Project I , Part II

Källen-Lehmann decomposition, CPT, causality and statistics.

Based on the results of the first part of the project, we now want to study the two point functions

$$\begin{aligned} G^{\pm}_{A\bar{B}}(x-y) &\equiv \langle 0|\theta(x^0-y^0)\mathcal{O}_A(x)\mathcal{O}^{\dagger}_{\bar{B}}(y) \pm \theta(y^0-x^0)\mathcal{O}^{\dagger}_{\bar{B}}(y)\mathcal{O}_A(x)|0\rangle \quad (1) \\ C^{\pm}_{A\bar{B}}(x-y) &\equiv \langle 0|\mathcal{O}_A(x)\mathcal{O}^{\dagger}_{\bar{B}}(y) \pm \mathcal{O}^{\dagger}_{\bar{B}}(y)\mathcal{O}_A(x)|0\rangle\,, \quad (2) \end{aligned}$$

show how they encapsulate the spectral content of the theory and derive the connection between spin and statistics. For that purpose you should follow these steps.

1. Assuming, for simplicity, that $\mathcal{O}$ can only interpolate for massive states[1], show

$$\langle 0|\mathcal{O}_A(x)\mathcal{O}^{\dagger}_{\bar{B}}(y)|0\rangle = \sum\!\!\!\!\!\!\int_{r,\vec{p},s_3} |Z_{\mathcal{O},r}|\,\Pi^{(s_r)}_{A\bar{B}}(p_r)e^{-ip_r(x-y)} \quad (3)$$

2. Argue, using Lorentz symmetry, that you can write

$$\Pi^{(s_r)}_{A\bar{B}}(p_r)e^{-ip_r(x-y)} = \hat{\Pi}^{(s_r)}_{A\bar{B}}(i\partial_x, M_r^2)e^{-ip_r(x-y)} \quad (4)$$

   where $\hat{\Pi}^{(s_r)}_{A\bar{B}}$ is a finite polynomial in ∂_x, while $M_r^2 \equiv p_r^2$. The differential operator $\hat{\Pi}^{(s_r)}_{A\bar{B}}$ is thus *local*.

3. Define the positive spectral density at spin s

$$\rho^{(s)}(\mu^2) = \int dr\,|Z_{\mathcal{O},r}|\,\delta(\mu^2 - M_r^2)\delta_{ss_r} \quad (5)$$

4. Show

$$\begin{aligned} \langle 0|\mathcal{O}_A(x)\mathcal{O}^{\dagger}_{\bar{B}}(y)|0\rangle &= \int d\mu^2 \sum_s \rho^{(s)}(\mu^2)\hat{\Pi}^{(s)}_{A\bar{B}}(i\partial_x,\mu^2)D^{(\mu)}(x-y) \quad (6) \\ &\equiv \int d\mu^2 \sum_s \rho^{(s)}(\mu^2)\hat{D}^{(s,\mu)}_{A\bar{B}}(x-y) \quad (7) \end{aligned}$$

   where

$$D^{(\mu)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle \quad (8)$$

[1] This way we avoid the slight complication of dealing with massless representations of Poincarè.

and ϕ is a free Klein-Gordon field of mass μ. The content of the above equation should be appreciated. It expresses the (non-time ordered) two point function of an arbitrary operator $\mathcal{O}$ with its complex conjugate $\mathcal{O}^\dagger$ in terms of a superposition with positive weight of two point functions $D^{(s,\mu)}_{A\bar{B}}$ with definite spin s and mass μ. Each (s,μ) pair defines an irreducible representation of the Poincarè group. This is thus another instance of partial wave decomposition. Notice also that $D^{(s,\mu)}_{A\bar{B}}$ is simply obtained by acting with a differential operator $\hat{\Pi}^{(s)}_{A\bar{B}}(i\partial_x,\mu^2)$ on the two point function $D^{(\mu)}$ of a real scalar field: all the partial waves can be *algebraically* constructed starting from the lowest one by applying polynomial differential operators.

5. Using point 3 of Part I show

$$\begin{aligned}\langle 0|\mathcal{O}^\dagger_{\bar{B}}(y)\mathcal{O}_A(x)|0\rangle &= \int d\mu^2 \sum_s \rho^{(s)}(\mu^2)\hat{\Pi}^{(s)}_{A\bar{B}}(-i\partial_x,\mu^2)D^{(\mu)}(-x+y) \quad (9)\\ &\equiv \int d\mu^2 \sum_s \rho^{(s)}(\mu^2)\hat{D}^{(s,\mu)}_{A\bar{B}}(-x+y) \quad (10)\end{aligned}$$

where you should notice the $(-)$ signs in the coordinate dependence and derivatives.

6. Using point 5 of Part I show

$$C^\pm_{A\bar{B}}(x-y) = \int d\mu^2 \sum_s \rho^{(s)}(\mu^2)\hat{\Pi}^{(s)}_{A\bar{B}}(i\partial_x,\mu^2)\left(D^{(\mu)}(x-y) \pm (-1)^{2j}D^{(\mu)}(-x+y)\right) \quad (11)$$

where $2j \equiv j_1 + j_2$ defines the "spin" of $\mathcal{O}$, so that $(-)^{2j}$ equals 1 and -1 for operators of respectively integer and half-integer spin. [2] You can now recall the commutator of the free scalar

$$\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle = D^{(\mu)}(x-y) - D^{(\mu)}(-x+y) \quad (12)$$

which, you have seen, vanishes for $(x-y)^2 < 0$, i.e. outside the light cone. That property establishes causality for the free scalar field. You can now see that, for integer j, it is $C^-_{A\bar{B}}$ that vanishes outside the light cone, while for half-integer j this property is satisfied by $C^+_{A\bar{B}}$. In particular, at fixed time, the anticommutator of half-integer spin operators vanishes. This establishes from prime principles the well know fundamental relation between spin and statistics.

