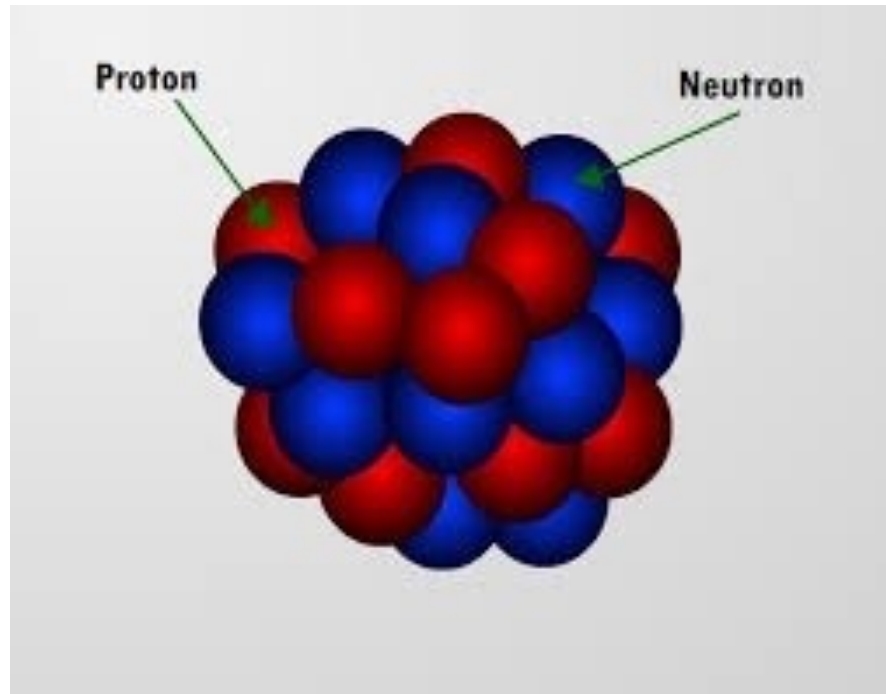


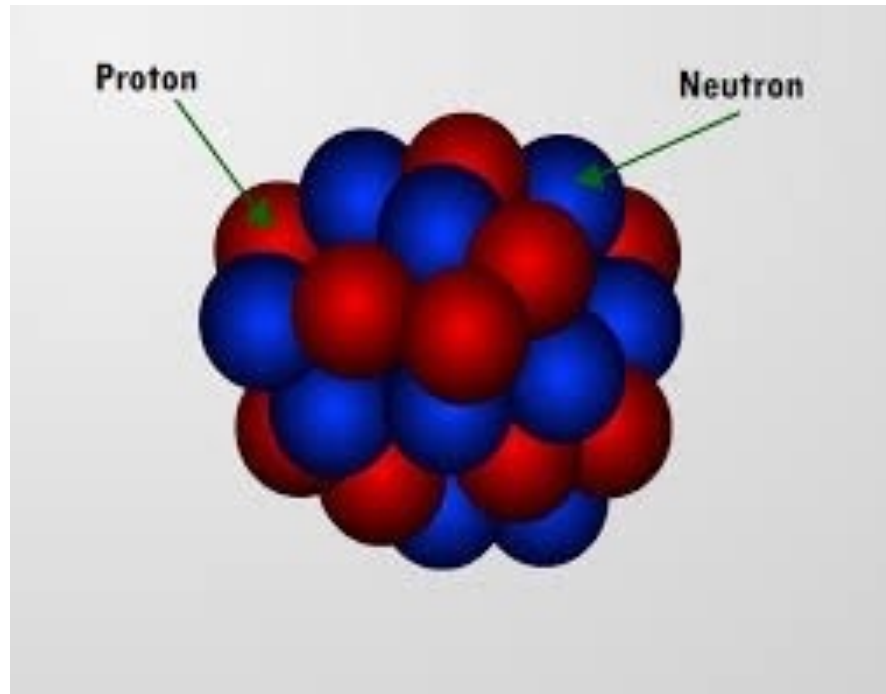
Random matrix theory

Ex: Atomic nuclei



Strongly coupled (QCD) protons and neutrons: chaotic, complex many body system!

