Random Matrix Theory: Wigner-Dyson statistics and beyond. Lecture notes given at SISSA (Trieste, Italy)

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(Dated: today)

PACS numbers: 72.15.Rn, 72.70.+m, 72.20.Ht, 73.23.-b

I. INVARIANT AND NON-INVARIANT GAUSSIAN RANDOM MATRIX ENSEMBLES

Any random matrix ensemble (RME) is determined through the probability distribution function (PDF) $P(\mathbf{M})$ that depends on the matrix entries M_{nm}. An important special class of random matrix ensembles is given by the PDF which is invariant under rotation of basis $\mathbf{M} \to \mathbf{TMT}^{-1}$:

$$P(\mathbf{M}) \propto \exp\left[-\mathrm{Tr}\, V(\mathbf{M})\right], \tag{1}$$

where $V(\mathbf{M})$ is and arbitrary function of $\mathbf{M}$ analytic at $\mathbf{M} = 0$. The invariance is ensured by the trace Tr in front of the matrix function $V(\mathrm{M})$. The RME with the PDF of the form Eq.(1) will be referred to as *invariant RME*.

In general the PDF Eq.(1) corresponds to a non-trivial correlation between fluctuating matrix entries. However, there is one extremely important case when
(i) the matrix $\mathbf{M}$ is Hermitean $\mathbf{M} \equiv \mathbf{H} = \mathbf{H}^\dagger$ and
(ii) $V(\mathbf{H}) = a\mathbf{H}^2$.
In this case

$$\exp\left[-\mathrm{TrV}(\mathbf{H})\right] = \exp\left[-a\sum_{n,m}|H_{nm}|^2\right] = \prod_{n,m}\exp\left[-a\,|H_{nm}|^2\right]$$

so that all matrix entries fluctuate independently around zero. This is the celebrated Gaussian random matrix ensemble of Wigner and Dyson (WD).

Note that Gaussian random matrix ensembles can be also *non-invariant*. The generic non-invariant Gaussian RME is determined by the PDF of the form:

$$P(\mathbf{H}) \propto \exp\left[-\sum_{n,m}\frac{|H_{nm}|^2}{A_{nm}}\right]. \tag{2}$$

In this case each matrix entry fluctuate independently of the other but with the variance which depends on the indices n, m that label the matrix entry. The simplest Gaussian non-invariant ensemble is the *Rosenzweig-Porter ensemble*[2] for which

$$A_{nm} = \begin{cases} a, & n \neq m \\ \Lambda\, a & n = m \end{cases} \tag{3}$$

It is remarkable that both the classic WD ensemble and the Rosenzweig-Porter ensemble allow for an exact solution[1,3].

Physically, an invariant random matrix ensemble describes extended (but phase-randomized) states, where the localization effects are negligible. In contrast to that any non-invariant ensemble accounts for a sort of structure of eigenfunctions (e.g. localization) in a given basis which may be not the case in a different rotated basis (remember about the extended states in the tight-binding model which are the linear combinations of states localized at a given site).

In particular the problem of localization in a quasi-1 wire can mapped onto the non-invariant *banded RME* with the variance matrix equal to:

$$A_{nm} = \exp\left[-|n-m|/B\right] \tag{4}$$

This model can be efficiently mapped onto a nonlinear supersymmetric sigma model and solved by the transfer matrix method[4].

Finally, we mention a *critical power-law banded random matrix ensemble* (CPLB-RME) for which the variance matrix is of the Lorenzian form:

$$A_{nm} = \frac{1}{1+\frac{(n-m)^2}{B^2}}. \tag{5}$$

This model (not yet solved) possesses a fascinating property of multifractality and is an extremely accurate model for describing the critical states at the Anderson localization transition point in dimensionality $d > 2$.

II. PARAMETRIZATION IN TERMS OF EIGENVALUES AND EIGENVECTORS

Consider a Hermitean $N \times N$matrix $\mathbf{H} = \mathbf{H}^\dagger$. The physical meaning is mostly contained in the eigenvalues E_n of this matrix and also in the unitary matrix $\mathbf{U} = (\mathbf{U}^\dagger)^{-1}$ whose n-th column is an n-th normalized eigenvector $\mathbf{\Psi}_n \equiv \{\Psi_n(r)\}$. Therefore it is sensible to parametrize the matrix $\mathbf{H}$ in the following way:

$$\mathbf{H} = \mathbf{U}\,\mathbf{E}\,\mathbf{U}^\dagger, \tag{6}$$

where $\mathbf{E} = \mathrm{diag}\{E_n\}$. Then instead of $N(N+1)/2$ independent entries of the Hermitean matrix $\mathbf{H}$ one will deal with $N(N-1)/2$ independent variables of the unitary matrix $\mathbf{U}$ plus N eigenvalues.

For invariant ensembles the PDF is independent of the eigenvector degrees of freedom and is determined only by the eigenvalues. In particular for the classic WD ensemble it reduces to:

$$P(\mathbf{H}) \propto \exp\left[-a\sum_n E_n^2\right]. \tag{7}$$

However the change of variables involves also computing the Jacobian of the transformation Eq.(179). The easiest way of computing it is to compute the form:

$$\mathrm{Tr}(d\mathbf{H})^2 = \mathbf{Tr}\left(2d\mathbf{f}\,\mathbf{E}\,d\mathbf{f}\,\mathbf{E} - 2\mathbf{E}^2\,(d\mathbf{f})^2 + (d\mathbf{E})^2\right) = 2\sum_{n>m}(E_n - E_m)^2\,|df_{nm}|^2 + \sum_n (dE_n)^2, \tag{8}$$

where

$$d\mathbf{f} = \mathbf{U}^\dagger\,d\mathbf{U} = -d\mathbf{U}^\dagger\,\mathbf{U} = -d\mathbf{f}^\dagger.$$

The set of f_{nm}, $(n > m)$, are $N(N-1)/2$ natural "coordinates" related to eigenvectors. Then the Jacobian of the transformation $d\mathbf{H} \to d\mathbf{f}\,d\mathbf{E}$ is given by

$$J \propto \begin{cases} \sqrt{D}, & \mathbf{H}\ is\ \ real \\ D, & \mathbf{H}\ is\ \ complex \end{cases} \qquad D = \Delta^2 = \prod_{n>m}(E_n - E_m)^2. \tag{9}$$

III. JOINT PROBABILITY DISTRIBUTION

According to Eqs.(7),(9) the entire joint probability distribution function of eigenvalues and eigenvectors for an arbitrary invariant ensemble Eq.(1) takes the form:

$$d\mathbf{H}P(\mathbf{H}) = d\mathbf{f}d\{E_n\}\,\exp\left[-\sum_{n=1}^N V(E_n)\right]\,|\Delta|^\beta, \tag{10}$$

where Δ is the Vandermond determinant:

$$\Delta_N = \begin{vmatrix} 1 & 1 & ... & 1 \\ E_1 & E_2 & ... & E_N \\ E_1^2 & E_2^2 & ... & E_N^2 \\ . & . & ... & . \\ E_1^N & E_2^N & ... & E_N^N \end{vmatrix} = \prod_{n>m}(E_n - E_m). \tag{11}$$

A remarkable property of this distribution is that it is independent of the eigenvectors and depends only on eigenvalues.

Since the Vandermond determinant is vanishing if any two eigenvalues coincide $E_n = E_m$ (two columns of the determinant are equal) the coincidence of two eigenvalues is statistically improbable. This is the basic property of the random matrix theory which is called *level repulsion*.

