,\psi]} \delta(F(A)) \det \Delta(A)}_{\text{independent of choice of } F}$$

... it if wasn't for the infinity

- but one can imagine regulating path integral (Ex lattice + finite volume) and then take a limit.

. We can thus declare the second factor to define the path integral of a gauge theory. Indeed one finds that for the specific case of axial gauge this path integral coincides with the one obtained by the Hamiltonian approach which in axial gauge can be easily carried out using the Dirac constrained quantization. (see Weinberg II).

Notice that the independence of the path integral from the gauge fixing function F^A extends obviously to all correlators of gauge invariant operators

$$\int D\psi DA\, e^{iS} \to \int D\psi DA\, \mathcal{O}_1(A,\psi)\cdots\mathcal{O}_n(A,\psi)\, e^{iS}$$

for $O_i(A_g, \psi_g) = O_i(A, \psi)$

· According to our procedure, non-gauge invariant operators are not univocally determined. They should thus not be considered observable. This is consistent with the interpretation of the gauge symmetry as a <u>redundancy</u> in the parametrization of the dynamical variables, rather than a genuine symmetry.

- Further useful manipulation of the path integral

⊙ bring gauge fixing in exponent

$$\underbrace{\int dB_a\, e^{-\frac{i}{2\xi}\int B_a^2}}_{\substack{\text{overall}\\ \text{constant}}}\ \underbrace{\int D\psi DA\, e^{iS}\, \delta(F_a(A)-B_a)\, \det\Delta}_{B\text{-indep}}$$

$$\Downarrow$$

$$\int D\psi DA\, \left|\mathrm{Det}\,\frac{\delta F^a}{\delta\alpha_b}\right|\, e^{iS-i\frac{1}{2\xi}(F_a)^2}$$

⊙ Write $\det\frac{\delta F^a}{\delta\alpha_b}$ as integral over grassmann variables $\bar\omega_b, \omega_a$

the Faddeev-Popov ghosts: scalar fields respecting Fermi-Dirac statistics (thus unphysical)

$$I_{gh} = \int dx\, dy\ \bar{\omega}_a(x)\ \frac{\delta F^a(x)}{\delta \alpha_b(y)}\ \omega_b(y)$$

$$\Downarrow$$

$$\operatorname{Det} \frac{\delta F}{\delta \alpha} = \int d\omega\, d\bar{\omega}\ e^{i I_{gh}}$$

so that finally

$$\boxed{\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}A\, \mathcal{D}\omega\, \mathcal{D}\bar{\omega}\ e^{iS - \frac{i}{2\xi}(F^a)^2 + i\bar{\omega}\frac{\delta F}{\delta \alpha}\omega}}$$

It is instructive to discuss this result and the resulting Feynman rules in Lorentz gauge

Ex $F^a = \partial^\mu A^a_\mu$

$$\frac{\delta F^a(x)}{\delta \alpha_b(y)} = \partial^\mu_x D^{ab}_\mu(x)\, \delta(x-y)$$

$$= \partial^\mu \left[\delta^{ab} \partial_\mu + f^{acb} A^c_\mu \right] \delta(x-y)$$

$$\Rightarrow I_{gh} = \bar{\omega}^a \left[\delta^{ab} \partial^2 + f^{acb} \partial^\mu A^c_\mu \right] \omega^b$$

- Vector propagator

$$-\frac{1}{4g^2} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2g^2 \xi} \left(\partial^\mu A^a_\mu \right)^2 = \mathcal{L}_{kin}$$

notice $\xi \to \xi g^2$

$$= \frac{1}{2g^2} \left[-\partial_\mu A^a_\nu \partial^\mu A^{\nu\, a} + \partial_\mu A^a_\nu \partial^\nu A^{\mu\, a} \left(1 - \frac{1}{\xi} \right) \right]$$

