Callan-Symanzik equation

- The renormalization group describe the dependence of the parameters on the subtraction scale μ. As the dependence on kinematic variables ($\equiv$ external momenta) is controlled by $d\,\mathrm{Log}\,E/\mu$ it is clear that the solution of the characterizes the asymptotic dependence on such variables.

Consider a renormalized n-poin correlator

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle \longrightarrow \langle \tilde{\phi}_1(p_1)\cdots\tilde{\phi}_n(p_n)\rangle$$
$$\equiv G(p_1\cdots p_n)$$

Notice $\phi_i = (\phi_0)_i/\sqrt{Z_i}$

$$\underset{\text{bare}}{\underbrace{(\phi_0)_i}}$$

$$\Rightarrow G(p_1 \cdots p_n) \equiv \prod_i \frac{1}{\sqrt{Z_i}} G_0(p_1 \cdots p_n)$$

$$G_0 = \prod_i \sqrt{Z_i}\, G(p_1 \cdots p_n)$$

More precisely:

$$G(p_1 \cdots p_n) = F(p_1 \cdots p_n, \lambda_a(\mu), \mu)$$

Now $\mu \frac{d}{d\mu} G_0 = 0 \quad = 0$

$$\left(\mu \frac{d}{d\mu} + \frac{1}{2} \sum_i \mu \frac{d}{d\mu} \ln Z_i\right) G = 0$$

$$\boxed{= \left(\mu \frac{\partial}{\partial\mu} + \sum_a \beta_a \frac{\partial}{\partial\lambda_a} + \sum_i \gamma_i\right) G = 0} \quad (*)$$

This is the Callan-Symanzik eq.
(in fact this is one form of CS, as written by Weinberg in 1973)

In (*) we have treated all couplings and masses together: λ_a

Specifically ex in $\lambda\phi^4 + m^2\phi^2$ or $g\phi^3 + m^2\phi^2$ it takes the form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma_m m^2\frac{\partial}{\partial m^2} + n\gamma\right] G = 0$$

or $$\Big(\underbrace{\mu\frac{d}{d\mu}}_{\text{total}} + n\gamma(\lambda(\mu))\Big) G = 0$$

- full μ-dependence purely due to wave function renormalization.
- Full solution is given using the previous result

$$Z(\mu',\mu) = e^{-\int_\mu^{\mu'}\frac{d\tilde{\mu}}{\tilde{\mu}}\gamma(\lambda(\tilde{\mu}))} \qquad \mu'\frac{d}{d\mu'}Z = -\gamma Z$$

Now $\quad \mathcal{Z}^{-n}(\mu,\tilde{\mu})\, G(p, \lambda(\mu), m^2(\mu), \mu) \equiv \hat{G}$

satisfies $\quad \mu \frac{d}{d\mu} \hat{G} = 0$

$$\hat{G}(p, \lambda(\tilde{\mu}), m^2(\tilde{\mu}), \tilde{\mu}) = \hat{G}(p, \lambda(\mu), m^2(\mu), \mu)$$

$$\Rightarrow G(p, \lambda(\mu), m^2(\mu), \mu) =$$

$$= \mathcal{Z}^{n}(\mu,\tilde{\mu})\, G(p, \lambda(\tilde{\mu}), m(\tilde{\mu}), \tilde{\mu})$$

- Consider some RG invariant quantity $\mathcal{O}$ which depends only on one energy parameter E.

$$\mathcal{O} = \mathcal{O}(E, \lambda(\mu), m(\mu), \mu)$$

$$\equiv E^{\Delta} f\left(\frac{E}{\mu}, \frac{m(\mu)}{E}, \lambda(\mu)\right)$$

$$\propto \left(\log E/\mu\right)^n$$

if $\mu \frac{d}{d\mu} \mathcal{O} = 0$ $\log E/\mu$ are mini=zed by choosing $\mu = E$

$$\Rightarrow \mathcal{O}(E) = E^{\Delta} f\left(1, \frac{m(E)}{E}, \lambda(E)\right)$$

$$= E^{\Delta} \lambda^{x}(E) \left[P_0\left(\frac{m(E)}{E}\right) + \lambda(E) P_1\left(\frac{m(E)}{E}\right) + \dots \right]$$

for $E \gg m$ we can normally further expand P_0 in a power series in m/E. So what we have is

- systematic resummation of $\lambda \ln E$ effects
- systematic expansion in $\frac{m}{E}$, including log corrections.