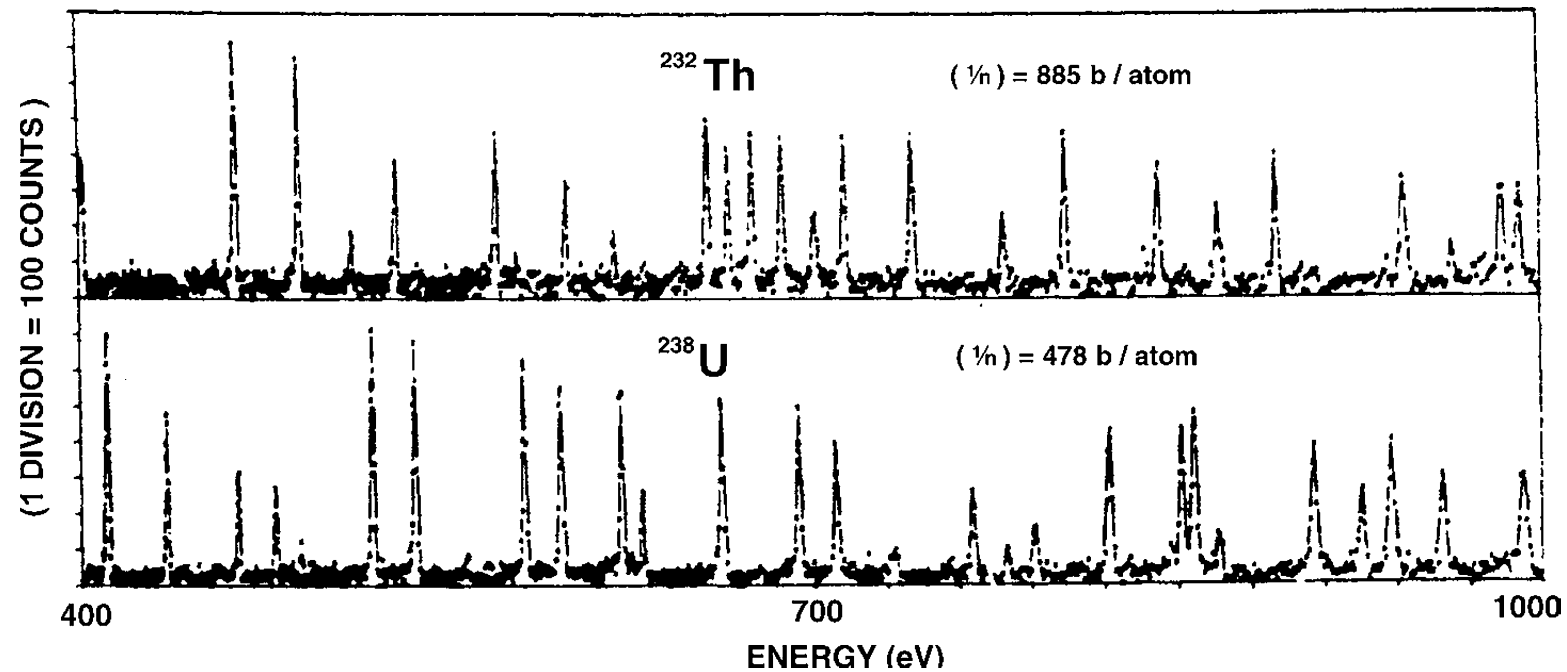
Ex: Atomic nuclei



Strongly coupled (QCD) protons and neutrons: chaotic, complex many body system!

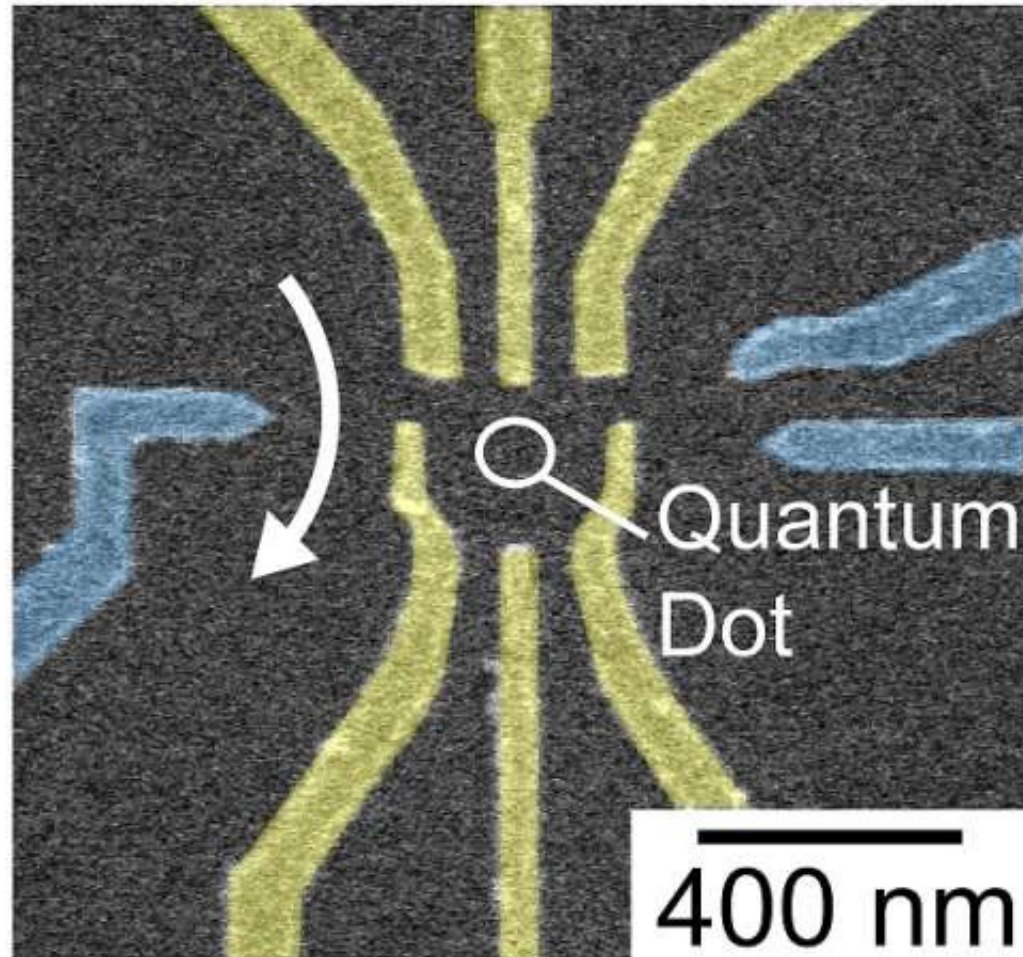
Note: non-MBL system! – Any other non-localized many body system is believed to be qualitatively similar, with energy level statistics well-described by random matrix theory