IV. LEVEL REPULSION: POOR MAN DERIVATION

In order to understand the physical origin of level repulsion let us consider a situation where occasionally two levels are very close to each other $|E_1 - E_2| \ll \Delta$, where Δ is the mean level separation. Then it is enough to consider only one block of the random matrix:

$$\begin{pmatrix} \varepsilon_1 & V \\ V^* & \varepsilon_2 \end{pmatrix}$$

The true energy levels of this two-level system are well known:

$$E_{1,2} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + |V|^2}. \tag{12}$$

The two-level correlation function which is the probability density to find a level at a distance ω from the given one, is given by:

$$R(\omega) = \int d\varepsilon_1 d\varepsilon_2 \mathcal{D}V \left[\delta(\omega - \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + |V|^2}) - \delta(\omega - \varepsilon_1 + \varepsilon_2)\right] P(\varepsilon_1, \varepsilon_2, V), \tag{13}$$

where

$$\mathcal{D}V = \begin{cases} dV, & \beta = 1 \\ d\Im V d\Re V, & \beta = 2 \end{cases} \tag{14}$$

Small energy difference $|E_1 - E_2| \ll \Delta$ implies that *both* $\varepsilon_1 - \varepsilon_2$ and $|V|$ are small. Then the PDF $P(\varepsilon_1, \varepsilon_2, V)$ can be considered independent of $\varepsilon_1 - \varepsilon_2$ and $|V|$. Thus integrating the δ-functions over $\varepsilon_1 - \varepsilon_2$ we arrive at:

$$R(\omega) = \int \mathcal{D}V \left[\sqrt{1 - \frac{|V|^2}{\omega^2}} - 1\right] \theta(\omega - |V|). \tag{15}$$

Apparently this integral is convergent and the power counting immediately leads to:

$$R(\omega) \propto \begin{cases} \omega, & \beta = 1 \\ \omega^2, & \beta = 2 \end{cases} \tag{16}$$

The simple analysis above illustrates two important points. One –physical– is that the level repulsion is nothing but the *avoided level crossing* which is well known in quantum mechanics and which is caused by the $|V|^2$ term in the square root in Eq.(12). The other one – formal– is that the pseudo-gap in $R(\omega)$ near $\omega = 0$ is the effect of the phase volume $\mathcal{D}V$ and the power of ω depends on the number of independent components of V which is 1 in case V is a real number and 2 if $V = \Re V + i\Im V$ is a complex number.

It is known that the algebra of real and complex numbers allows only one further step of generalization. This is the algebra of quaternions:

$$\tau_0 = 1, \quad \tau_{1,2,3} = i\sigma_{1,2,3}, \tag{17}$$

where $\sigma_{1,2,3}$ are 2×2 Pauli matrices. It appears that this generalization makes sense in the context of random matrices too. Namely, one can consider random matrices which entries are *real quaternions*, i.e.:

$$V = \sum_{i=0}^{3} \xi_i \tau_i, \tag{18}$$

with real components ξ_i. This generalization corresponds to $\beta = 4$ in Eqs.(44,51).

V. TIME-REVERSAL SYMMETRY AND THE DYSON SYMMETRY CLASSES

It turns out the the parameter β is related with the time-reversal symmetry. In order to see this we note that the time-reversal operator $\mathcal{T}$ should obey a basic property

$$\mathcal{T}^2 = \alpha\, 1, \qquad |\alpha| = 1$$

(time reversal applied twice leaves the wave function unchanged). As the time reversal operator should involve the complex conjugation of wave function one may write:

$$\mathcal{T} = K\,C, \tag{19}$$

where C is the complex conjugation operator and K is an operator such that

$$K\,C\,K\,C = K\,K^* = \alpha\,1. \tag{20}$$

But K must be a unitary operator (as the norm of the wave function must be conserved). That is why

$$K^*\,K^T = 1. \tag{21}$$

From these two conditions one finds:

$$K = \alpha\,K^T = \alpha\,(\alpha\,K^T)^T = \alpha^2\,K. \tag{22}$$

Thus we conclude that

$$\alpha^2 = 1 \qquad \Rightarrow \alpha = \pm 1. \tag{23}$$

For spinless particles (or particles with even spin) we have:

$$\mathcal{T}^2 = 1, \tag{24}$$

and K can always be chosen to be a unity operator $K = 1$.

However, for particles with half-integer spin

$$\mathcal{T}^2 = -1, \tag{25}$$

and K is not an identity operator. In particular for spin-$\frac{1}{2}$ particles K is a 2×2 matrix. Using Eqs.(20),(21) one can show that up to a phase factor $e^{i\theta}$ the matrix K is equal to:

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{26}$$

The physical meaning of this operator is very simple: it flips the spinor.

The time-reversal symmetry

$$\mathcal{T} H \mathcal{T}^\dagger = H$$

in the cases Eq.(24) and Eq.(25) implies, respectively

$$H = H^* \tag{27}$$

and

$$H = -K\,H^*\,K = K\,H^*\,K^\dagger. \tag{28}$$

In the first case time reversal symmetry requires the Hamiltonian matrix $\mathbf{H}$ to be *real*, which corresponds to $\beta = 1$. In the second case one can do a simple algebra exercise and show that the condition Eq.(28) is fulfilled if the Hamiltonian matrix $\mathbf{H}$ has entries of the form Eq.(18) with *real* coefficients ξ_i. As was already mentioned this case corresponds to $\beta = 4$.

It is remarkable that Eq.(28) leads to a two-fold degeneracy of energy levels known as the Kramers degeneracy. To prove this statement we assume that the wave vector ψ corresponds to the eigenstate with the energy E, i.e

$$H\,\psi = E\,\psi, \quad H^*\,\psi^* = E\,\psi^*.$$

Multiplying the second of these equations by K and using Eq.(28)one obtains:

$$K\,H^*\,\psi^* = -K\,H^*\,K\,(K\,\psi^*) = +H\,(K\,\psi^*) = E\,(K\,\psi^*). \tag{29}$$

The last equality implies that $K\,\psi^*$ is also an eigenvector corresponding the eigenvalue E. The two eigenvectors are different. Indeed, if $K\,\psi^* = \lambda\,\psi$ then $K^2\,\psi = \lambda^*\,K\,\psi^* = |\lambda^2|\,\psi = -\psi$ and ψ must be zero. This completes the proof of the two-fold degeneracy.

Thus we see that the case $\beta = 1$ (known as the *orthogonal ensemble*) corresponds to the particles with an even spin and a Hamiltonian that preserves the time reversal symmetry. The case $\beta = 4$ (known as the *symplectic ensemble*)corresponds to particles with an odd spin and a *spin-dependent* Hamiltonian that preserves time reversal symmetry. In particular it applies to a system with the *spin-orbit* interaction. The case $\beta = 2$ (known as the *unitary ensemble*)does not assume any definite relationship between H and H^*, and thus the time-reversal symmetry must be broken (e.g. by magnetic field or magnetic impurities).

According to the initial idea of Wigner and Dyson all systems with complex interactions do not possess any symmetry but possibly time-reversal symmetry and thus should be classified according to one of the three symmetry classes discussed above.

VI. EXTENSION OF THE DYSON SYMMETRY CLASSES

It has been discovered relatively recently (Altland and Zirnbauer, 1995) that the Dyson list of symmetry classes can be naturally extended from 3 to 10 symmetry classes if one introduces, in addition to the time-reversal symmetry, also the *particle-hole* symmetry. This requires a certain quasi-relativistic description where there exist both particles and anti-particles (holes). In application to condensed matter physics such situation realizes in superconductivity which basic description (the Bogolyubov-de Gennes equation) is in terms of the two coupled Schroedinger-type equations for particles and holes. The block-matrix form of such a Hamiltonian reads as follows:

$$H = \begin{pmatrix} \mathcal{H} & \Delta \\ \Delta^\dagger & -\mathcal{H}^* \end{pmatrix}. \tag{30}$$

The off-diagonal part Δ could be even or odd under the transposition:

$$\Delta^T = \pm\Delta. \tag{31}$$

The first choice (sign $+$) corresponds to the *singlet* superconductive paring of fermions, which is odd under spin permutations and thus must be even under orbital permutation. The second choice (sign $-$) corresponds to the triplet superconductivity which is odd under orbital permutation.

This Hamiltonian acts on a wave function

$$\Psi = \begin{pmatrix} p \\ h \end{pmatrix}, \quad p = \begin{pmatrix} p\uparrow \\ p\downarrow \end{pmatrix}, \quad h = \begin{pmatrix} h\uparrow \\ h\downarrow \end{pmatrix}. \tag{32}$$

Note that here the block 2×2 matrix in the *particle-hole* space which should not be confused with the spinor space we considered in connection with the time reversal symmetry in the previous section. Thus inclusion of *both* time-reversal and the particle-hole symmetry requires to consider the product Hilbert space of spinor and particle-hole components of the wave functions. One can check that the Hamiltonian of the form Eq.(30) obeys the symmetry relation:

$$H = K\,H^*\,K = -K\,H^T\,K^\dagger, \tag{33}$$

where

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ph}, \quad K^2 = -1. \tag{34}$$

for the singlet paring and

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ph}, \quad K^2 = +1 \tag{35}$$

for the triplet paring.