$\Downarrow$ Fourier

$$e^{iS} \Rightarrow \frac{i}{2g^2} A^a_\nu(-p) \left[p^2 \eta^{\mu\nu} - p^\nu p^\mu \left(1 - \frac{1}{\xi} \right) \right] A_\mu(p)$$

$$= \frac{i}{2g^2} A_\nu(-p) \Delta_{\mu\nu} A_\mu(p)$$

$$\int e^{i\frac{1}{2} q \Delta q + J q} =$$

$$\int e^{i/2 (q + i\Delta^{-1}J)\Delta(q + i\Delta^{-1}J) - \frac{i}{2} J \Delta^{-1} J}$$

$$\Rightarrow \langle A^a_\mu A^b_\nu \rangle = -i g^2 \delta^{ab} (\Delta^{-1}_{\mu\nu})$$

$$= \frac{-i g^2 \delta_{ab}}{p^2} \left[\eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right]$$

— o —

· Faddeev-Popov ghost

$$\int Dc D\bar{c} \; e^{+\bar{c}\Delta c + \bar{c}\eta + \bar{\eta} c}$$

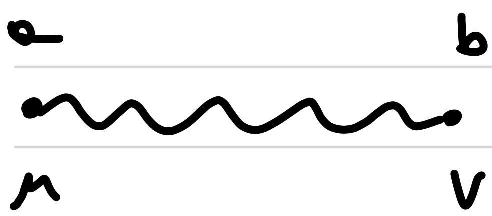
$$\langle c_a \bar{c}_b \rangle = -\Delta^{-1}_{ab}$$

In our case $\Delta_{ab} = + i \partial^2 \cdot \mathbb{1}_{ab}$

$$-\Delta^{-1}_{ab} = \frac{-\mathbb{1}_{ab}}{i\partial^2} = \frac{+i\mathbb{1}_{ab}}{\partial^2} = \frac{-i\delta_{ab}}{p^2}$$

Going to the canonical basis for A^a_μ : $A^a_\mu \to g A^a_\mu$

We thus have

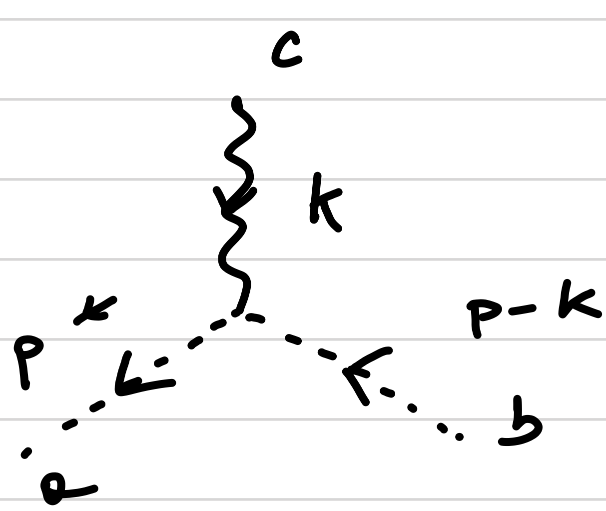


$$= -\frac{i\,\eta_{\mu\nu}\,\delta^{ab}}{p^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

$$= -\frac{i}{p^2}\,\delta^{ab}$$

- Vertex $ig\,\bar{\omega}^a f^{acb}\partial^\mu\left(A^c_\mu \omega^b\right)$

$$\Rightarrow -ig\left(\partial^\mu \bar{\omega}^a\right)\left(A^c_\mu \omega^b\right) f^{acb}$$



$$p^\mu f^{acb} g$$

For the next application our action

$$(*)\quad S = S_0(A,\psi) - \frac{1}{2\xi}(F_a)^2 + \bar{\omega}_a \frac{\delta F^a}{\delta \alpha_b}\omega_b$$

can more conveniently be traded for

$$S = S_0 + \frac{\xi}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F^a}{\delta \alpha_b}\omega_b$$

given B_a is a non propagating auxilliary field with a purely gaussian path integral

$\Rightarrow$ integrate out B gives $(*)$

• <u>BRST symmetry</u>

Consider $\quad\quad\theta \equiv$ fermionic parameter

$$\delta_\theta \psi = i\, T_a\, \theta\, \omega_a\, \psi$$

$$\delta_\theta A_{a\mu} = \theta\, D_\mu\, \omega_a = \theta\left[\partial_\mu \omega_a + f_{abc} A_{b\mu} \omega_c\right]$$

$$\delta \bar{\omega}_a = -\theta\, B_a$$

$$\delta \omega_a = -\frac{1}{2}\theta\, f_{abc}\, \omega_b\, \omega_c$$

$$\delta B_a = 0$$

• Notice that for ψ, A this is just an infinitesimal transformation with Lie parameters $\alpha_a = \theta\, \omega_a$

$\Rightarrow$ The $S_0(A,\psi)$ part of the action is <u>invariant</u>.