Energy levels in nuclei

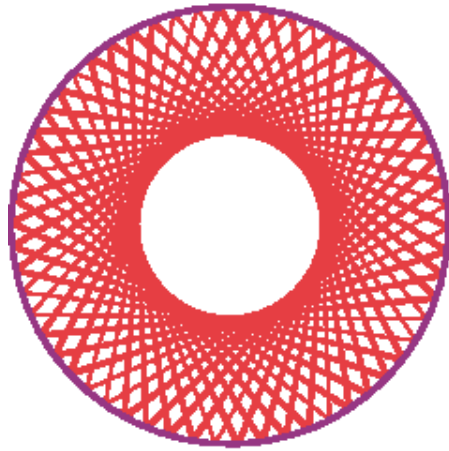


Ex: Electrons in a quantum dot

Chaotic motion of single or few electrons in a gate-confined region of a 2d electron gas



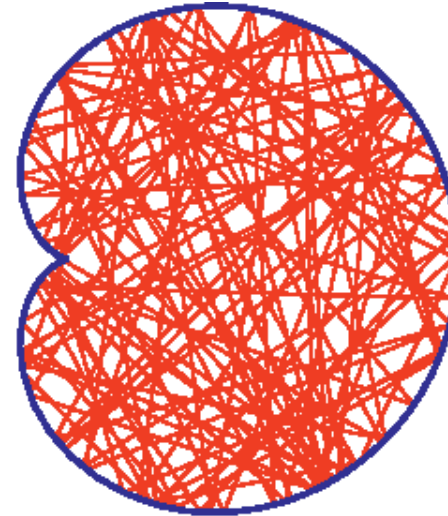
Ex: Quantum billiards



(a)

