AQFT - Exercise Set 2

Exercise 1

Compute explicitly the wave functions in the helicity basis[1] for a massive spin 1/2 particle and its anti-particle, mediated by the Dirac (4-component) spinor

$$u(p,\lambda) = \langle 0|\Psi(0)|p,\lambda;r\rangle \text{ and } v^*(p,\lambda) = \langle 0|\Psi^*(0)|p,\lambda;\bar{r}\rangle \tag{5}$$

with normalization $u^\dagger u = v^\dagger v = 2E$. The states with r and $\bar{r}$ correspond to particle and antiparticle. In the following we will assume the theory to be invariant under both charge conjugation and parity. The particle/anti-particle states are then related via charge conjugation

$$U(C)\,|p,\lambda;r\rangle = |p,\lambda;\hat{r}\rangle \tag{6}$$

whose action on the Dirac spinor is given by

$$\Psi^c(x) = U(C)^\dagger \Psi(x) U(C) = -i\gamma^2 \Psi^*(x). \tag{7}$$

You may want to also use parity transformation for particles at rest

$$U(P)^\dagger \Psi(t,\vec{x}) U(P) = \gamma^0 \Psi(t,-\vec{x}). \tag{8}$$

Exercise 2

Consider a massive spin-2 particle state in the helicity basis $|p,\lambda\rangle$ with $p^2 = m^2$ and $\lambda = -2,-1,0,1,2$. A symmetric rank-2 tensor can interpolate for those states

$$\psi^{\mu\nu}(x) = \varepsilon^{\mu\nu}(p,\lambda)e^{-ipx} = \langle 0|h_{\mu\nu}(0)|p,\lambda\rangle\,. \tag{9}$$

- Find the polarizations in the rest frame $\bar{p}^\mu = (m,0,0,0)$, $\varepsilon^{\mu\nu}(\bar{p},\lambda)$.
- What are the constraint that they satisfy?
- Show that the constraints in an arbitrary reference frame can be written as

$$\partial_\mu \psi^{\mu\nu} = 0, \qquad \psi^\mu_{\ \mu} = 0. \tag{10}$$

- Prove that the Fierz-Pauli equation

$$\Box\psi_{\mu\nu} - \partial_\sigma\partial_\mu\psi^\sigma_\nu - \partial_\sigma\partial_\nu\psi^\sigma_\mu + \partial_\mu\partial_\nu\psi + \eta_{\mu\nu}(\partial_\lambda\partial_\sigma\psi^{\lambda\sigma} - \Box\psi) + m^2(\psi_{\mu\nu} - \eta_{\mu\nu}\psi) = 0, \tag{11}$$

  with $\psi = \psi^\mu_\mu$, leads to the constraints above.

[1] Let us remind the definition of the helicity basis for massive single particle state. Consider the state of the particle in its rest frame where $\bar{p}^\mu = (m,0,0,0)$: $|\bar{p},\lambda\rangle$ where, in this frame, λ is defined for convention as the projection of the angular momentum along the third axis

$$J^3\,|\bar{p},\lambda\rangle = \lambda\,|\bar{p},\lambda\rangle\,, \tag{1}$$

and it takes values $\lambda = -j,...,j$ with j the spin of the particle. A state with generic momentum p^μ is then defined as

$$|p,\lambda\rangle = U(R(\theta,\phi))U(B(\eta))\,|\bar{p},\lambda\rangle\,, \tag{2}$$

where

$$B(\eta) = e^{-i\eta K^3}, \tag{3}$$

is the boost that fixes the energy of the particle to be equal to p^0, $cosh\eta = p^0/m$, and the rotation

$$R(\theta,\phi) = e^{i\phi J^3}e^{-i\theta J^2}e^{-i\phi J^3}, \tag{4}$$

adjust the direction of motion to be $\hat{p}$. What is the physical meaning of λ for a state with generic momentum p^μ?