HW12

The Nobel prize worth computation

In this homework you are supposed to compute the beta-function in a non-Abelian Yang-Mills theory. Consider a theory of N_f fermions in the fundamental representation of $SU(N_c)$ gauge group with the following Lagrangian[1] written in terms of bare fields and couplings (hence, the subscript 0)

$$\mathcal{L} = -\frac{1}{4}\sum_{a=1}^{N_c^2-1} F^a_{0\mu\nu}F_0^{a\mu\nu} + i\sum_{i=1}^{N_f}\bar{\psi}_{0i}(\not{D}_0 - m)\psi_{0i} + \sum_{a=1}^{N_c^2-1}\bar{\omega}_0^a\partial_\mu(D_0^\mu\omega_0)^a + \frac{1}{2\xi_0}\partial_\mu A_0^{a\mu}\partial_\nu A_0^{a\nu}, \quad (1)$$

where covariant derivatives are defined as

$$D_{0\mu}\psi_{0i} = \partial_{0\mu}\psi_{0i} - ig_0 A^a_{0\mu}(t^a\psi_0)_i, \quad (2)$$
$$D_{0\mu}\omega_0^a = \partial_\mu\omega_0^a + g_0 f^{abc}A^b_{0\mu}\omega_0^c, \quad (3)$$

with $SU(N_c)$ generators satisfying

$$[t^a, t^b] = if^{abc}t^c. \quad (4)$$

Feynman rules for this theory are listed below:

- Fermions

$$i \;\text{———}\; j = \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}\delta_{ij}, \quad (5)$$

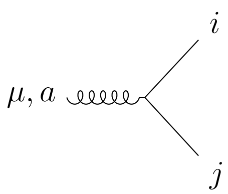
- Gluons[2]

$$\mu, a \;\sim\sim\sim\; \nu, b = \frac{-i\delta^{ab}}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right), \quad (6)$$

- Ghosts

$$a \cdots\cdots b = \frac{-i\delta^{ab}}{p^2 + i\varepsilon}, \quad (7)$$

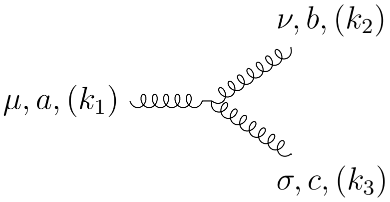
- Fermion-gluon iteraction



$$= ig(t^a)_{ij}\gamma^\mu \quad (8)$$

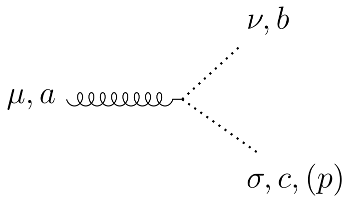
[1]The sum over flavor and color indices is written explicitly only once, in what follows it is omitted.
[2]Computations can be somewhat simplified if the $\xi = 1$ gauge is chosen.

- 3-gluon interaction (outgoing momenta k_1, k_2, k_3)



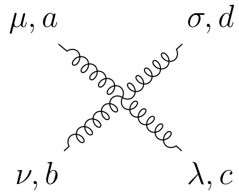
$$= -g f^{abc} \left[(k_1 - k_2)_\sigma \eta_{\mu\nu} + (k_2 - k_3)_\mu \eta_{\nu\sigma} + (k_3 - k_1)_\nu \eta_{\mu\sigma} \right] \tag{9}$$

- Ghost-gluon interaction (outgoing momentum p)



$$= g f^{abc} p_\mu \tag{10}$$

- 4-gluon interaction



$$\begin{aligned} & -ig^2 \Big[\quad f^{abe} f^{cde} \left(\eta_{\mu\lambda} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\lambda} \right) \\ = & \qquad\quad f^{ace} f^{bde} \left(\eta_{\mu\nu} \eta_{\lambda\sigma} - \eta_{\mu\sigma} \eta_{\nu\lambda} \right) \\ & \qquad\quad f^{ade} f^{bce} \left(\eta_{\mu\lambda} \eta_{\nu\sigma} - \eta_{\mu\nu} \eta_{\lambda\sigma} \right) \quad \Big]. \end{aligned}$$

Based on the results we shall discuss in class during the next lecture (which you find in section 12.1 of Collins), renormalization is carried out by rescaling couplings

$$g_0^2 = Z_g g^2, \quad \xi_0 = Z_\xi \xi, \tag{11}$$

and fields

$$A^a_{0\mu} = \sqrt{Z}_3 A^a_\mu, \quad \omega^a_0 = \sqrt{\tilde{Z}}_2 \omega^a, \quad \bar{\omega}^a_0 = \sqrt{\tilde{Z}}_2 \bar{\omega}^a, \quad \psi_0 = \sqrt{Z}_2 \psi, \tag{12}$$

where Z_2 will be different for different representations of the gauge group. In terms of renormalized fields with explicitly written counter terms the Lagrangian becomes