7. Define now the Feynman propagator

$$G_{A\bar{B}}(x-y) \equiv \langle 0|\theta(x^0-y^0)\mathcal{O}_A(x)\mathcal{O}^\dagger_{\bar{B}}(y) + (-1)^{2j}\theta(y^0-x^0)\mathcal{O}^\dagger_{\bar{B}}(y)\mathcal{O}_A(x)|0\rangle \quad (13)$$

[2] You should not make confusion for our calling both s and j the spin: the first is truly the spin labelling an irreducible representation of Poincarè, the second is the maximal spin that can be created by the action of $\mathcal{O}$ on the vacuum.

Using eqs. (6,9) above end the result of point 5 in Part I, now show

$$\begin{aligned} G_{A\bar{B}}(x-y) &= \int d\mu \sum_s \Big\{ \rho^{(s)}(\mu^2)\hat{\Pi}^{(s,\mu)}_{A\bar{B}}(i\partial_x) G^{(\mu)}(x-y) + \\ &+ [\theta_+, \hat{\Pi}^{(s,\mu)}_{A\bar{B}}(i\partial_x)] D^{(\mu)}(x-y) + [\theta_-, \hat{\Pi}^{(s,\mu)}_{A\bar{B}}(i\partial_x)] D^{(\mu)}(-x+y) \Big\} \end{aligned} \tag{14}$$

where $\theta_\pm \equiv \theta(\pm(t_x - t_y))$ and $G^{(\mu)}$ is the time ordered propagator of a Klein-Gordon field of mass μ. By using the locality of $\Pi^{(s,\mu)}_{A\bar{B}}(i\partial_x)$ ($\equiv$ polynomial in ∂), $D^{(\mu)}(x) = D^{(\mu)}(-x)$ for spacelike x, and that derivatives of $\theta(t)$ are derivatives of $\delta(t)$ you can argue that the second line of the above equation is a distribution localized at $x - y = 0$. In other words it is given by spacetime derivatives of $\delta^{(4)}(x-y)$. It is what in QFT jargon is called a contact term. In particular it can be eliminated by modifiying the definition of the time-ordered product. In fact these contact terms are often not Lorentz covariant and inconvenient to carry out perturbation theory. The definition of time ordering that is provided by the path integral has indeed these terms removed. Dropping the contact terms we can thus write

$$\langle 0| T\left(\mathcal{O}_A(x)\mathcal{O}^\dagger_{\bar{B}}(y)\right)|0\rangle = \int d\mu \sum_s \rho^{(s)}(\mu^2)\hat{\Pi}^{(s,\mu)}_{A\bar{B}}(i\partial_x) G^{(\mu)}(x-y) \tag{15}$$

This equation should be compared to eqs. (6,9,11): all the different 2-point functions of the operator $\mathcal{O}$ are written in terms of the corresponding 2-point functions of the Klein-Gornon field, by applying the same differential operator $\hat{\Pi}^{(s,\mu)}_{A\bar{B}}$ and by summing over (s,μ) with a positive definite spectral density $\rho^{(s)}(\mu^2)$.

It is interesting to express the above result in momentum space

$$\begin{aligned} \int d^4x e^{ip(x-y)} \langle 0| T\left(\mathcal{O}_A(x)\mathcal{O}^\dagger_{\bar{B}}(y)\right)|0\rangle &\equiv \langle 0| T\left(\mathcal{O}_A(p)\mathcal{O}^\dagger_{\bar{B}}(-p)\right)|0\rangle \\ &= \int d\mu \sum_s \rho^{(s)}(\mu^2) \frac{\hat{\Pi}^{(s,\mu)}_{A\bar{B}}(p)}{p^2 - \mu^2 + i\epsilon} \end{aligned} \tag{16}$$

We can now try and say something about $\rho^{(s)}(\mu^2)$. It is rather clear that the presence in the spectrum of a single particle state with mass M and spin s will correspond to a delta function contribution $\delta(\mu^2 - M^2)$ to $\rho^{(s)}(\mu^2)$. By the above equation this will imply a pole $p^2 = M^2$ in the time-ordered 2-point function, in agreement with what we found when discussing LSZ reduction. In general $\rho^{(s)}(\mu^2)$ will have the form

$$\rho^{(s)}(\mu^2) = \sum_i C^{(s)}_i \delta(\mu^2 - M^{(s)2}_i) + \rho^{(s)}_{cont}(\mu^2) \tag{17}$$

with $C_i^{(s)} > 0$, where $M_i^{(s)}$ represent the masses of the asymptotic states of spin s while $\rho_{cont}^{(s)}(\mu^2)$ represents the contribution of multiparticle states, which has obviously a continuous distribution in μ^2 starting from a lowest threshold value.

Two final questions

1. Show that for the case of $s = 1$ and $\mathcal{O}_A \to V_\nu = (1/2, 1/2)$ the projector $\hat{\Pi}_{A\bar{B}}^{(s,\mu)}$ becomes

$$\hat{\Pi}_{\lambda\nu}^{(1,\mu)}(p) = -\eta_{\lambda\nu} + \frac{p_\lambda p_\nu}{\mu^2} \tag{18}$$

2. If an asymptotic state contributes a $\delta(\mu^2 - M^2)$ to the spectral density, how do you expect the latter to change if the state developes a long but finite lifetime when turning on a new weak interaction? Of course you will no longer have a delta funtion density...but things should not change too much. So?