Symmetric shapes
Periodic classical orbits

Solvable quantum
wavefunctions

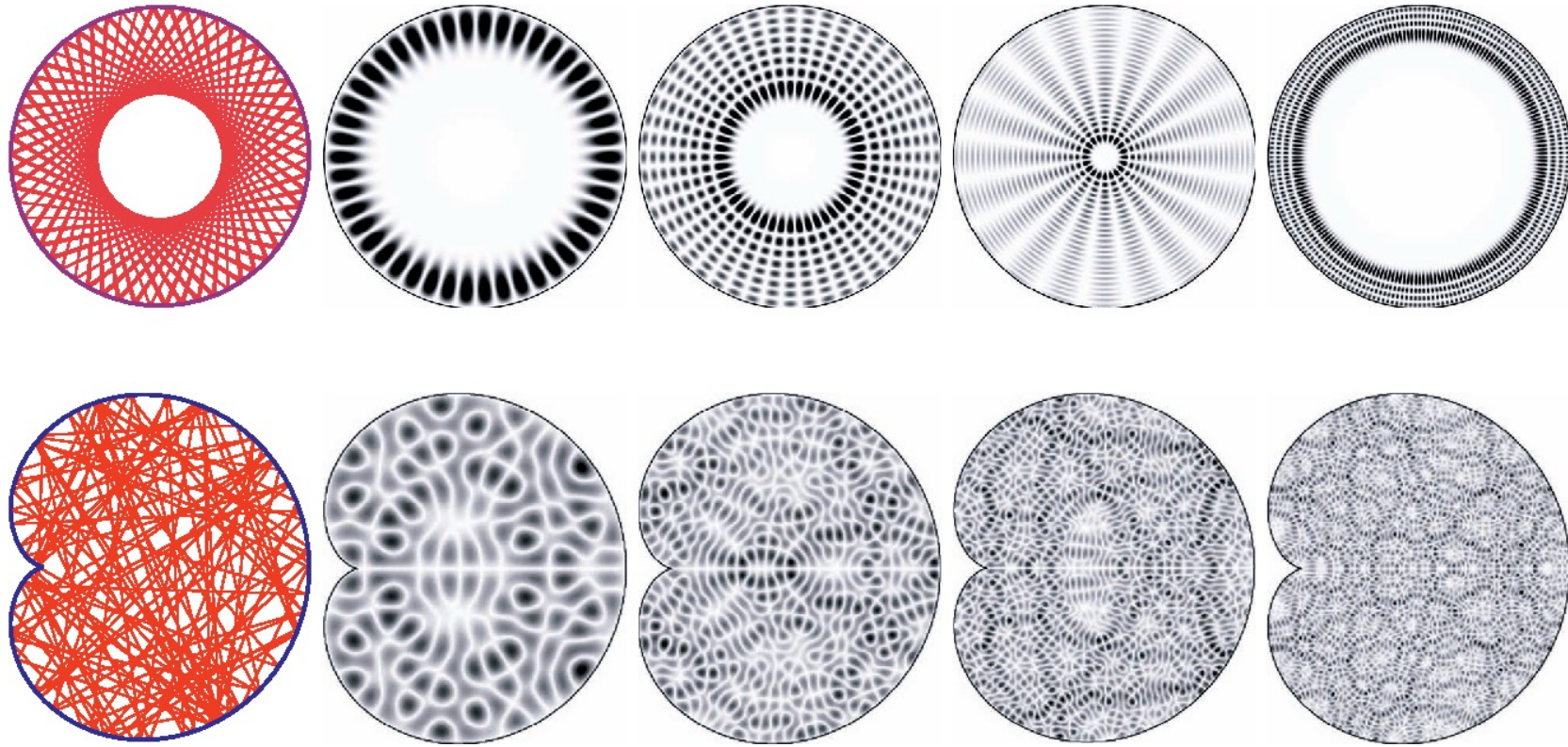


(b)