Eq.(33) is the same type of constraint as Eq.(28) but with the opposite sign of the r.h.s. This change of sign reflects the fact that the charge-conjugation operator $\mathcal{C}$ is *anti-unitary*. The two cases of the singlet and triplet paring with the corresponding behavior

$$K^2 = \alpha, \quad \alpha = \pm 1 \tag{36}$$

are analogous to the two realizations of the time-reversal transformations discussed in the previous section. However, the consequences of the time-reversal symmetry (TRS) and the particle-hole (PH) symmetry for the spectrum of random matrices are different: the TRS with $\mathcal{T}^2 = -1$ implies the Kramers degeneracy (each level is doubly degenerate) while PH symmetry with $K^2 = -1$ implies that the spectrum is exactly symmetric with respect to $E = 0$ (for each level $E_n > 0$ there is a corresponding level $E_m = -E_n < 0$). This is related with the change of sign in Eq.(33) compared to Eq.(28). The triplet pairing with $K^2 = +1$ does not necessarily lead to a degeneracy, as one might think. The reason is that the linear dependence of the two eigenvectors with the same eigenvalues $K\,\psi^* = \lambda\,\psi$ does not lead to a contradiction $K^2\,\psi = \lambda^*\,K\,\psi^* = |\lambda^2|\,\psi = +\psi$, as in the case $K^2 = -1$.

If we denote the behavior of the system with respect to each of the two the symmetry transformations as 0 (no symmetry), $+1$ ($\alpha = +1$) and -1 ($\alpha = -1$) then we obtain $3 \times 3 = 9$ possible combinations of (p, p'), (where $p, p' = 0, \pm 1$) and 9 respective symmetry classes. The 10-th class appear because when neither of the two symmetries is present (the $(0, 0)$ case) the symmetry with respect to their product $\mathcal{TC}$ may be present or absent. So the $(0, 0)$ case (and mathematic says that only this case) is actually split into two classes.

The symmetry under the product $\mathcal{TC}$ can always be cast (in some special basis) as a symmetry constraint:

$$H = -\sigma_z\,H\,\sigma_z. \tag{37}$$

The corresponding matrix can be shown to be *block off-diagonal*

$$H = \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}. \tag{38}$$

This is a new type of the symmetry, which symmetry constraint *Eq.*(37) contains H (rather than $H^* = H^T$) in the r.h.s.. It is the simple consequence of the product $\mathcal{TC}$, where the symmetry constrains Eq.(28),(33) for $\mathcal{T}$, $\mathcal{C}$ contain H^* in the corresponding r.h.s.

The symmetry under $\mathcal{L} = \mathcal{TC}$ is known as the *sublattice*, or *chiral* symmetry. The reason for the first nickname is that the off-diagonal structure of Eq.(38) appears in one of the simplest models of disorder: the one-dimensional chain with on-site energies $\varepsilon_n = 0$ and the hopping to the nearest neighbor $t_{n,n\pm1}$ containing a random part. If one introduces two sublattices A (containing even sites) and B (containing odd sites), then the random hopping will connect only different sublattices resulting in the block off-diagonal terms $H_{AB} = V_{AB}$. There will be no terms H_{AA} or H_{BB}. Indeed, the diagonal entries of H_{AA} and H_{BB} are zero because the on-site energy is zero while the off-diagonal entries are zero due to the absence of the hopping integrals other than between the nearest neighbors (which belong to different sublattices).

The presence of the chiral symmetry usually favors delocalization. For instance in the one-dimensional disordered chain discussed above the localization radius tends to infinity as the energy of the eigenstate approaches zero.

VII. LEVEL REPULSION: CLASSICAL AND QUANTUM ANALOGY

A. Classical plasma with logarithmic interaction

One that for the Gaussian invariant ensemble one can rewrite the JPDF Eq.(10) in the following way:

$$J\,P(\mathbf{H}) \propto \exp[-\beta\,\mathcal{L}], \qquad \mathcal{L} = -\sum_{n>m} \ln|E_n - E_m| + a' \sum_n E_n^2, \tag{39}$$

where we introduces the Dyson symmetry parameter:

$$\beta = \begin{cases} 1, & for \quad real & symmetric & \mathbf{H} \\ 2, & for \quad complex & Hermitean & \mathbf{H} \\ 4, & for \quad real-quaternionic & Hermitean & \mathbf{H} \end{cases} \tag{40}$$

Note that by a proper choice of energy units the parameter $a' = a\beta^{-1}$ can be set equal to $\frac{1}{2}$ which will be always assumed throughout the lecture notes. Thus there is only one important parameter β in the classic WD random matrix theory.

Looking at Eq.(39) one concludes that the PDF in the ($\mathbf{f}$, $\mathbf{E}$) representation coincides with the partition function of classical particles repelling each other *logarithmically*, in a harmonic *confinement potential*. The Dyson symmetry parameter β plays a role if an inverse temperature.

The above derivation which lead to Eq.(39) can be repeated for an arbitrary invariant RME. The corresponding energy functional $\mathcal{L}$ of the logarithmically repelling particles will differ from Eq.(39) only by the confinement potential

which will be no longer harmonic but rather $\beta^{-1} V(E_n)$. The basic property of the PDF which dependends only on the set of eigenvalues E_n (but not the eigenvector variables $\mathbf{f}$) is retained for all the invariant RME making the corresponding eigenfunction statistics trivial.

This is no longer true once the invariance under basis rotation is broken. The latter circumstance is what makes non-invariant ensembles difficult to solve but at the same time having a rich variety of eigenfunction statistics.

B. Quantum analogy

Besides the analogy with logarithmically repelling *classical* particles at *finite temperature* β^{-1} living in one dimension (1d) there is also an important analogy with the system of *quantum* particles in 1d. To facilitate this analogy let us remind that the Jacobian in Eq.(9) can be expressed as the power of the Vandermond determinant Eq.(11)

$$J \propto |\Delta|^{\beta}. \tag{41}$$

The property of the Vandermond determinant is that

$$\sum_{n=1}^{N} \frac{\partial^2 \Delta_N}{\partial E_n^2} = 0. \tag{42}$$

Another property is that it changes sign upon any permutation of two E_n and E_m.

These two properties imply that Δ_N can be considered as the many-body wave function $\Psi(\{E_n\})$ of the system of N free fermions with the Hamiltonian consisting only of kinetic energy:

$$\mathcal{H}_K = -\sum_{n=1}^{N} \frac{\partial^2}{\partial E_n^2} \tag{43}$$

Moreover, as the energy of the corresponding many-body state is minimal possible for kinetic energy $\mathcal{E} = 0$, this is a *ground state* of this free fermionic system.

So we come to the statement that the Jacobian Eq.(9) at $\beta = 2$ is the probability density for the ground state of the free fermion system in the entire space.

$$J \sim |\Psi_0(E_n)|^2 \propto \prod_{n>m} |E_n - E_m|^{\beta}. \tag{44}$$

Note that the system of fermions in an infinite space is not well defined, as it expands indefinitely. Formally this is seen from the fact that the wave function Eq.(44) is not normalizable. In order to fix this pathology one has to consider a full probability distribution function Eq.(10) which includes also the confinement potential $V(E_n)$.

For the harmonic confinement potential the property Eq.(42) can be generalized in the following way:

$$\frac{1}{2m} \sum_{n=1}^{N} \frac{\partial^2}{\partial E_n^2} \left[\Delta_N \, e^{-\frac{m}{2} \sum_{n=1}^{N} E_n^2} \right] = \left[-\mathcal{E}_N + \frac{m}{2} \sum_{n=1}^{N} E_n^2 \right] \Delta_N \, e^{-\frac{m}{2} \sum_{n=1}^{N} E_n^2}. \tag{45}$$

Now we see that

$$\Psi_0 \propto \Delta_N \, e^{-\frac{m}{2} \sum_{n=1}^{N} E_n^2}, \tag{46}$$

is an eigenfunction of the free fermions with mass m in a harmonic confinement potential $V(E) = \frac{m}{2} E^2$. It corresponds to a certain positive energy $\mathcal{E}_N$ which arises due to confinement of fermions.

Now suppose that this property is valid also for arbitrary β and check that the wave function

$$\Psi_{0,\beta}(\{E_n\}) = \prod_{n>m} |E_n - E_m|^{\beta/2} \operatorname{sgn}(E_n - E_m) \, e^{-\frac{m}{2} \sum_{n=1}^{N} E_n^2} \tag{47}$$

is the eigenfunction of a certain Hamiltonian. Note that the coefficient m can be done arbitrary small by a proper choice of E_n units. So, for simplicity of further derivation we consider the case $m \to 0$.