- N-point correlators and scattering amplitudes depend on several kinematic variables: in general they are not characterized by a single scale

$\Rightarrow$ RG less useful.

However two notable exceptions that fall in the same class as $\mathcal{O}(E)$ above

1) inclusive cross-section (if finite)

2) Correlators in region where

$$P_i \cdot P_j \sim Q^2 \qquad \forall\ i j$$

Consider the case of n-point correlator in ϕ^4

$$p_i p_j = I_{ij} Q^2$$

$$\hat{G}(I_{ij} Q^2, \lambda(\mu), m^2(\mu), \mu) =$$

$$= \hat{G}(I_{ij} Q^2, \lambda(Q), m^2(Q), Q)$$

$$= \frac{\lambda(Q)^{\frac{n}{2}-1}}{Q^{3n-4}} f\left(I_{ij}, \lambda(Q), \frac{m^2(Q)}{Q^2}\right)$$

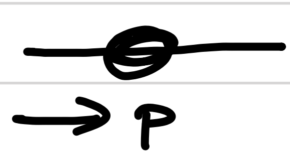
$$= \frac{\lambda(Q)^{\frac{n}{2}-1}}{Q^{3n-4}} \left[f_0\left(I_{ij}, \frac{m^2}{Q^2}\right) + \lambda(Q) f_1 \ldots \right]$$

tree-level diagrams , 1-loop diagrams

and thus

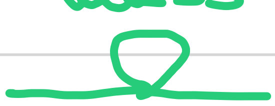
$$G(I_i;\, Q^2,\, \lambda(\mu),\, m^2(\mu),\, \mu) = Z^n(\mu, Q)\, \hat{G}(I_i;\, Q^2,\, \dots)$$

$$e^{-n\int_Q^{\mu} \gamma(\lambda(\tilde{\mu}))\frac{d\tilde{\mu}}{\tilde{\mu}}} \qquad \frac{\lambda(Q)^{\frac{n}{2}-1}}{Q^{3n-4}} \left[f_0\left(I_i;\, \frac{m^2}{Q^2}\right) + \dots \right.$$

In particular for the two point function  $p^2 \equiv Q^2$ only invariant

$$G_2(Q^2,\, \lambda(\mu),\, m^2(\mu),\, \mu) =$$

$$= \quad e^{-2\int_Q^{\mu} \gamma(\lambda)\frac{d\tilde{\mu}}{\tilde{\mu}}} \cdot \frac{1}{Q^2 - m^2(Q^2)} \left[1 + O(\lambda) \right]$$

As explained in class, this is actually not true: there appear $\log Q/m$ term in the mass part. These are due to tadpole diagrams  which do not depend on the external momentum. The result would however work in theories like QED.

Fixed points

a point $\lambda = \lambda_*$ where $\beta(\lambda_*) = 0$ is a fixed point of the RG evolution. If either $m = 0$ or $\lambda \to \lambda_*$ is reached as $\mu \to \infty$ where we can neglect m, then the Callan-Symanzik eq. takes a particularly simple scaling form corresponding to the existence of scale invariance.