Generic situation:
Chaotic classic dynamics

Quantum eigenstates
superpose essentially all equal
energy states randomly

Ex: Quantum billiards



Regular,
solvable

Chaotic

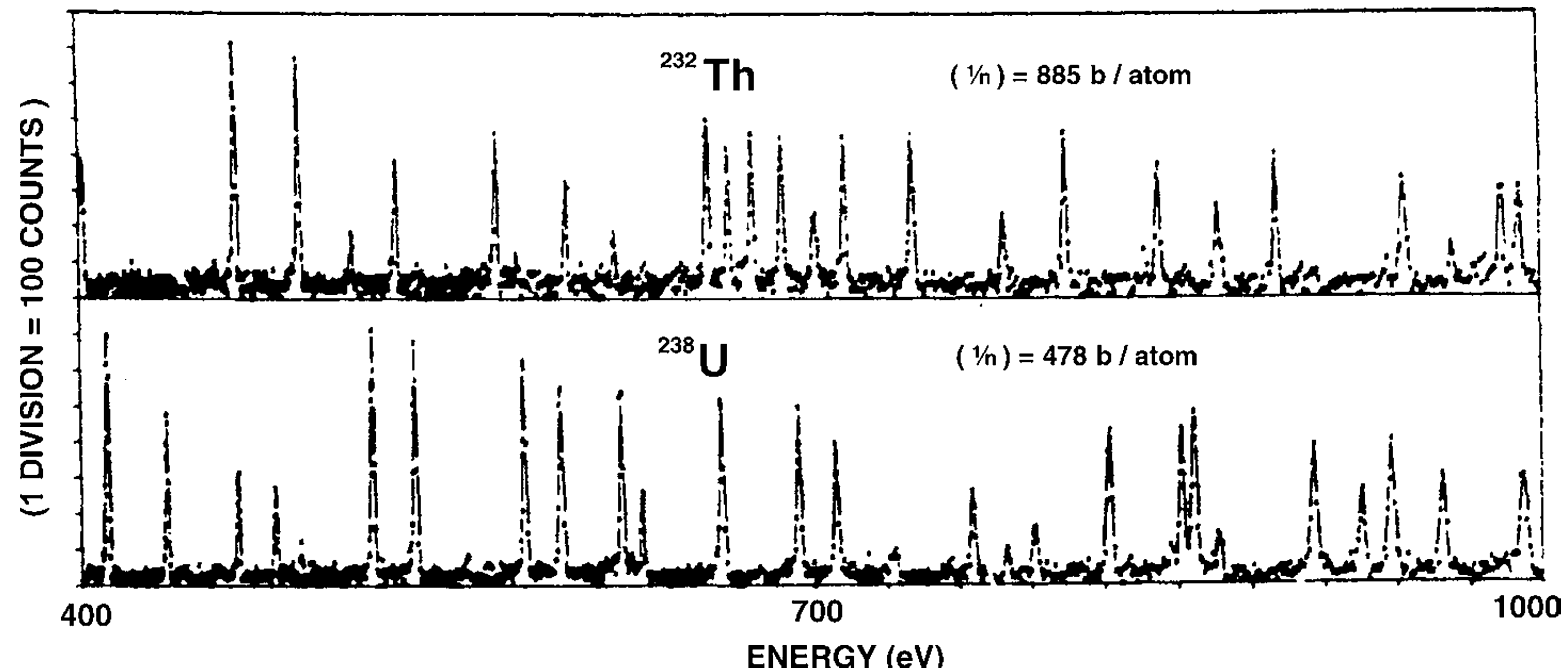
Classical orbits

Quantum eigenstates

Other instances of random matrices

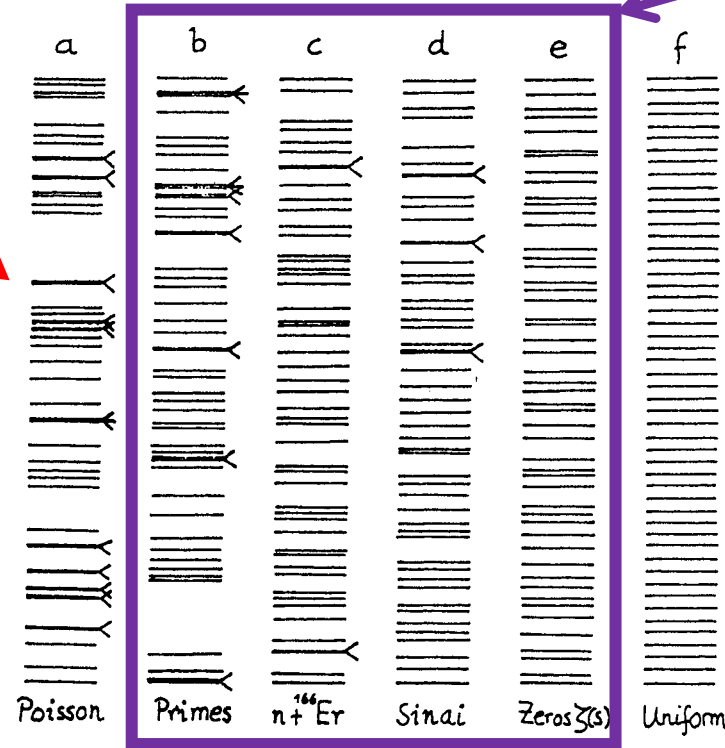
- Sound waves in disordered solids
- Matrix J_{ij} of exchange couplings in spin glasses
- Couplings in neural networks or genomic regulatory networks (usually non-Hermitian!)
- Relations between random matrices and directed polymers / KPZ universality class
- ...

Energy levels in nuclei



Other random eigenvalues

Random,
uncorrelated
distribution
(Poisson)



Neither similar to
random
Poissonian nor
uniformly spaced
distribution

Figure 1.2. Some typical level sequences. From Bohigas and Giannoni (1984). (a) Random levels with no correlations, Poisson series. (b) Sequence of prime numbers. (c) Slow neutron resonance levels of the erbium 166 nucleus. (d) Possible energy levels of a particle free to move inside the area bounded by $1/8$ of a square and a circular arc whose center is the mid point of the square; i.e. the area specified by the inequalities, $y \geq 0$, $x \geq y$, $x \leq 1$, and $x^2 + y^2 \geq r$. (Sinai's billiard table.) (e) The zeros of the Riemann zeta function on the line $\text{Re } z = 1/2$. (f) A sequence of equally spaced levels (Bohigas and Giannoni, 1984).

Energy levels with no correlation

Poissonian processes : e.g.

- Random points thrown on a line
- Spectrum of Anderson localized particles in a large volume
- Radioactive decay processes: emission time of a particle

Energy levels with no correlation

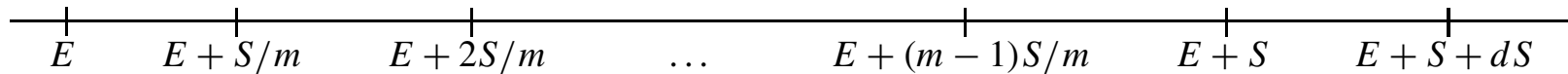
Poissonian processes : e.g.

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- Radioactive decay processes: emission time of a particle

Average density ρ of points (or emission rate $1/\tau$)

Probability that there is no point in interval s :

$$P(\text{no level in interval } [0,s]) = e^{-s\rho} \quad \left(= \lim_{m \rightarrow \infty} \prod_{r=0}^{m-1} \left(1 - \rho \frac{s}{m} \right) = \lim_{m \rightarrow \infty} \left(1 - \rho \frac{s}{m} \right)^m \right)$$