- Consider now the remaining part

$$\frac{\xi}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F^a}{\delta \alpha_b} \omega_b$$

- now notice $\delta F_a = \theta \frac{\delta F^a}{\delta \alpha_b} \omega_b$

given F^a is just a functional of A_μ and the BRST transformation of A_μ is a gauge transf with parameter $\alpha_b = \theta \omega_b$

$\Rightarrow$ the variation gives

$$B_a^2 \to 0 \qquad B_a F_a \to \cancel{B_a \left(\theta \frac{\delta F^a}{\delta \alpha_b} \omega_b\right)} \qquad \bar{\omega} \frac{\delta F}{\delta \alpha} \omega \to \cancel{-\theta B_a \left(\frac{\delta F}{\delta \alpha_b} \omega_b\right)} + {}$$

$$+ \bar{\omega}\, \delta\left(\frac{\delta F}{\delta \alpha} \omega\right)$$

$$= \bar{\omega}_a \delta \left(\delta F_a \right)$$

Now, remarkably, it turns out that the BRST we defined is <u>nilpotent</u>

$$\delta^2 = 0$$

$\Rightarrow$ the last term also vanishes

The nilpotency of δ. Some more formalism. Define, following Weinberg,

$$\delta_\theta F = \theta\, s F$$

Let us check that $s^2 = 0$

- $s\psi = i T_a \omega_a \psi$

$$\delta_\theta s\psi = i T_a (\delta_\theta \omega_a)\psi + i T_a \omega_a \delta_\theta \psi$$

$$= i T_a \left(-\frac{1}{2}\theta f_{abc}\omega_b\omega_c\right)\psi$$

$$+ i T_a \omega_a\, i T_b \theta \omega_b \psi$$

$$= \theta \left\{ -\frac{i}{2} f_{abc}\omega_b\omega_c T_a + \omega_a\omega_b T_a T_b \right\} \psi$$

$$= \theta \left\{ -\frac{i}{2} f_{abc}\omega_b\omega_c T_a + \frac{1}{2}\omega_a\omega_b [T_a, T_b] \right\} \psi$$

$$= \theta \left\{ -\frac{i}{2} f_{abc}\omega_b\omega_c T_a + \frac{i}{2} f_{abc}\omega_a\omega_b T_c \right\} \psi$$

$$= 0$$

$$\delta_\theta s\psi \equiv \theta s^2\psi = 0$$

• $\delta_\theta s A_{a\mu} = \delta_\theta \left[\partial_\mu \omega_a + f_{abc} A_{b\mu} \omega_c\right]$

$$= -\frac{\theta}{2} f_{abc} \cancel{\partial_\mu(\omega_b \omega_c)} - \frac{\theta}{2} f_{abc} A_{b\mu} f_{cde} \omega_d \omega_e$$

$$+ \theta f_{abc} \left(\cancel{\partial_\mu \omega_b} + f_{bde} A_{d\mu} \omega_e\right) \omega_c$$

$$= \theta \left\{ -\frac{1}{2} f_{abc} f_{cde} - f_{acd} f_{cbe} \right\} A_{b\mu} \omega_d \omega_e$$

$$= \theta \left\{ -\frac{1}{2} f_{abc} f_{dec} + f_{adc} f_{bec} \right\} A_{b\mu} \omega_d \omega_e$$

$$= -\frac{\theta}{2} \left\{ f_{abc} f_{dec} + f_{adc} f_{ebc} + f_{aec} f_{bdc} \right\} \times A_{b\mu} \omega_d \omega_e$$

where we used antisymmetry in de

Now $\{\dots\} = 0$ by the Jacobi identity

$$\Rightarrow \quad s^2 A_{a\mu} = 0$$

- $s\bar{\omega}_a = -\theta B_a \qquad s B_a = 0$

$$\Rightarrow \quad s^2 \bar{\omega}_a = 0 \qquad \text{trivially}$$
$$s^2 B_a = 0$$

