Project I

Källen-Lehmann decomposition, CPT, causality and statistics

This is a guided project through the derivation of the Källen-Lehmann spectral formula for operators transforming as arbitrary Lorentz tensors. In the course of the derivation a connection between CPT, causality and statistics will emerge. The project consists of two steps. The first step here described consists of a set of preliminary results.

Notation

In what follows it will be convenient to choose a basis of states described by a collective label r that runs over irreducible representations of the Poincarè group: r includes in particular the mass m_r and angular momentum s_r of the state, but also includes other quantum numbers such as the number of particles in the state, their charges, etc. The 3-momentum $\vec{p}$ and angular momentum s_3 along $\hat{z}$, shall be kept as independent quantum numbers. To make clear what we mean, the completeness relation will take the form

$$\mathbb{I} = \int dr \int d\Omega_r \sum_{s_3=-s_r}^{s_r} |r, \vec{p}, s_r\rangle \langle r, \vec{p}, s_3| \tag{1}$$

where

$$d\Omega_r = \frac{d^3p}{(2\pi)^3 2p_r^0}, \qquad p_r^0 = \sqrt{m_r^2 + (\vec{p})^2}, \qquad p_r^\mu = (p_r^0, \vec{p}). \tag{2}$$

Of course, for states of zero angular momentum ($s_r = 0$) the sum over s_3 reduces to just one term corresponding to $s_3 = 0$. We stress that the Poincarè group only acts on $\vec{p}$ and s_3, while the label r is Poincarè invariant: that is the main advantage of the above apparently complicated notation. Eq. (1) corresponds to the normalization condition

$$\langle r, \vec{p}, s_3 | r', \vec{p}', s_3' \rangle = \delta(r - r')\delta_{s_3\, s_3'}(2\pi)^3 2p_r^0\, \delta^{(3)}(\vec{p} - \vec{p}') \tag{3}$$

In what follows we shall often condense eq. (1) into

$$\mathbb{I} = \sum\!\!\!\!\!\!\int_{r,\vec{p},s_3} |r, \vec{p}, s_r\rangle \langle r, \vec{p}, s_3| \tag{4}$$

Step 1

Consider the action of the CPT transformation performed by the anti-unitary operator Θ

$$\Theta |r, \vec{p}, s_3\rangle = \eta_r (-1)^{s_r - s_3} |\bar{r}, \vec{p}, -s_3\rangle\,, \tag{5}$$

where $|\eta_r| = 1$ and $\bar{r}$ corresponds to an identical state but with all the charges reversed. The projector on the subspace of states with total 4-momentum $P^\mu = k^\mu$ (here simply indicated by k) and total angular momentum s is written as

$$\mathbb{P}_{k,s} = \sum\!\!\!\!\!\!\int_{r,\vec{p},s_3} \delta_{s\, s_r} \delta^{(4)}(p_r - k)\, |r, \vec{p}, s_3\rangle \langle r, \vec{p}, s_3|\,. \tag{6}$$

Of course, the sum over r includes states and their CPT conjugates. Consider now a local operator $\mathcal{O}_A(x)$ transforming in some irreducible (j_-, j_+) representation of the Lorentz group. The index A then collectively indicates a set $\alpha_1, ..., \alpha_{2j_-}; \dot\beta_1, ..., \dot\beta_{2j_+}$ of left and right spinorial indices. Under CPT the operator satisfies

$$\Theta\mathcal{O}_A(x)\Theta^\dagger = \eta_\mathcal{O}\mathcal{O}^\dagger_{\bar A}(-x), \tag{7}$$

with $|\eta_\mathcal{O}| = 1$ and where we indicate the index of the hermitian conjugate operator by $\bar A$, to account for the fact that it transforms like the conjugate representation (j_+, j_-). Consider furthermore the matrix element

$$\langle 0|\,\mathcal{O}_A(0)\,|r, \vec p, s_3\rangle = \sqrt{Z_{\mathcal{O},r}}\,\Psi^{(s_r)}_{A,s_3}(p_r) \tag{8}$$

where $\Psi^{(s_r)}_{A,s_3}(p_r)$ is a suitably normalized relativistic wave function. (According to the discussion in class, the normalization can be fixed by going to the rest frame $p^\mu \to \bar p^\mu = (m, 0, 0, 0)$ and choosing $\Psi^{(s)}_{A,s_3}(\bar p)$ such that $\Pi^{(s)}_{A\bar B}(\bar p) = \sum_{s_3}\Psi^{(s)}_{A,s_3}(\bar p)\Psi^{(s)}_{\bar B,s_3}(\bar p)^*$ is the projector on the subspace of spin s. Notice that, since $\mathcal{O}_A \sim (j_-, j_+)$, there is at most one subspace of any given spin, and indeed only for $|j_+ - j_-| \le s \le j_+ + j_-$). For any p_r define then

$$\Pi^{(s)}_{A\bar B}(\bar p_r) = \sum_{s_3}\Psi^{(s_r)}_{A,s_3}(p_r)\Psi^{(s_r)}_{\bar B,s_3}(p_r)^*. \tag{9}$$

We want to derive the spectral decomposition of the two-point function $\langle 0|\,\mathcal{O}_A(x)\mathcal{O}^\dagger_{\bar B}(y)\,|0\rangle$. For this purpose, you should first prove the following preliminary results

1. $\mathbb{P}_{k,s}\mathbb{P}_{k',s'} = \delta^{(4)}(k-k')\delta_{s,s'}\mathbb{P}_{k,s}$, (as befits a projector)

2. $\langle 0|\,\mathcal{O}_A(0)\mathbb{P}_{k,s}\mathcal{O}^\dagger_{\bar B}(0)\,|0\rangle = \langle 0|\,\mathcal{O}^\dagger_{\bar B}(0)\mathbb{P}_{k,s}\mathcal{O}_A(0)\,|0\rangle$,

3. $\Pi^{(s)}_{A\bar B}(p)$ transforms covariantly, namely

$$\Pi^{(s)}_{A\bar B}(\Lambda p) = D_{AA'}(\Lambda)D_{\bar B\bar B'}(\Lambda)^*\Pi^{(s)}_{A'\bar B'}(p) \tag{10}$$

4. $\Pi^{(s)}_{A\bar B}(-p) = (-1)^{2j_- + 2j_+}\Pi^{(s)}_{A\bar B}(p)$.
   Hint: by point 4, $\Pi^{(s)}_{A\bar B}(p)$ is a covariant tensor transforming as the representation $\mathcal{O}_A\mathcal{O}^\dagger_{\bar B} \sim (j_-, j_+)\otimes(j_+, j_-)$. In spinorial notation we thus have the equivalence

$$\Pi^{(s)}_{A\bar B} \sim \Pi^{(s)}_{\alpha_1,\ldots,\alpha_{2j_-+2j_+};\dot\beta_1,\ldots,\dot\beta_{2j_++2j_-}}. \tag{11}$$

   Again, according to point 3, this expression should be covariantly built out of covariant objects, that is out of $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$ and $p_{\alpha\dot\beta} = p_\mu\sigma^\mu_{\alpha\dot\beta}$. From this, you should be able to prove 4.

5. In the definition of $\mathbb{P}_{k,s}$, we have been a bit sloppy, as we have not specified whether the states in the sum are "in" or "out". For single particle states there is no distinction, but here we are summing over all states. Can you argue that indeed $\mathbb{P}_{k,s}$ can be equally well written using "in" and using "out" states?