AQFT - Exercise Set 1

Exercise 1: $SL(2,\mathbb{C})$ representations.

- Recast the Lorentz algebra spanned by the $J_{\mu\nu} = -J_{\nu\mu}$ generators in terms of their spinorial form where they carry only dotted and/or undotted indices.
- Recast the Pauli-Lubanski pseudo-vector W^μ in its spinorial form $W_{\alpha\dot{\alpha}}$, writing it explicitly in terms of the spinorial forms of P_μ and $J_{\mu\nu}$.
- Show that a symmetric tensor corresponds to the representation $(1,1)\oplus(0,0)$.

Exercise 2: Partial wave expansion for scalar particles.

Consider a system of two distinguishable scalar particles. A complete set of states for this system is the one given by the tensor product of two single-particle states $|p1,p2\rangle = |p_1\rangle \otimes |p_2\rangle$. The normalization of the single-particle states is assumed to be

$$\langle p'|p\rangle = (2\pi)^3\, 2E_{\mathbf{p}}\, \delta^{(3)}(\mathbf{p'} - \mathbf{p}). \tag{1}$$

A different choice of basis can be defined using the method of induced representations as it is usually done for single particle states. To this purpose, let us choose a convenient reference frame. For the two-particle system, the preferred choice is the center of mass frame where the total momentum is $P^\mu = p_1^\mu + p_2^\mu = (\sqrt{s},\mathbf{0})$. A basis of states in this frame decomposes into irreducible representations of the little group of P^μ: $SO(3)$. We can then define a basis of states with vanishing 3-momentum by assigning the angular momentum J and its projection on the third axis m: $|P,J,m\rangle$. We are not explicitly taking into account the existence of internal quantum numbers as they don't modify the construction.

- Using rotational covariance, argue that

$$\langle p_1,p_2|P,j,m\rangle = \mathcal{N}\delta^{(4)}(P - p_1 - p_2)Y^J_m(\hat{\mathbf{p}}) \tag{2}$$

  where Y^J_m is the spherical harmonic with quantum numbers (J,m), $\hat{\mathbf{p}}$ is the direction of the 3-momentum of the first particle and $\mathcal{N}$ is a rotation-invariant prefactor.

- Asking for the normalization

$$\langle P',J',m'|P,J,m\rangle = (2\pi)^4\delta^{(4)}(P-P')\delta_{J,J'}\delta_{m,m'} \tag{3}$$

  find the prefactor $\mathcal{N}$ (up to an irrelevant phase).

- Write down explicitly the map between the two bases of states, that is find the coefficients $C_{J,m}$ of the decomposition

$$|p_1,p_2\rangle = \sum_{J=0}^{\infty}\sum_{m=-J}^{J} C_{J,m}\,|P,J,m\rangle \tag{4}$$

The $\{|P,J,m\rangle\}$ basis of states is convenient as it allows us to exploit the Lorentz invariance of the S-matrix and write down a useful representation for $2\to2$ scattering amplitude. Indeed, note that the previous arguments apply equally well to *in* and *out* states.

- Using the previous decomposition (4), prove that the $2 \to 2$ scattering amplitude can be written as

$$\langle out; p_3, p_4 | p_1, p_2; in \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) i\mathcal{M}(s, \cos\theta) \tag{5}$$

with

$$\mathcal{M}(s, \cos\theta) = \sum_{J=0}^{\infty} \mathcal{A}_J(s) P_J(\cos\theta) \tag{6}$$

where θ is the angle which defines the direction of $\mathbf{p}_3$ relative to the one of $\mathbf{p}_1$ and P_J are the Legendre polynomials.

(*For the solution of this point you may want to use the completeness relations and the sum rules for the spherical harmonics that you can find on Wikipedia: en.wikipedia.org/wiki/Spherical_harmonics.*)