To this end we take the sum of second derivatives of the wave function applying the kinetic energy operator Eq.(43) to Eq.(47) with $m \to 0$. The result appears to be proportional to $\Psi_{0,\beta}(\{E_n\})$:

$$-\mathcal{H}_K \, \Psi_{0,\beta} = \frac{\beta}{2} \left(\frac{\beta}{2} - 1 \right) |\Delta_N|^{-2} \sum_{n=1}^{N} \left(\frac{\partial \Delta_N}{\partial E_n} \right)^2 \Psi_{0,\beta}. \tag{48}$$

Thus at any $\beta \neq 2$ the system of fermions equivalent to an invariant random matrix theory is interacting with the interaction Hamiltonian:

$$\mathcal{H}_{\rm int} = \frac{\beta}{2}\left(\frac{\beta}{2}-1\right)|\Delta_N|^{-2}\sum_{n=1}^{N}\left(\frac{\partial\Delta_N}{\partial E_n}\right)^2. \tag{49}$$

Now if we use the property of the Vandermond determinant:

$$\sum_{n=1}^{N}\left(\frac{\partial\Delta_N}{\partial E_n}\right)^2 = 2|\Delta_N|^2\sum_{n=1}^{N}\frac{1}{(E_n-E_m)^2} \tag{50}$$

we finally obtain the total Hamiltonian of an equivalent system of fermions:

$$\hat{H} = -\frac{1}{2}\sum_{n=1}^{N}\frac{\partial^2}{\partial E_n^2} + \frac{\beta}{2}\left(\frac{\beta}{2}-1\right)\sum_{n>m}^{N}\frac{1}{(E_n-E_m)^2}. \tag{51}$$

This is the celebrated Calogero-Sutherland Hamiltonian[5] with the inverse square interaction. For $\beta = 2$ the interaction constant vanishes and the entire level repulsion is due to fermionic nature of the fictitious particles. For $\beta = 1$ there is some attraction on top of the free fermionic mutual avoiding, while for $\beta = 4$ the interaction is repelling. This additional interaction explains why the level repulsion for $\beta = 4$ is stronger then for $\beta = 2$ and for $\beta = 1$ it is weaker than for $\beta = 2$.

VIII. PLASMA MODEL AND THE WIGNER SEMI-CIRCLE

The model of classical particles in one dimension with logarithmic repulsion Eq.(39) can be represented by a continuous energy functional:

$$\mathcal{L} = -\frac{1}{2}\int dE\int dE'\,\varrho(E)\,\varrho(E')\,\ln|E-E'| + \frac{1}{\beta}\int dE\,\varrho(E)\,V(E), \tag{52}$$

expressed through the exact density

$$\varrho(E) = \sum_n \delta(E-E_n). \tag{53}$$

Now we make two *assumptions*:
(i) replace $\varrho(E)$ by an *ensemble average* value $\rho(E)$ and
(ii) neglect the thermal fluctuations by minimizing the energy functional Eq.(52) (with ϱ replaced by ρ) instead of computing the partition function

$$\sum_{\text{config.}\{E_n\}} {\rm e}^{-\beta\,\mathcal{L}}. \tag{54}$$

As a result one gets a kind of *mean field* approximation which is justified by the long-range, logarithmic nature of interaction.

Minimizing Eq.(52) with respect to $\rho(E)$ and differentiating both sides with respect to E one obtains:

$$\int_{-\infty}^{+\infty}\rho(E')\,\frac{dE'}{E-E'} = \frac{1}{\beta}\frac{dV}{dE} \equiv f(E). \tag{55}$$

The physical meaning of this equation is very simple: the force acting upon the given "particle" from all other particles should be balanced by the confining force. This is the condition of the plasma equilibrium.

From the mathematical viewpoint Eq.(55) is a *strongly singular* integral equation. Its solution is well known[6]. For an even function $V(E) = V(-E)$ it reads:

$$\rho_0(E) = \frac{1}{\pi^2}\sqrt{D^2-E^2}\int_{-D}^{D}\frac{f(E')}{\sqrt{D^2-E'^2}}\,\frac{dE'}{E'-E}, \tag{56}$$

where the principle value of the integral is assumed in Eq.(55) and Eq.(56), namely

$$\frac{1}{E'-E} \to \frac{1}{2}\left(\frac{1}{E'-E-i0}+\frac{1}{E'-E+i0}\right). \tag{57}$$

This definition allows to make an analytic continuation of Eq.(56) for E in the complex plane with the cut along the real axis with $|E|>D$, where the bandwidth D should be chosen from the condition that the total number of eigenvalues is equal to the size of matrix N:

$$\int_{-D}^{D}\rho_0(E)\,dE = N. \tag{58}$$

Namely, $\rho(E)$ can be represented as a sum of a function $\rho_+(E)$ which is regular in the upper half-plane $\Im E>0$ and a function $\rho_-(E)$ which is regular in the lower half-plane $\Im E<0$:

$$\rho_0(E)=\rho_+(E)+\rho_-(E), \qquad \rho_\pm(E)=\frac{1}{2\pi^2}\sqrt{D^2-E^2}\int_{-D}^{D}\frac{f(E')}{\sqrt{D^2-E'^2}}\,\frac{dE'}{E'-E\mp i0}. \tag{59}$$

It is important that along the cut $|E|>D$ the analytic function $\sqrt{D^2-E^2}=\pm i\sqrt{E^2-D^2}$ has different signs just above and just below the cut. This means that for $|E|>D$

$$\rho_+(E)+\rho_-(E) = -\frac{1}{\pi}\sqrt{E^2-D^2}\int_{-D}^{D}dE'\,\frac{f(E')}{\sqrt{D^2-E'^2}}\,\delta(E-E')=0. \tag{60}$$

On the other hand, for real E beyond the cut ($|E|<D$) one obtains:

$$\rho_+(E)-\rho_-(E) = \frac{2\pi i}{2\pi^2}\sqrt{D^2-E^2}\int_{-D}^{D}\frac{f(E')}{\sqrt{D^2-E'^2}}\,\delta(E-E')=\frac{i}{\pi}\,f(E). \tag{61}$$

Now we are in a position to check that Eq.(56) is really a solution of Eq.(55) for real E beyond ($|E|<D$) the cut. Indeed, the integral over the real axis in Eq.(55) can be closed either through the upper complex half-plane of E' or through the lower half-plane. We use the first option for the part containing $\rho_+(E')$ and the second option for the part containing $\rho_-(E')$. Each of the two contour integrals allows for the evaluation using the residue theorem. Then omitting the terms which do not have poles in the corresponding half-plane we obtain:

$$\int_{-\infty}^{+\infty}\rho_0(E')\,\frac{dE'}{E-E'}=\frac{1}{2}\int_{\text{upper}}\frac{\rho_+(E')}{E-E'+i0}\,dE'+\frac{1}{2}\int_{\text{lower}}\frac{\rho_-(E')}{E-E'-i0}\,dE'=-\pi i\,[\rho_+(E)-\rho_-(E)]=f(E).$$

This concludes the proof that Eq.(56) is indeed a solution of the integral equation Eq.(55). The beauty of the proof is that it is based only on the analytic properties of the solution.

For the Gaussian ensemble where $f(E')=E'$ the integral in Eq.(56) is actually independent of E (**show this using the definition of the principle value of the integral**)

$$\int_{-D}^{D}\frac{E'}{\sqrt{D^2-E'^2}}\,\frac{dE'}{E'-E}=\int_{-D}^{D}\frac{1}{\sqrt{D^2-E'^2}}\,dE'=\pi. \tag{62}$$

and the average density is the celebrated *semi-circle*:

$$\rho_0(E)=\frac{1}{\pi}\sqrt{2N-E^2}. \tag{63}$$

IX. PROBABILITY OF HAVING A HOLE IN SPECTRUM AND THE WIGNER SURMISE

One of the most popular statistics of eigenvalues of complex quantum systems is the *the level spacing distribution* $P(\omega)$: the probability density to have a level at a distance ω from a given level and no other levels between them. For ω much smaller than the mean level spacing $\Delta=\rho^{-1}$, it is improbable that in between of the two close levels there is yet another one or several levels. Then the requirement of having no levels in between of the two is unimportant and the leading term in $P(\omega)$ is the same as in the two-level correlation function $R(\omega)\propto\omega^\beta$ at $\omega\ll\Delta$. However,

for $\omega \gg \Delta$ the two statistics dramatically differ: $R(\omega)$ tends to a constant whereas $P(\omega)$ is very small due to a small probability to have no levels in between of the two levels separated by a large distance. Basically the $P(\omega)$ for $\omega \gg \Delta$ is limited by the probability of having a hole of the size ω in the spectrum. Let us find this probability using the plasma analogy.