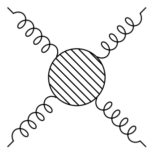
$$\begin{aligned} \mathcal{L} \;=\; & -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\xi} \partial_\mu A^{a\mu} \partial_\nu A^{a\nu} - \partial_\mu \bar{\omega}^a \partial_\mu \omega^a + i \bar{\psi}_i \not{\partial} \psi_i \qquad (13) \\ & -g f^{abc} \partial_\mu A^a_\nu A^b_\mu A^c_\nu - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu - g f^{abc} \partial_\mu \bar{\omega}^a A^b_\mu \omega^c + g A^a_\mu \bar{\psi} t^a \gamma^\mu \psi_i \\ & -\frac{\delta_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{\tilde{\delta}_3}{2\xi} \partial_\mu A^{a\mu} \partial_\nu A^{a\nu} - \tilde{\delta}_2 \partial_\mu \bar{\omega}^a \partial_\mu \omega^a + i \delta_2 \bar{\psi}_i \not{\partial} \psi_i \\ & -g \delta_{3A} f^{abc} \partial_\mu A^a_\nu A^b_\mu A^c_\nu - \delta_{4A} \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu - g \tilde{\delta}_1 f^{abc} \partial_\mu \bar{\omega}^a A^b_\mu \omega^c + g \delta_1 A^a_\mu \bar{\psi} t^a \gamma^\mu \psi_i \end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{Z_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\xi Z_\xi}\partial_\mu A^{a\mu}\partial_\nu A^{a\nu} - \tilde{Z}_2 \partial_\mu \bar{\omega}^a \partial_\mu \omega^a + i Z_2 \bar{\psi}_i \not{\partial} \psi_i \\
& - g Z_{3A} f^{abc} \partial_\mu A^a_\nu A^b_\mu A^c_\nu - \frac{g^2}{4} Z_{4A} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu - g \tilde{Z}_1 f^{abc} \partial_\mu \bar{\omega}^a A^b_\mu \omega^c + g Z_1 A^a_\mu \bar{\psi} t^a \gamma^\mu \psi_i,
\end{aligned}
\tag{14}
$$

where we used

$$
Z_{(\bullet)} = 1 + \delta_{(\bullet)} \ . \tag{15}
$$

All coefficients $\delta_{(\bullet)}$ can be fixed by explicit computations, similar to other theories we considered before (QED or Yukawa). For instance, δ_{4A} is chosen in such a way as to make the gluon 4-pt function finite



(16)

Notice that even though there are a priori 8 (by the number of operators in the Lagrangian) renormalization constants, there are non-trivial relations between them due to gauge invariance (or, more precisely, to the Ward indentities of BRST), reducing the number of independent ones to 4. Indeed, comparing (**??**) and (**??**) with the Lagrangian in terms of bare quantities (**??**) we get

$$
\frac{g_0^2}{g^2} = \frac{Z_1^2}{Z_2^2 Z_3} = \frac{Z_{3A}^2}{Z_3^3} = \frac{Z_{4A}}{Z_3^2} = \frac{\tilde{Z}_1^2}{\tilde{Z}_2 Z_3}. \tag{17}
$$

The relations in eq. (**??**) provide several different ways of computing the running of g^2. The very first equality looks identical to the one in QED. However, unlike in QED it is not enough to compute the wave function renormalization for A^a_μ, i.e. Z_3, in order to find the beta-function. Can you explain why?

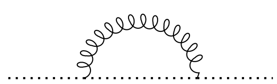
No fermions: $N_f = 0$

It seems that for the case of pure gluodynamics the most efficient way of finding the beta-function is from

$$
\frac{g_0^2}{g^2} = \frac{\tilde{Z}_1^2}{\tilde{Z}_2 Z_3}, \tag{18}
$$

however, you are free to make your own choice. In this case diagrams that have to be computed as as follows:

- Ghost wave function renormalization $\tilde{Z}_2$



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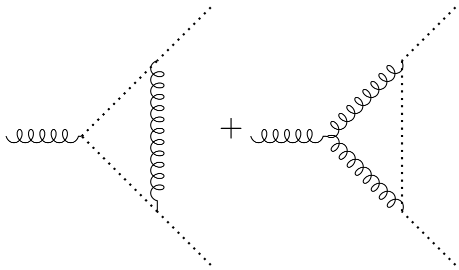
- Ghost-gluon interaction vertex $\tilde{Z}_1$



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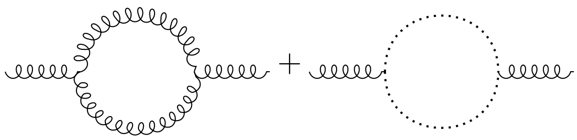
- Gluon wave function renormalization Z_3



(21)

Adding fermions: $N_f \neq 0$

- What would change if now N_f fermions are introdiced?
- Compute the beta-function in this case.