- $\delta_\theta\, s\omega_a = \delta_\theta \left(-\frac{1}{2} f_{abc}\, \omega_b \omega_c \right)$

$$= -\frac{\theta}{4} \Big\{ f_{abc} f_{bde}\, \omega_d \omega_e \omega_c - f_{abc} f_{cde}\, \omega_b \omega_d \omega_e$$

$$= \frac{\theta}{2} f_{acb} f_{deb}\, \omega_d \omega_e \omega_c$$

$$= \frac{\theta}{2} \frac{1}{3} \Big[f_{acb} f_{deb} + f_{aeb} f_{cdb} + f_{adb} f_{ecb} \Big] \omega_d \omega_e \omega_c$$

$$= 0 \quad \text{by Jacobi}$$

• We have thus proven $s^2=0$ when acting on elementary fields. We still need to prove this extends to any product of fields

Ex two fields

$$\delta_\theta \phi_1\phi_2 = \theta s\phi_1 \phi_2 + \phi_1 \theta s\phi_2$$

$$= \theta\left(s\phi_1 \phi_2 + (-)^{F_1} \phi_1 s\phi_2 \right)$$

$$s(\phi_1\phi_2) \equiv s\phi_1 \phi_2 + (-1)^{F_1} \phi_1 s\phi_2$$

$$\Rightarrow s^2(\phi_1\phi_2) = s^2\phi_1 + (-1)^{F_1+1} s\phi_1 s\phi_2$$

$$(-1)^{F_1} s\phi_1 s\phi_2 + \phi_1 s^2\phi_2 =$$

$$= (-1)^{F_1}\left\{ - s\phi_1 s\phi_2 + s\phi_1 s\phi_2 \right\} = 0$$

• n-fields: by induction

- Thus for any functional $Y[\phi]$

$$s^2 Y[\phi] = 0$$

- Now let us focus on the last remaining term in $\delta_\theta \mathcal{L}$

$$\bar{\omega}_a \delta_\theta \frac{\delta F_a}{\delta \alpha_b} \omega_b$$

$$\equiv \bar{\omega}_a \delta_\theta [s F_a] = \bar{\omega}\, \theta\, s^2 F_a = 0$$

- Indeed using all the previous results we can write

$$\frac{3}{2} B_a^2 + B_a F_a + \bar{\omega}_a \frac{\delta F^a}{\delta \alpha_b} \omega_b$$

$$= -s\left[\frac{3}{2}\bar{\omega}_a B_a + \bar{\omega}_a F_a\right] \equiv s Y$$

which is obviously BRST invariant as a consequence of the nilpotency of s

- In the end the action has the form

$$S = S_0[A,\psi] + sY$$

$S_0[A,\psi]$: gauge inv $\equiv$ non-trivially BRST invariant

sY: BRST exact $\equiv$ trivially BRST inv.

- BRST charge: by Noether we thus have a conserved J^μ_{BRST} with Ward id.

$$i\partial_\mu \langle J^\mu_{BRST}(y)\,\phi_1(x_1)\cdots\phi_n(x_n)\rangle =$$

$$\delta(y-x_1)\langle s\phi_1(x_1)\cdots\rangle + \delta(y-x_2)(-1)^{F_1}\langle\phi_1 s\phi_2\cdots\phi_n\rangle$$
$$+\ldots+\delta(y-x_n)(-1)^{F_1+\ldots F_{n-1}}\langle\phi_1\cdots\phi_{n-1}\,s\phi_n\rangle$$