As for any fluctuation, the probability of having a hole is given by the energy cost $\delta\mathcal{L}$ of this configuration relative to the equilibrium one:

$$P(\omega) \propto \exp(-\beta\,\delta\mathcal{L}). \tag{64}$$

One can cast the energy difference in the following way:

$$\Delta\mathcal{L} = \frac{1}{2}\int_C dE \int_C dE'\ \delta\rho(E)\,\delta\rho(E')\ \ln|E-E'| - \frac{1}{2}\int_{-\omega/2}^{\omega/2} dE \int_{-\omega/2}^{\omega/2} dE'\ \rho_0(E)\,\rho_0(E')\ \ln|E-E'|, \tag{65}$$

where the integrals in the first term run over the real axis *outside the gap region* and in the second term they run *over the gap region*; $\rho_0(E)$ is the equilibrium density without the gap and $\delta\rho(E) = \rho_\omega(E) - \rho_0(E)$ with $\rho_\omega(E)$ being the solution of the integral equation Eq.(55) with the *additional condition* that there is a gap for $|E| < \omega/2$.

The solution with the gap can also be constructed and and looks as follows:

$$\rho_\omega(E) = \frac{2|E|}{\pi^2\sqrt{E^2-(\omega/2)^2}}\sqrt{\bar{D}^2-E^2}\int_{\omega/2}^{\bar{D}}\frac{f(E')}{\sqrt{\bar{D}^2-E'^2}}\frac{\sqrt{E'^2-(\omega/2)^2}}{E'^2-E^2}\,dE'. \tag{66}$$

It is important that for the *steep confinement* $V(E) \sim |E|^\alpha$ ($\alpha > 1$) there is a scale separation, namely the integral in Eq.(66) varies slowly as a function of E with the typical scale of $D \sim N^{1/\alpha}$. In the large N limit one can disregard this dependence and consider

$$\rho_\omega(E) = \rho_0\,\frac{|E|}{\sqrt{E^2-(\omega/2)^2}}. \tag{67}$$

One can immediately recognize the gapped density of states with the square-root divergency near the gap edges similar to the one for a BCS superconductor.

Now by making a re-scaling $E \to sx$, $E' \to sx'$ and observing that the double integral in the first term is convergent for $\rho_\omega(E)$ of the form Eq.(67) we immediately obtain that $\Delta\mathcal{L} = \omega^2\,(a + b\ln\omega)$. More detailed inspection show that the coefficient $b = 0$.

Indeed, the coefficient b is proportional to

$$\left(\int_{\mathcal{C}}\delta\rho(E)\,dE\right)^2 - \left(\int_{-\omega/2}^{\omega/2}\rho_0\,dE\right)^2. \tag{68}$$

On the other hand, the conservation of the total number of levels requires:

$$\int_{\mathcal{C}}\delta\rho(E)\,dE = \int_{-\omega/2}^{\omega/2}\rho_0. \tag{69}$$

Raising the l.h.s. and the r.h.s. of the last equation to second power one proves the statement $b = 0$.

The final result for $\Delta\mathcal{L}$ reads:

$$\Delta\mathcal{L} = \frac{\pi^2}{16}\,(\rho_0\,\omega)^2 \approx 0.62\,(\rho_0\,\omega)^2. \tag{70}$$

This implies that the spacing distribution function for large level separations $s = (\omega/\Delta) \gg 1$ is given by:

$$P(s) \propto \exp\left(-\frac{\pi^2\beta}{16}\,s^2\right). \tag{71}$$

Note that the "Gaussian" form of $P(s)$ has nothing to do with the quadratic confinement potential (Gaussian invariant ensemble). In fact $P(s)$ has the same asymptotic form Eq.(71)for all *steep* confinement potentials.
check that it has the same form for the confinement potential $V(E) = E^4$.
Finally we mention a famous *interpolation formula* for $P(s)$ known as the $Wigner Surmise$:

$$P(s) = A(\beta)\,s^\beta \exp\left[-B(\beta)\,s^2\right], \qquad s = \frac{\omega}{\Delta}. \tag{72}$$

The coefficients $A(\beta)$ and $B(\beta)$ are found from two conditions: the normalization to the total probability 1 and the condition that the mean level spacing in the units of s is one:

$$\int_0^\infty P(s)\,ds = 1, \qquad \int_0^\infty s\,P(s)\,ds = 1. \tag{73}$$

These conditions result in:

$$A(\beta) = 2B(\beta)^{\frac{\beta}{2}+1}\,\Gamma\left(\frac{\beta}{2}+1\right), \qquad B(\beta) = \begin{cases} \frac{\pi}{4} \approx 0.78\ (0.62) & \beta = 1 \\ \frac{4}{\pi} \approx 1.27\ (1.24) & \beta = 2 \\ \frac{64}{9\pi} \approx 2.26\,(2.48) & \beta = 4 \end{cases} \tag{74}$$

For comparison we give in the brackets the exact values of $B_{\rm exact}(\beta) = \pi^2\beta/16$. One can see that they are rather close to the approximate values of the Wingner Surmise, especially for $\beta = 2$.

X. LEVEL COMPRESSIBILITY, NORMALIZATION SUM RULE AND NORMALIZATION ANOMALY

The *two-level correlation function* (TLCF) is formally defined as a correlation function of the exact density of states Eq.(53):

$$R_N(E,E') = \frac{\langle \varrho(E)\,\varrho(E')\rangle}{\rho(E)\,\rho(E')} \equiv \rho^{-1}(E)\delta(E-E') + 1 + Y_N(E,E'). \tag{75}$$

The δ-function in Eq.(75) is the self-correlation coming from one and the same level n in the sum in Eq.(53). The 1 term gives the asymptotic value of TLCF at energy separations $|E-E'| \gg \Delta$ when the average of two densities of states can be decoupled. The function $Y(E,E')$ gives then a regular contribution to the TLCF which decreases to zero as $|E-E'|$ increase.

There is an important *normalization sum rule* that applies to the TLCF. Indeed, consider

$$\int dE'\,\rho(E)\,\rho(E')\,R_N(E,E') = \int dE'\,\langle \varrho(E)\,\varrho(E')\rangle = \langle \varrho(E)\int dE'\,\varrho(E')\rangle \tag{76}$$

The total number of states is equal to the number of degrees of freedom N and *does not fluctuate*:

$$\int dE'\,\varrho(E') = N. \tag{77}$$

This normalization condition leads to

$$\int dE'\,\rho(E)\,\rho(E')\,R_N(E,E') = N\,\langle \varrho(E)\rangle = N\,\rho(E),$$

which implies that

$$\int_{-\infty}^{+\infty} dE'\,\rho(E')\,Y_N(E,E') = -1, \qquad \int_{-\infty}^{+\infty} dE'\,\rho(E')\,[R_N(E,E')-1] = 0. \tag{78}$$

This is the *normalization sum rule*.

Note that the sum rule can only be proven if N is finite and integration in Eq.(78) are extended over *all* energies. Taking the limit $N \to \infty$ could be a dangerous procedure as in this case one has to worry about the commutativity of the limits $N \to \infty$ and *limits of integration*$\to \infty$. The sum rule is certainly satisfied if the limit $N \to \infty$ is done *after* doing the integral. A simple example below shows that it can be violated if the limit $N \to$ is taken *prior* of doing the integral.

Consider an ensemble of diagonal random matrices with independently fluctuating random elements each having a distribution

$$P(\varepsilon_n) = \begin{cases} N^{-1}, & |\varepsilon_n| < N/2 \\ 0 & |\varepsilon_n| > N/2 \end{cases} \tag{79}$$

The TLCF for this ensemble can be computed straightforwardly:

$$\langle \varrho(E)\, \varrho(E')\rangle = \sum_{n\neq m} \int_{-N/2}^{N/2} \frac{d\varepsilon_n}{N} \int_{-N/2}^{N/2} \frac{d\varepsilon_m}{N}\, \delta(E-\varepsilon_n)\, \delta(E'-\varepsilon_m) + \sum_n \int_{-N/2}^{N/2} \frac{d\varepsilon_n}{N}\, \delta(E-\varepsilon_n)\, \delta(E'-\varepsilon_n),$$

so that for $|E| < N/2$ and $|E'| < N/2$ one obtains:

$$R(E,E') = \delta(E-E') + \frac{N^2-N}{N^2}, \qquad \rho(E) = 1, \quad Y_N(E,E') = -\frac{1}{N}. \tag{80}$$

At a finite N the normalization sum rule Eq.(78) is obviously fulfilled:

$$\int_{-N/2}^{N/2} dE' \left(-\frac{1}{N}\right) = -1. \tag{81}$$

However, if one takes the limit $N \to \infty$ in Eq.(80) before integrating, one obtains $Y_\infty = 0$, and the normalization sum rule will we violated:

$$\int_{-N/2}^{N/2} dE' \lim_{N\to\infty} Y_N(E,E') = 0. \tag{82}$$

This mechanism of violation of sum rules in the thermodynamic limit is called the *anomaly* and is well known in the field theory.