Energy levels with no correlation

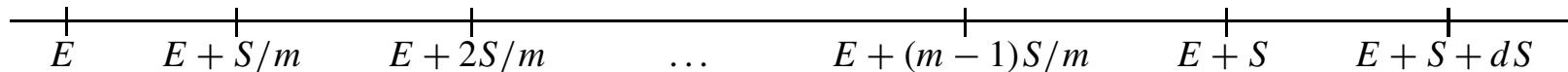
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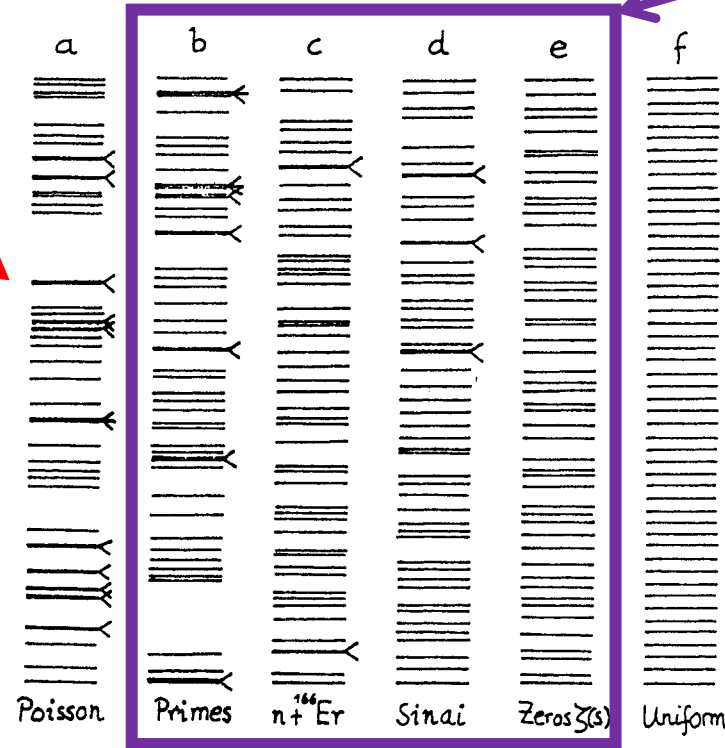
Probability density to find a spacing s between

adjacent levels:

$$p(s)ds = -\frac{dP}{ds}ds = e^{-s\rho}\rho ds; \quad p(s=0) = \rho = \text{finite!}$$

Other random eigenvalues

Many close levels for random, uncorrelated distribution (Poisson)



Neither similar to random Poissonian nor uniformly spaced distribution

Figure 1.2. Some typical level sequences. From Bohigas and Giannoni (1984). (a) Random levels with no correlations, Poisson series. (b) Sequence of prime numbers. (c) Slow neutron resonance levels of the erbium 166 nucleus. (d) Possible energy levels of a particle free to move inside the area bounded by $1/8$ of a square and a circular arc whose center is the mid point of the square; i.e. the area specified by the inequalities, $y \geq 0$, $x \geq y$, $x \leq 1$, and $x^2 + y^2 \geq r$. (Sinai's billiard table.) (e) The zeros of the Riemann zeta function on the line $\text{Re } z = 1/2$. (f) A sequence of equally spaced levels (Bohigas and Giannoni, 1984).

Level spacings in nuclei

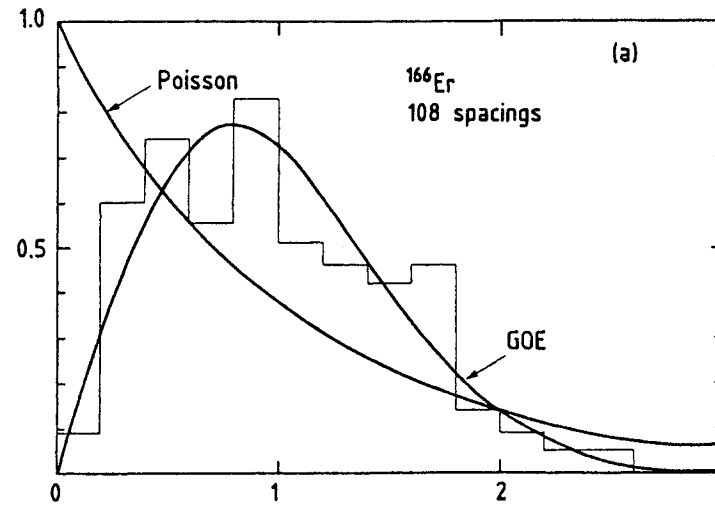


Figure 1.3. The probability density for the nearest neighbor spacings in slow neutron resonance levels of erbium 166 nucleus. The histogram shows the first 108 levels observed. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from The American Physical Society, Liou et al., Neutron resonance spectroscopy data, *Phys. Rev. C* 5 (1972) 974–1001.