Integrating it over $\mathbb{R}^3 \times \underset{[t_1-\varepsilon,\, t_1+\varepsilon)}{\underset{\downarrow}{\Delta T}}$

and defining $Q = \int J^{\rho}_{BRST}\, d^3x$

We have (remember T-ordering ... which we omitted indicating)

$$\langle 0|\, T\left(i[Q,\phi_1]_{\pm}\, \phi_2 \cdots \phi_n\right)|0\rangle =$$

$$= \langle T\left(s\phi_1\, \phi_2 \cdots \phi_n\right)\rangle$$

as a matter of fact this eq. is valid even for $|0\rangle \to |\psi_1\rangle$

$$\langle 0| \to \langle \psi_2|$$

● It is about now time to reconsider the issue of the independence of physical quantities from the choice of gauge fixing procedure. We have already argued for such independence for the correlators of gauge invariant operators. We were a bit superficial as concerns boundary conditions though. Let us reconsider the whole matter. Consider a matrix element for an in principle finite time evolution

$$\langle \Psi_f | e^{-iH(t_f-t_i)} | \Psi_i \rangle \equiv$$

$$\int \mathcal{D}\phi \; e^{i(S_0 + sY)} \; \Psi_f^*(\phi)\, \Psi_i(\phi)$$

I can consider $\psi_{i,f}$ physical if $\langle\psi_f|\psi_i\rangle$ does not vary when I change $\Upsilon \to \Upsilon + \Delta\Upsilon$

$$\Rightarrow \langle\psi_f| \; i[Q, \Delta\Upsilon]_+ |\psi_i\rangle = 0$$

(we neglect to write e^{iH} given $[Q, H] = 0$)

this condition should be satisfied $\forall \Delta\Upsilon$

$$\Rightarrow \quad Q|\psi_i\rangle = \langle\psi_f| Q = 0$$

- Now have a situation structurally similar to what encountered in the case of Gupta-Bleuler:

The role of the gauge redundancy is played in the quantum theory by the BRST cohomology.

- Q conserved

  $Q^2 = 0$

- "physics" sits in the <u>Kernel</u> of $\underline{Q}$

  the image of Q is trivially in the kernel.

- dynamics $\quad \underbrace{S_0}_{Q\text{- closed}} + \underbrace{[Q, Y]}_{Q\text{- exact}}$

  $[Q, S_0] \neq 0$

  $S_0 \neq [Q, \ldots]$

cohomology :

$$S_0 + [Q, Y_1] \sim S_0 + [Q, Y_2]$$

- states $|\psi\rangle + Q|q\rangle$

$$Q|\psi\rangle = 0$$
$$|\psi\rangle \neq Q|\dots\rangle$$

$$|\psi\rangle + Q|q_1\rangle \sim |\psi\rangle + Q|\psi_2\rangle$$

$$|\psi_1'\rangle = |\psi_1\rangle + Q|q_1\rangle \qquad Q|\psi_1\rangle = 0$$
$$|\psi_2'\rangle = |\psi_2\rangle + Q|q_2\rangle \qquad Q|\psi_2\rangle = 0$$