Notice that in this one

$$\mathcal{Z}(\mu', \mu) = e^{-\int_{\mu}^{\mu'} \gamma(\lambda) \frac{d\tilde{\mu}}{\tilde{\mu}}}$$

$$= e^{-\int \gamma(\lambda_*) \, d\ln\tilde{\mu}} = \left(\frac{\mu}{\mu'}\right)^{\gamma_*}$$

from whence

$$G_2(Q^2) \sim \left(\frac{Q}{\mu}\right)^{2\gamma} \frac{1}{Q^2}$$

$$= Q^{-2+2\gamma}$$

$$\Rightarrow \langle \phi(x)\phi(0)\rangle = \frac{1}{x^{2+2\gamma}}$$

corresponding to $[\phi] = 1+\gamma$

anomalous dimension

Renormalization of composite operators

- local operators should be properly viewed as operator valued distributions: $\phi_f = \int \phi(x) f(x)\, dx$
  f smooth

indeed $[\phi, \dot{\phi}] = i\delta(x-y)$

$\langle \phi \phi \rangle = \frac{1}{(x-y)^2}$

etc.

- Because of this, expressions like $\phi(x)^2$ are ill defined. The problem is clearly a UV one and it can be solved by proper renormalization.
- In general we can treat the insertion of composite operator $\mathcal{O}(x)$ by the source method :

$$\int \mathcal{D}\phi \, e^{-S} \longrightarrow \int \mathcal{D}\phi \, e^{-S + \sum_i c_i(x) \mathcal{O}_i(x)}$$

sources

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$$

$$= \frac{\delta}{\delta c_1(x_1)} \cdots \frac{\delta}{\delta c_n(x_n)} Z$$

For the purpose of this discussion we content ourself to the case where only one of these n insertion is composite. In that case we can work at linear order in the c_i associated to composites.

<u>Ex</u> $\mathcal{O}(x) = \phi(x)^2$

$$\mathcal{L} \to \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi + c\phi^2$$

sources

Viewed in this term the problem is just one of renormalization: We need to subtract divergences in $Z[c]$. We can do so by treating the c_i has all other couplings ... with the added peculiarity that they are position dependent. Focussing on at most a single insertion of the composite we can study the renormalization of

$$\Delta\mathcal{L} = \sum_i c_i \mathcal{O}_i$$

working at linear order in the c_i $\Rightarrow$

imagining O_i are written in terms of renormalized elementary fields

Ex $\phi^2, \phi^3, \phi^4, \ldots \partial_\mu\phi\partial^\mu\phi, \partial_\mu\phi\partial_\nu\phi, \ldots$

$$\Delta\mathcal{L} \to \sum_i c_i Z^{-1}_{ij} O^0_j$$

$\hookrightarrow$ bare

$$= \sum_j c^0_j O^0_j$$

$$O^R_i(\mu) \equiv Z^{-1}_{ij} O^0_j$$

$$Z_{ji} O^R_i(\mu) = O^0_j$$

$$\mu\frac{d}{d\mu} O^R_i = \underbrace{\left(\mu\frac{d}{d\mu} Z^{-1}_{ij}\right) Z_{jk}}\, O^R_k$$

$$= -\gamma_{ik} O^R_k$$

The $O^R_i(\mu)$, as usual, should be

interpreted as a basis of operators optimized to perform pert. theory at $E \sim \mu$.

• **Notice** by power counting if $\mathcal{L}$ is renormalizable ($[\lambda_i] \geq 0$)

then given $[c_i] = (4 - \Delta_i)$

$Z_{ij}^{-1} \neq 0$ only for $c_j > c_i$

Ex ϕ^2 in $\lambda\phi^4$

$$\mathcal{O}_i = \mathbb{1}, \ \phi^2/2, \ \phi^4/4!, \ \Box\phi^2, \ (\partial\phi)^2, \ \ldots$$

$$\begin{array}{ccccc} c_0 & c_2 & c_4, & c_4' & c_4'' \\ \downarrow & \downarrow & \downarrow & & \\ [c_0]=4 & 2 & 0 & & \end{array}$$

$$\left(c_0 + c_2 m^2 P_{02} + c_4 m^4 P_{04}\right) \cdot \mathbb{1}$$

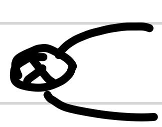
$$\left(c_2 P_{22} + c_4 m^2 P_{24}\right) \phi_0^2/2 + \ldots$$