A remarkable property of the Wigner-Dyson level statistics is that in this case (as well for all invariant RM ensembles, even with shallow confinement potentials) the normalization sum rule is not violated and the anomaly does not occur. Let us show how this property follows from the plasma model Eq.(52). To this end we note that the density-density correlation function is given by the variational derivative of the mean density with respect to the confinement potential:

$$\langle \varrho(E)\, \varrho(E')\rangle - \langle \varrho(E)\rangle\, \langle \varrho(E')\rangle = -\frac{\delta\rho(E)}{\delta V(E')} \approx \rho_0^2\, [R_\infty(E-E') - 1], \tag{83}$$

where we assume that the mean density $\rho(E) \approx \rho_0$ does not change much at a scale of the mean level spacing Δ. In this case one can approximately consider the TLCF as a function of the energy difference. Then integrating by parts in Eq.(56) and neglecting the energy dependence of the square roots we obtain:

$$-\frac{\delta\rho(E)}{\delta V(E')} = -\frac{1}{\pi^2\beta}\, \frac{1}{(E-E')^2}, \tag{84}$$

where the regularization

$$(E-E')^{-2} \to \frac{1}{2}\, [(E-E'+i0)^{-2} + (E-E'-i0)^{-2}]$$

is assumed. Using this regularization one can immediately check that

$$\int_{-\infty}^{+\infty} [R_\infty(E) - 1]\, dE = 0,$$

as it is required by the sum rule Eq.(78) at $\rho(E) = \text{const}$. Thus we see that for the plasma model doing the limit $N \to \infty$ and doing the integral commute.

The absence of the anomaly is related with the incompressible character of the system of logarithmically repelling particles. This is the reason why approximations made in deriving the plasma model Eq.(52) did not affect the regular fulfillment of the normalization sum rule. Below we define the *level compressibility* and show that it is zero if the normalization sum rule is not violated. To this end we define the *level number variance*:

$$\Sigma(\bar{n}) = \langle n^2\rangle - \bar{n}^2, \tag{85}$$

where n is the fluctuating number of levels in an energy interval δ_E that contains on the average $\bar{n}$ levels. Writing

$$n = \int_{E_0-\delta E/2}^{E_0+\delta E/2} \rho(E)\, dE \equiv \int_{\delta_E} \rho(E)\, dE$$

we obtain:

$$\Sigma(\bar{n}) = \int_{\delta_E} \rho(E)\, dE \int_{\delta_E} \rho(E')\, [R_N(E,E') - 1]\, dE'.$$

Now we assume that N is large, the confinement is steep and thus $D \gg |\delta E| \gg \Delta$. This allows to consider $\rho(E) = \rho_0$ and $R_N(E,E') \approx R_\infty(E-E')$. Then one can integrate over E at a fixed $E - E'$ and arrive at:

$$\Sigma(\bar{n}) = \bar{n} \int_{-\bar{n}}^{\bar{n}} [R_\infty(s) - 1]\, ds - \int_{-\bar{n}}^{\bar{n}} |s|\, [R_\infty(s) - 1]\, ds, \tag{86}$$

where $s = (E - E')\rho_0$. The *level compressibility* is defined as

$$\chi(\bar{n}) = \frac{d\Sigma}{d\bar{n}} = \int_{-\bar{n}}^{\bar{n}} [R_\infty(s) - 1]\, ds. \tag{87}$$

In the case where the anomaly does not occur we have

$$\lim_{\bar{n}\to\infty} \int_{-\bar{n}}^{\bar{n}} [R_\infty(s) - 1]\, ds = \lim_{N\to\infty} \int_{-\infty}^{+\infty} [R_N(s) - 1]\, ds,$$

which in view of the normalization sum rule Eq.(78) implies incompressible nature of the system of energy levels:

$$\lim_{\bar{n}\to\infty} \chi(\bar{n}) = 0. \tag{88}$$

The opposite is also true: if Eq.(88) is fulfilled, the limits $\bar{n} \to \infty$ and $N \to \infty$ commute and there is no anomaly.

Now we use the absence of the anomaly and the normalization sum rule to compute the level number variance at $\bar{n} \gg 1$. To this end we cast Eq.(86) in the following way

$$\Sigma(\bar{n}) = \bar{n} \int_{-\infty}^{+\infty} [R_\infty(s) - 1]\, ds - 2\bar{n} \int_{\bar{n}}^{\infty} [R_\infty(s) - 1]\, ds - 2 \int_0^{\bar{n}} |s|\, [R_\infty(s) - 1]\, ds,$$

The first integral vanishes because of the sum rule Eq.(78), the second term requires only the knowledge of TLCF at large distances $s \gg 1$ and can be computed using Eq.(84) and appears to be a constant of order 1. The third integral is logarithmic and this allows to compute the leading logarithmic term also using Eq.(84) which is valid at $s \gg 1$. Cutting the logarithmic divergency at $s \sim 1$ we obtain for $\bar{n} \gg 1$:

$$\Sigma(\bar{n}) = \frac{2}{\pi^2 \beta} \ln \bar{n} + O(1). \tag{89}$$

This variance is considerably smaller than for independently fluctuating levels (diagonal RME) where it is distributed according to Poisson law:

$$\Sigma(\bar{n}) = \bar{n}. \tag{90}$$

XI. ORTHOGONAL POLYNOMIALS AND ENERGY LEVEL STATISTICS FOR $\beta = 2$.

As has been already mentioned level statistics in the Gaussian invariant RME with $\beta = 2$ can be exactly mapped onto the system of non-interacting fermions in one dimension in the parabolic confinement potential $V(E) = E^2$. Let us check this statement. To this end we recall that the ground state *many-body* wavefunction of non-interacting fermions is the *Slatter determinant*:

$$\Psi(\{E_n\}) = \begin{vmatrix} \varphi_0(E_1) & \varphi_0(E_2) & ... & \varphi_0(E_N) \\ \varphi_1(E_1) & \varphi_1(E_2) & ... & \varphi_1(E_N) \\ ... & ... & ... & ... \\ \varphi_{N-1}(E_1) & \varphi_{N-1}(E_2) & ... & \varphi_{N-1}(E_N) \end{vmatrix}. \tag{91}$$

The *one-particle* eigenfunctions $\varphi_n(E) = H_n(E)\, e^{-E^2/2}$ in the parabolic confinement potential $V(E)$ obeying the Schroedinger equation:

$$-\frac{1}{2} \frac{\partial^2}{\partial E^2} \varphi_n(E) + \frac{1}{2} E^2 \varphi_n(E) = \mathcal{E}_n \varphi_n(E), \tag{92}$$

are related to the Hermite *orthogonal polynomials* $H_n(E)$. These are the polynomials of the n-th order satisfying the orthogonality relation:

$$\int_{-\infty}^{+\infty} \exp[-V(x)]\, H_n(x)\, H_m(x) = h_n\, \delta_{nm}, \qquad V(x) = x^2, \quad h_n = 1. \tag{93}$$

It is important that the Hermite polynomials obey the *three-term* recursive relation:

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x). \tag{94}$$

Using this relation one can show that the Slatter determinant Eq.(91) reduces to:

$$\begin{vmatrix} \varphi_0(E_1) & \varphi_0(E_2) & ... & \varphi_0(E_N) \\ \varphi_1(E_1) & \varphi_1(E_2) & ... & \varphi_1(E_N) \\ ... & ... & ... & ... \\ \varphi_{N-1}(E_1) & \varphi_{N-1}(E_2) & ... & \varphi_{N-1}(E_N) \end{vmatrix} = \text{const} \begin{vmatrix} 1 & 1 & ... & 1 \\ E_1 & E_2 & ... & E_N \\ ... & ... & ... & ... \\ E_1^{N-1} & E_2^{N-1} & ... & E_N^{N-1} \end{vmatrix} \exp\left[-\frac{1}{2}\sum_{\mathrm{n=1}}^{\mathrm{N}} \mathrm{E_n^2}\right]. \tag{95}$$

Indeed, the exponential factors $e^{-\frac{1}{2}E^2}$ in all the $\varphi_n(E)$ can be taken out of the determinant using the rule of multiplication of determinant by a factor which is equivalent to multiplication of all the elements in a column by this factor. So we obtain the exponential factor in the r.h.s. of Eq.(95).