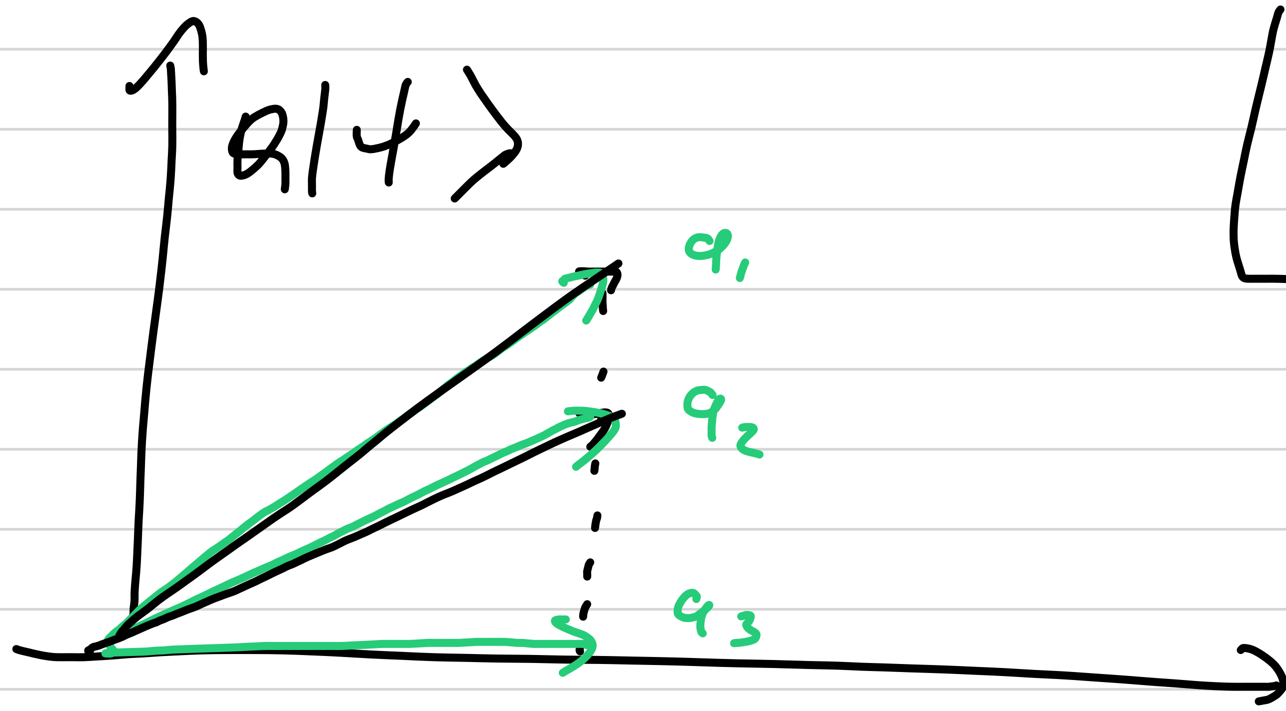
$$\langle\psi_1'|\psi_2'\rangle = \langle\psi_1|\psi_2\rangle$$

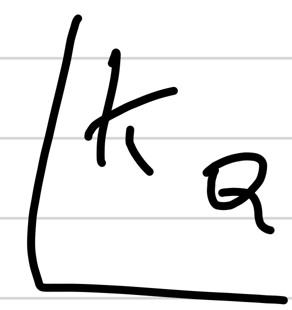
- the same argument goes through for correlators

$$\langle\psi_1'|O_1\dots O_n|\psi_2'\rangle = \langle\psi_1|O_1\dots O_n|\psi_2\rangle$$

provided $[Q, O_i]_\pm = 0$

cohomology $\equiv$ Kernel_Q modulo image of Q





$$|q_1\rangle \sim |q_2\rangle \sim |q_3\rangle$$

- Notice that since $[H, Q] = 0$ time evolution is unambiguously defined on the cohomology: Q-equivalent states evolve in the same Q-equivalent class

- **BRST closed states evolve unitarily.**
- Notice that $F_a(A, \psi)$ should be a real expression. this is because we have only one real Lie parameter for each index in order to set $F_a = 0$.

therefore the condition of reality of $\bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b$ is

$$\bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b = \omega_b^{\dagger} \frac{\delta F_a}{\delta \alpha_b} \bar{\omega}_a^{\dagger}$$

$$= - \bar{\omega}_a^{\dagger} \frac{\delta F_a}{\delta \alpha_b} \omega_b^{\dagger}$$

- the choice $\begin{bmatrix} \omega_b^{\dagger} = \omega_b \\ \bar{\omega}_a^{\dagger} = -\omega_a \end{bmatrix}$ $(*)$

  is consistent with the reality of S and with $[iQ, A_\mu] = \partial_\mu \omega \dots$ assuming $Q = Q^{\dagger}$

According to (*) S is real and the corresponding hamiltonian H hermitean therefore $\left(e^{-iHt}\right)^{+} = e^{iHt}$

$$U(t)^{+} U(t) = \mathbb{1}$$

. the time evolution operator is unitary

. As the S matrix is a combination of time evolution operators we similarly have $S^{+}S = \mathbb{1}$

• Time evolution is unitary on the cohomology. If we can prove the cohomology has a positive metric we are done. That can indeed by proven in perturbation theory. We just need to study the states of the free theory.