Level spacings in nuclei

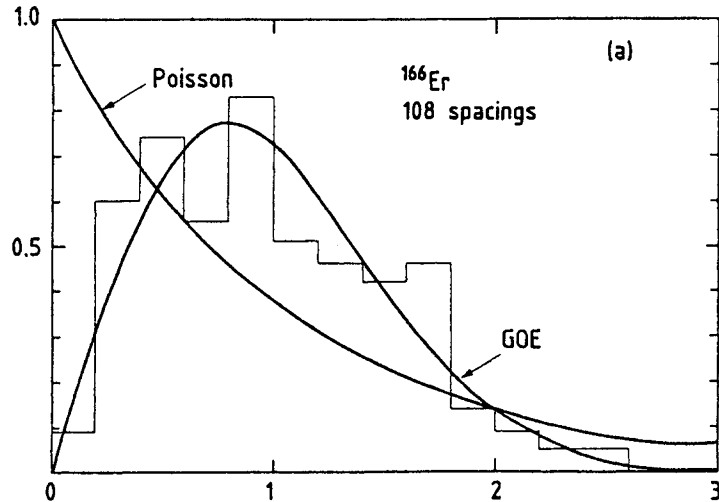


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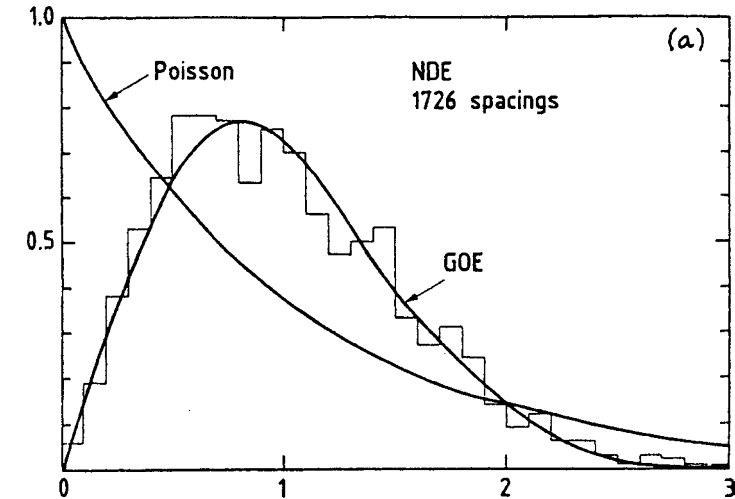


Figure 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consists of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232) and U(238); (1146 levels); (ii) proton resonances of Ca(44) ($J = 1/2+$), Ca(44) ($J = 1/2-$), and Ti(48) ($J = 1/2+$); (157 levels); and (iii) (n, γ)-reaction data on Hf(177) ($J = 3$), Hf(177) ($J = 4$), Hf(179) ($J = 4$), and Hf(179) ($J = 5$); (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity). For each of the 30 sequences the average quantities (e.g. the mean spacing, spacing/mean spacing, number variance μ_2 , etc., see Chapter 16) are computed separately and their aggregate is taken weighted according to the size of each sequence. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from Kluwer Academic Publishers, Bohigas O., Haq R.U. and Pandey A., Fluctuation properties of nuclear energy levels and widths, comparison of theory with experiment, in: *Nuclear Data for Science and Technology*, Bökhoff K.H. (Ed.), 809–814 (1983).

Riemann zeta function: normalized spacing of zeroes

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \prod_p (1 - p^{-z})^{-1}$$

Riemann conjecture:

Zeroes: $z_n = 1/2 + i\xi_n$

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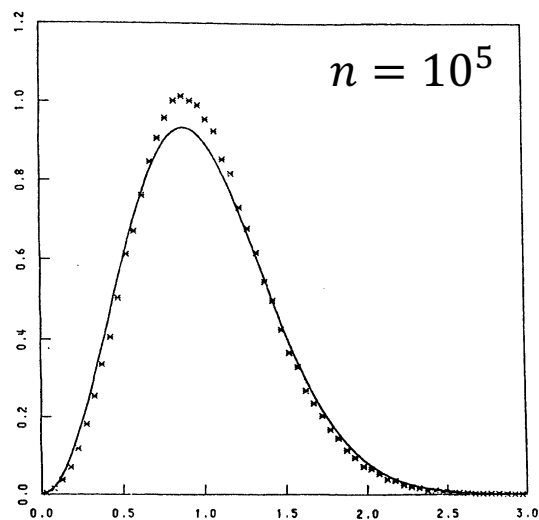


Figure 1.12. Plot of the density of normalized spacings for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function on the critical line, $1 < n < 10^5$. The solid curve is the spacing probability density for the Gaussian unitary ensemble, Eq. (6.4.32). From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