Next choosing $H_0(E) = 1$, $H_1(E) = x$ one can find all the other polynomials using the recursion relation Eq.(94). In particular, $H_2 = 2x^2 - 2$. This polynomial should be plugged in the third line of the determinant in the l.h.s. of Eq.(95). Note, however, that the constant term -2 can be omitted as its inclusion is equivalent to an addition of the first line to the third line in the determinant which according to the basic property of a determinant does not change its value. This process can be continued. For instance in $H_3 = 2xH_2 - 4H_1$ which stands in the fourth line of the determinant one can omit $-4H_1$, as H_1 stands in the second line. Then expressing $H_2 = 2xH_1 - H_0 = 2x^2 - 1$ one can put in the fourth line $2x(2x^2 - 1)$ instead of H_3. Finally, observing that the term $-2x$ can be considered as a linear combination of $4x^3$ and the first line in the determinant and omitting this term we conclude that instead of H_3 on can put in the fourth line of the determinant just one term $4x^3$.

The determinant in the r.h.s. is the famous *Vandermond determinant* which is equal to:

$$\begin{vmatrix} 1 & 1 & ... & 1 \\ E_1 & E_2 & ... & E_N \\ ... & ... & ... & ... \\ E_1^{N-1} & E_2^{N-1} & ... & E_N^{N-1} \end{vmatrix} = \prod_{n>m} (E_n - E_m). \tag{96}$$

Now we see that:

$$|\Psi(\{E_n\})|^2 = \text{const} \exp\left[-\sum_n E_n^2\right] \prod_{n>m} (E_n - E_m)^2. \tag{97}$$

This is exactly the probability distribution functions for the eigenvalues of the Gaussian RME with $\beta = 2$.

It turns out that the theory of orthogonal polynomials[7] is the powerful method to solve *any* orthogonal random matrix ensemble. What one has to do for that is to generate a set of orthogonal polynomials $p_n(x)$ obeying the orthogonality relation Eq.(93) and to be able to compute the large-N asymptotic behavior of the "wavefunctions":

$$\varphi_n(E) = \varphi_n(E) = h_n^{-1/2}\, p_n(E)\, e^{-V(E)/2}. \tag{98}$$

The generation of orthogonal polynomials is possible for any confinement potential $V(x)$ using the Gram-Schmidt orthogonalization procedure. According to this procedure one computes the Gram-Schmidt determinant:

$$G_n = \begin{vmatrix} a_0 & a_1 & ... & a_n \\ a_1 & a_2 & ... & a_{n+1} \\ ... & ... & ... & ... \\ a_n & a_{n+1} & ... & a_{2n} \end{vmatrix}, \tag{99}$$

where a_n are the moments:

$$a_n = \int_{-\infty}^{+\infty} \exp[-V(x)]\, x^n\, dx. \tag{100}$$

Then the orthogonal polynomial of n-th power p_n is given by:

$$p_n(x) = \frac{q_n}{G_{n-1}} \begin{vmatrix} a_0 & a_1 & ... & a_n \\ a_1 & a_2 & ... & a_{n+1} \\ ... & ... & ... & ... \\ a_{n-1} & a_n & ... & a_{2n-1} \\ 1 & x & ... & x^n \end{vmatrix}, \tag{101}$$

where q_n is the coefficient in front of x^n in p_n.

It follows from this generic procedure that any set of orthogonal polynomials $p_n(x)$ should obey the three-term recursion relation similar to Eq.(94). For the confinement potential $V(x)$ being an even function of x and the choice $q_n = 1$ (for the Hermite polynomials the standard definition corresponds to $q_n = 2^n$) it reads:

$$p_{n+1}(x) = xp_n(x) - C_{n+1}\, p_{n-1}(x), \qquad C_{n+1} = \frac{G_n\, G_{n-2}}{G_{n-1}^2}, \quad p_0(x) = 1, \quad p_1(x) = x. \tag{102}$$

We note that this recursion relation generates orthogonal but not *ortho-normal* polynomials. The price of having $q_n = 1$ is that the normalization constant h_n in Eq.(93) is not unity and is related to the coefficient C_n as follows:

$$h_n = h_0 \prod_{m=2}^{n+1} C_m, \quad h_0 = a_0. \tag{103}$$

The recursive relation Eq.(102) appears to be the most convenient way of generating orthogonal polynomials for *any* confinement potential. It works also for non-classical polynomials for which there are no second-order differential equations (similar to the Schroedinger equation Eq.(92) in the case of Hermite polynomials) which the "wavefunctions" Eq.(98) should obey.

The efficiency of the orthogonal polynomials in the problem of level statistics is largely due to the *Christoffel-Darboux formula*:

$$K_N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\, \varphi_n(y) = \sqrt{\frac{h_N}{h_{N-1}}}\, \frac{\varphi_{N-1}(x)\, \varphi_N(y) - \varphi_{N-1}(y)\, \varphi_N(x)}{y-x}. \tag{104}$$

This formula can be proven by induction using the three term recursive relation Eq.(102) and a relation Eq.(103) between h_n and C_n.

The Christoffel-Darboux formula is important because the mean density of states and the two level correlation function at $x \neq y$ are given by:

$$\rho(E) = \sum_{n=0}^{N-1} \varphi_n(x)^2 = K_N(E,E), \tag{105}$$

$$\rho(E)\, \rho(E')\, R_N(E,E') = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \left[\varphi_n^2(E)\, \varphi_m^2(E') - \varphi_n(x)\, \varphi_n(y)\, \varphi_m(y)\, \varphi_m(x)\right] = K_N(E,E)\, K_N(E',E') - K_N^2(E,E'). \tag{106}$$

Eqs.(105),(106) can be proven formally without any reference to systems of non-interacting fermions. However, it is instructive to see how they follow from the fermionic second quantization formalism. Indeed, the density of non-interacting fermions and the density-density correlation function are given by:

$$\langle 0|\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|0\rangle, \qquad \langle 0|\hat{\Psi}^\dagger(x)\hat{\Psi}(x)\, \hat{\Psi}^\dagger(y)\hat{\Psi}(y)|0\rangle,$$

where $\langle 0|...|0\rangle$ is the quantum-mechanical averaging over the ground state. According to the rules of second quantization the fermionic field operator $\hat{\Psi}(x)$ is given by the expansion over the single-particle wavefunctions:

$$\hat{\Psi}(x) = \sum_{n=0}^{N-1} \varphi_n(x)\, c_n,$$

where c_n and $c_n^\dagger$ are the fermionic creation and annihilation operators obeying the anti-commutation relation

$$c_n^\dagger\, c_m + c_m\, c_n^\dagger = \delta_{nm}.$$

The averages over the ground state can be computed using the Wick theorem:

$$\langle 0|c_n^\dagger c_m|0\rangle = \delta_{nm}, \qquad \langle 0|\, c_{n_1}^\dagger c_{n_2} c_{n_3}^\dagger c_{n_4}|0\rangle = \delta_{n_1 n_2}\,\delta_{n_3 n_4} - \delta_{n_1 n_4}\,\delta_{n_2 n_3}. \tag{107}$$

The two terms in Eq.(106) follow from the two terms in Eq.(107) which correspond to two possible parings of $c^\dagger$ and c.

Eq.(106) can be conveniently represented in terms of the determinant:

$$\rho(E)\,\rho(E')\,R_N(E,E') = \begin{vmatrix} K_N(E,E) & K_N(E,E') \\ K_N(E',E) & K_N(E',E') \end{vmatrix}. \tag{108}$$

Using the fermionic analogy and the Wick theorem one can prove that any multi-point level density correlation function can be represented in the form of a similar determinant:

$$\rho(E_1)\rho(E_2)...\rho(E_n)\,R(E_1,E_2...E_n) = \begin{vmatrix} K_N(E_1,E_1) & K_N(E_1,E_2) & ... & K_N(E_1,E_n) \\ K_N(E_2,E_1) & K_N(E_2,E_2) & ... & K_N(E_2,E_n) \\ ... & ... & ... & ... \\ K_N(E_n,E_1) & K_N(E_n,E_2) & ... & K_N(E_n,E_n) \end{vmatrix}. \tag{109}$$

We see that the analogy with non-interacting fermions allows to express any multi-point level density correlation function in terms of only one single kernel $K_N(x,y)$. The latter according to the Christoffel-Darboux theorem is a product of only two "wave functions" which require the knowledge of only two orthogonal polynomials $p_N(x)$ and $p_{N-1}(x)$. Thus the problem of energy level statistics is reduced to the problem of finding the asymptotic behavior of orthogonal polynomials of high order.