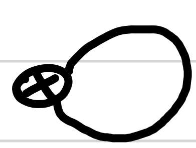
From whence

$$[\phi^2] = \frac{\delta}{\delta c_2} = m^2 P_{02} + P_{22}\,\phi_0^2$$

$$[\phi^4] = \frac{\delta}{\delta c_4} = m^4 P_{04} + m^2 P_{24}\,\phi_0^2 + P_{44}\,\phi_0^4 + \dots$$

<u>Ex</u> $[\phi^2]$ at 1-loop

c_2 

• I_{c2}  $= \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2}$

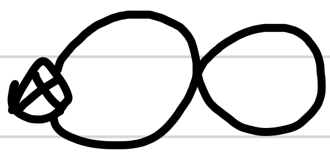
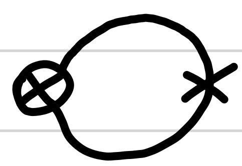
$$\frac{1}{2}\left(\frac{1}{4\pi}\right)^{d/2} \frac{1}{\Gamma(d/2)} \int \frac{t^{\frac{d}{2}-1}\,dt}{t+m^2}$$

$$\frac{1}{2}\left(\frac{1}{4\pi}\right)^{d/2} \frac{\Gamma(1-d/2)}{\Gamma(1)} (m^2)^{d/2-1}$$

$$\frac{1}{2}\left(\frac{1}{4\pi}\right)^{d/2} \frac{1}{1-d/2}\,\Gamma(2-d/2)\, m^2 (m^2)^{\frac{d}{2}-2}$$

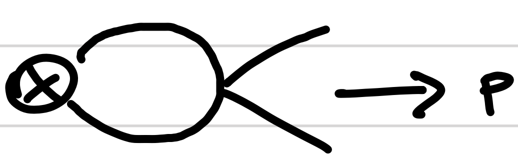
$$\text{C.T.}\quad (-)\left(\frac{1}{16\pi^2}\right)(-1)\,\frac{m^2}{4-d} = \frac{1}{16\pi^2}\,\frac{m^2}{\varepsilon}$$

At the next order we have

 $+$  $\rightsquigarrow$ mass C.T.

$\Rightarrow$ Computation left as an exercise

P_{22} instead



$$-\frac{\lambda\mu^{\varepsilon}}{2}\int\frac{d^d e}{(2\pi)^d}\,\frac{1}{e^2(e+k)^2}$$

$$= -\frac{\lambda\mu^{\varepsilon}}{2}\,\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} = -\frac{\lambda}{16\pi^2}\,\frac{1}{\epsilon}$$

$$\text{C.T.} = +\,\lambda/16\pi^2\ \frac{1}{\epsilon}$$

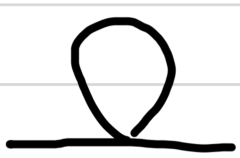
$$P_{22} = 1 + \frac{\lambda}{16\pi^2}\,\frac{1}{\epsilon}$$

$$c_2 \left(1+\frac{\lambda}{16\pi^2}\frac{1}{\varepsilon}+\cdots\right)\phi_0^2$$

$$[\phi^2] = \left(1+\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}+\cdots\right)\phi_0^2$$

$$\mu\frac{d}{d\mu}[\phi^2] = -\frac{\lambda}{16\pi^2}[\phi^2]$$

This is related to γ_m



$$= -\frac{\lambda}{2}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2}$$

$$= (-)(-)\frac{\lambda}{16\pi^2}\frac{m^2}{\epsilon}$$

$$\Delta\mathcal{L} = \frac{\lambda}{16\pi^2}\frac{m^2}{\epsilon}\phi^2$$

$$m_0^2 = m^2\left(1+\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}\right)$$

$$\mu\frac{d}{d\mu}m^2 = \frac{\lambda}{16\pi^2}m^2$$

$$m_0^2 \frac{\Phi_0^2}{2} = \frac{m^2}{2} Z_m \phi_0^2 = \frac{m^2}{2} [\phi^2]$$

$$\Rightarrow \begin{cases} [\phi^2] = Z_m \phi_0^2 \\ m^2 = Z_m^{-1} m_0^2 \end{cases}$$