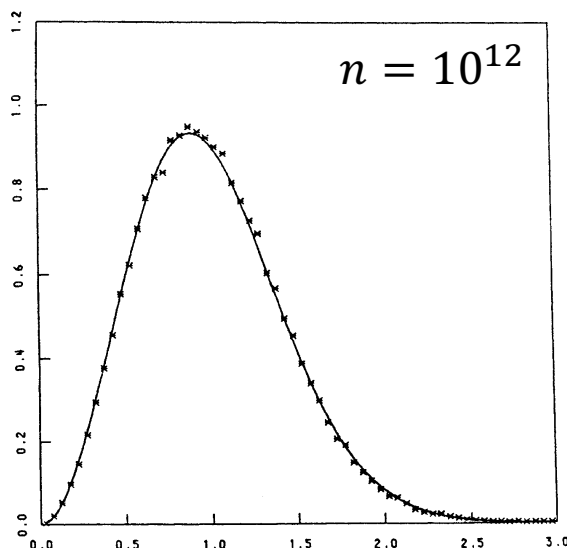


Figure 1.13. The same as Figure 1.12 with $10^{12} < n < 10^{12} + 10^5$. Note the improvement in the fit. From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

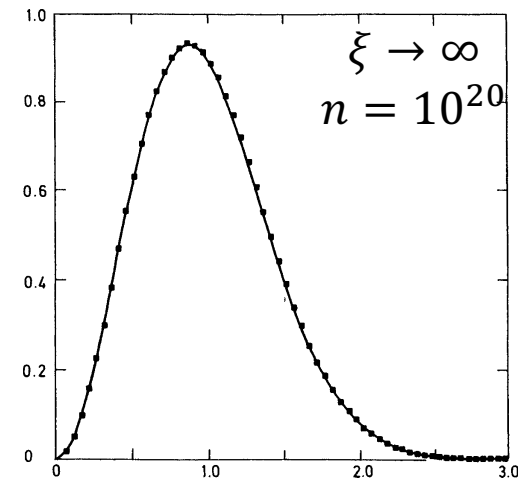


Figure 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20} th zero. From Odlyzko (1989). Copyright © 1989, American Telephone and Telegraph Company, reprinted with permission.

Riemann zeta function: normalized spacing of zeroes

Riemann conjecture:
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$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} = \prod_p (1 - p^{-z})^{-1}$$

$$s_n := \frac{\xi_{n+1} - \xi_n}{\Delta \xi}$$

Zeroes are spaced
like levels of complex
random matrices!
WHY??

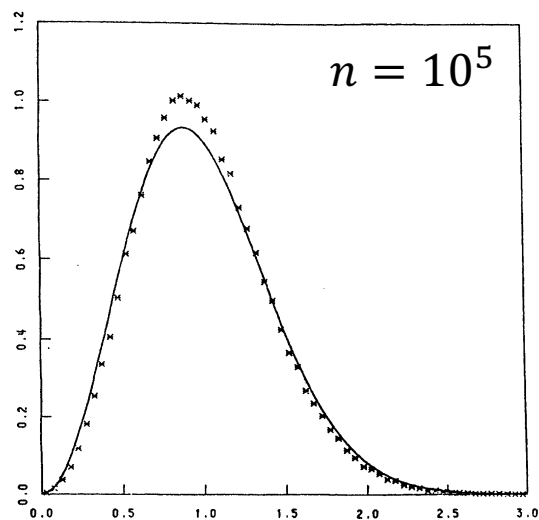


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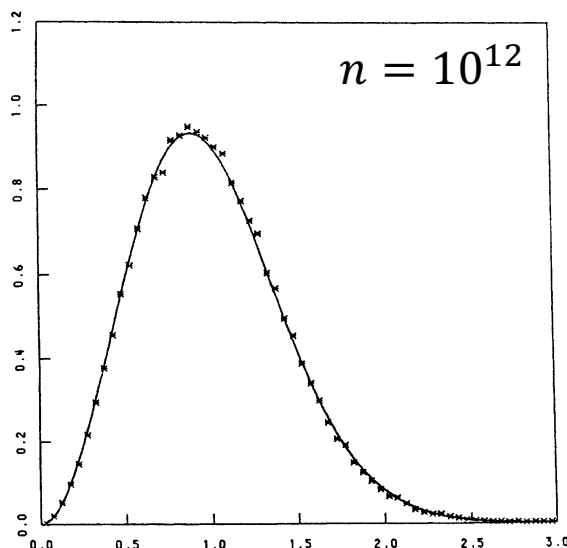


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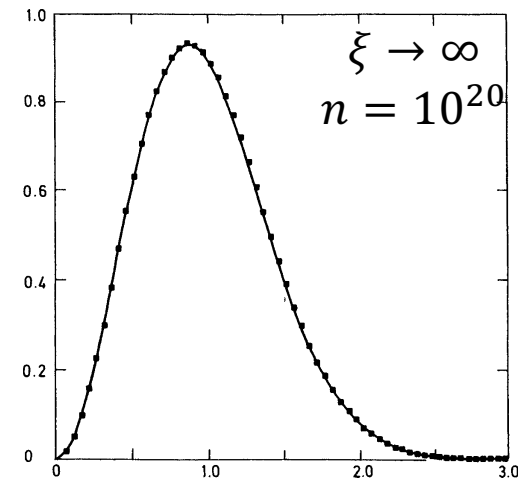


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