Canonical formalism in free case

choose $F_a = \partial_\mu A_a^{\mu}$

$$\mathcal{L}_{ghost} = -\partial_\mu \bar{\omega}_a \partial^\mu \omega_a$$

$$\left.\begin{array}{l} \pi_\omega = +\dot{\bar{\omega}} \\ \pi_{\bar{\omega}} = -\dot{\omega} \end{array}\right\} \Rightarrow \begin{array}{l} \{+\dot{\bar{\omega}}(x), \omega(y)\} =: \delta^3(x-y) \\ \{-\dot{\omega}(x), \bar{\omega}(y)\} =: \delta^3(x-y) \end{array}$$

all other $\{..,..\}$ vanish

- can check that these reproduce time evolution for $H = \pi_\omega \pi_{\bar{\omega}}$

Now following Weinberg:

$$A_\mu = \int d\Omega_k \left(e^{-ikx} a_\mu(k) + h.c.\right)$$

$$\omega = \int d\Omega_k \left(e^{-ikx} c(k) + e^{ikx} c^*(k)\right)$$

$$\bar{\omega} = \int d\Omega_k \left(e^{-ikx} b(k) + e^{ikx} b^*(k)\right)$$

$$c^*(k) = c^+(k) \qquad b^*(k) = -b^+(k)$$

$$\omega = \int d\Omega_k \left(e^{-ikx} c(k) + e^{ikx} c^+(k) \right)$$

$$\bar{\omega} = \int d\Omega_k \left(e^{-ikx} b(k) - e^{ikx} b^+(k) \right)$$

$$\{ c^+(k), b(p) \} = -(2\pi)^3 2\omega_k \, \delta^3(p-k)$$

$$\{ c(k), b^+(k) \} = -(2\pi)^3 2\omega_k \, \delta^3(p-k)$$

All other $\{ ,$

- Single particle states [photon + b + c]

$$\varepsilon^\mu a_\mu^+(k) |0\rangle \qquad \langle 0| a_\mu \varepsilon^{\mu *}$$

$$b^+(k) |0\rangle \qquad \langle 0| b(k)$$

$$c^+(k) |0\rangle \qquad \langle 0| c(k)$$

Lets us compute, modulo $(2\pi)^3 2\omega_k \delta^3(p-k)$ the metric

$$\langle Q_\mu Q_\nu^+ \rangle = -\eta_{\mu\nu}$$

$$\langle b\, b^+ \rangle = \langle c\, c^+ \rangle = 0$$

$$\langle b\, c^+ \rangle = \langle c\, b^+ \rangle = -1$$

$$g_{\alpha\beta} = \left(\begin{array}{cccc|cc} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & \multicolumn{2}{c}{0} \\ & & & 1 & & \\ \hline & & & & 0 & -1 \\ \multicolumn{4}{c|}{0} & -1 & 0 \end{array}\right)$$

diagonalizing the ghost part

$$g = \mathrm{diag}(-,+,+,+,-,+)$$

$$\mathrm{Tr}\, g = 2$$ consistent with just 2 physical states

- diagonal ghosts $(b^+ + c^+)|0\rangle$
  $(b^+ - c^+)|0\rangle$

$$\langle b+c | b+c \rangle = \langle 0 | (b+c)(b^+ + c^+) | 0 \rangle$$

$$= \langle b c^+ + c b^+ \rangle = -2$$

$$\langle b-c | b-c \rangle = -\langle b c^+ + c b^+ \rangle = 2$$

- ghost-antighost state

$$c^+(k_1)\, b^+(k_2) | 0 \rangle = c_1^+ b_2^+ | 0 \rangle$$

$$\langle 0 | b_1 \, c_2$$

$$\langle 0 | b_1 \, c_2 \, c_1^+ b_2^+ \rangle =$$

$$= - \langle b_1 \, c_1^+ \, c_2 \, b_2^+ \rangle = -1$$