XII. WKB QUASI-CLASSICAL APPROXIMATION AND THE ONE-DIMENSIONAL "WIGNER CRYSTAL".

The semicircle law for the mean level density follows immediately and trivially from the free-fermion representation. Indeed, the density of one-dimensional fermions is directly related with the Fermi-momentum p_F by:

$$\frac{2p_F}{2\pi} = \rho, \qquad \hbar = 1. \tag{110}$$

When the density varies slowly at a scale of the Fermi wavelength, one can apply Eq.(110) *locally* thus relating the local Fermi-momentum $p_F(x)$ with the local density $\rho(x)$. The local Fermi momentum corresponds to the momentum of the highest occupied state in a parabolic potential

$$p_F = p_N(x) = \sqrt{2mE_{\rm kin}} = \sqrt{2\mathcal{E}_n - x^2} = \sqrt{2N+1-x^2}, \qquad m = 1. \tag{111}$$

Then the local density is obtained immediately from Eq.(110):

$$\rho(x) = \frac{1}{\pi}\,p_N(x) = \frac{1}{\pi}\,\sqrt{2N+1-x^2}. \tag{112}$$

This is the celebrated *semicircle law*.

In order to obtain the *density-density correlation function*, or the *two-level correlation function* one should work a little bit harder determining the large N asymptotic behavior of φ_n and applying Eq.(106).

In the case of Hermite polynomials the problem of the large-N asymptotic behavior can be solved quite easily. The reason is that there is the second-order differential equation (the Schroedinger equation) which the wavefunctions $\varphi_N(x)$ must obey. As is well known the solutions to the Schroediger equation corresponding to large quantum numbers bear the properties of classical motion in the corresponding potential. In quantum mechanics this corresponds to the "quasi-classical", or WKB approximation[8].

According to this approximation the wave function at $x^2 < 2N$ is proportional to:

$$\varphi_N(x) \propto [p_N(x)]^{-1/2} \begin{cases} \cos\left[\int_0^x p_N(x')\,dx'\right] & N \;\; is \;\; even \\ \sin\left[\int_0^x p_N(x')\,dx'\right] & N \;\; is \;\; odd \end{cases} \tag{113}$$

At large N we obtain:

$$p_N(x) \approx \sqrt{2N}, \qquad \int_0^x p_N(x')\,dx' \approx x\sqrt{2N}.$$

Then the kernel $K_N(x,y)$ is easily calculated using the Christoffel-Darboux formula Eq.(104):

$$K_N(x,y) = \text{const}\,\frac{\sin\left(\sqrt{2N}\,(x-y)\right)}{x-y}, \qquad K_N(x,x) = \text{const}\,\sqrt{2N}.$$

The normalization constant const $= 1/\pi$ is most easily found from the comparison of $K_N(x,x) = \rho(x)$ and the semi-circle mean level density Eq.(63) which at large N reduces to $\rho(x) \approx \sqrt{2N}/\pi = \rho_0$. Now, introducing mean level spacing $\Delta = \rho^{-1} \approx \pi/\sqrt{2N}$ we arrive at:

$$K_N(x,y) \to K(s) = \rho_0\,\frac{\sin(\pi\,s)}{\pi s}, \qquad s = \frac{x-y}{\Delta}, \qquad N \to \infty. \tag{114}$$

The two level correlation function Eq.(106) is then equal to:

$$R_\infty(x,y) = \delta(s) + 1 - \frac{\sin^2(\pi\,s)}{(\pi s)^2} = 1 - \frac{1}{2\pi^2\,s^2} + \frac{\cos(2\pi\,s)}{2\pi^2\,s^2}. \tag{115}$$

One can see that the TLCF given by Eq.(115) has all the asymptotic limits right. It is proportional to s^2 at $s \ll 1$ and its envelope corresponds to Eq.(84) for $s \gg 1$. In addition to that it obeys the normalization sum rule Eq.(78). However, in Eq.(115) there is a term that oscillates with the period of the mean level spacing Δ. This term evades the consideration based on the 2×2 matrix (small $s \ll 1$) and the effective continuous plasma model (large $s \gg 1$).

Let us discuss the physical meaning of this term using the plasma model analogy but without the continuous approximation. It is well known that plasma of particles with the long-range repulsion in a confinement potential tend to develop a crystal order known as Wigner crystal. Such Wigner crystal of electrons have been observed on top of the helium surface. Our case is special, as it is one-dimensional. According to the Mermin theorem the crystal order cannot survive in one dimensions at a finite temperature because thermal fluctuations destroy the long-range order. However, local crystal order may exist. The last oscillating them in Eq.(115) reflects exactly this order. The short-range nature of this order manifests itself in the fast s^{-2} decay of oscillations at large distances.

So far in this section we have considered the $\beta = 2$ case. As in the plasma analogy β plays a role of inverse temperature, one would expect the oscillating term to decay slower for $\beta = 4$ and faster for $\beta = 1$. This expectation is in fact true.

One can show using the more sophisticated application of the orthogonal polynomial machinery that in the limit $N \to \infty$ the two-level correlation functions for the orthogonal ($\beta = 1$) ensemble R_∞^{orth} and that for the symplectic ($\beta = 4$) ensemble R_∞^{symp} can also be expressed in terms of the kernel $K_N(s)$, Eq.(114):

$$Y_\infty^{\text{orth}}(s) = -K^2(s) - \frac{dK(s)}{ds}\int_s^\infty K(x)\,dx, \tag{116}$$

$$Y_\infty^{\text{symp}}(s/2) = -K^2(s) + \frac{dK(s)}{ds}\int_0^s K(x)\,dx. \tag{117}$$

The $s/2$ in the argument of $Y_\infty^{\text{symp}}(s/2)$ appears because of the Kramers degeneracy: for the same total number of levels N the mean level spacing between doubly degenerate levels is two times longer. Accordingly, the $\delta(s)$ function in $R(s)$ enters with the pre-factor of 2. The asymptotic behavior of these functions for $s \gg 1$ is the following:

$$Y_\infty^{\text{orth}}(s) = -\frac{1}{\pi^2\,s^2} + \frac{\cos(2\pi s)}{2\pi^4 s^4}, \tag{118}$$

$$Y_\infty^{\text{symp}}(s) = -\frac{1}{4\pi^2\,s^2} + \frac{\cos(2\pi\,s)}{4\,s} + \frac{\cos(4\pi\,s)}{2(2\pi\,s)^4}. \tag{119}$$

One can see that the leading oscillating term decrease as $s^{-4/\beta}$ as was expected. In addition to that, in the symplectic ensemble $\beta = 4$, the sub-leading second-harmonic term appears which was absent to all orders in $1/s$ for $\beta = 1, 2$.

The spectral correlations of a quantum system show up in the time-dependence of response to external time-dependent perturbations. For such applications one need to know the Fourier-transform $F(t)$ of the two-level correlation (cluster) function $Y_\infty(s)$. It appears to be amazingly simple for the unitary ensemble $\beta = 2$:

$$F^{\rm unit} = \begin{cases} |t| - 1, & |t| < 1 \\ 0, & |t| > 1. \end{cases} \tag{120}$$

with the jump of the first derivative at $t = 1$ that leads to the oscillations with the period 1 which amplitude decreases as $1/s^2$. For the orthogonal ensemble $\beta = 1$ there is a jump only in the third derivative:

$$F^{\rm ortho} = \begin{cases} 2|t| - 1 - |t| \, \ln(1 + 2|t|), & |t| < 1 \\ 1 - |t| \, \ln\left(\frac{2|t|+1}{2|t|-1}\right), & |t| > 1. \end{cases} \tag{121}$$

For the symplectic ensemble $\beta = 4$ there are *two* singular points: $|t| = 1$ and $|t| = 2$ which correspond to two oscillating terms in Eq.(119):

$$F^{\rm symp} = \begin{cases} \frac{1}{2} \, |t| - 1 - \frac{1}{4} \, |t| \, \ln|1 - |t|\,|, & |t| < 2 \\ 0, & |t| > 2. \end{cases} \tag{122}$$

XIII. WIGNER-DYSON LEVEL STATISTICS AND THE LUTTINGER LIQUID.

The large-s asymptotics of the two-level correlation function containing both the non-oscillating and the oscillating terms which decay as a certain power-law can be written in a compact form which involves only one single function $G(s)$:

$$Y_\infty^{\rm unit}(s) = -\frac{1}{4\pi^2} \, \frac{\partial^2 G(s)}{\partial s^2} + \cos(2\pi s) \, e^{G(s)}, \tag{123}$$

$$Y_\infty^{\rm ortho}(s) = -\frac{1}{2\pi^2} \, \frac{\partial^2 G(s)}{\partial s^2} + 2\cos(2\pi s) \, e^{2G(s)}, \tag{124}$$

$$Y_\infty^{\rm symp}(s) = -\frac{1}{8\pi^2} \, \frac{\partial^2 G(s)}{\partial s^2} + \frac{\pi}{\sqrt{8}} \, \cos(2\pi s) \, e^{G(s)/2} + \frac{1}{8} \, \cos(4\pi s) \, e^{2G(s)}, \tag{125}$$

where

$$G = -\ln(2\pi^2 s^2). \tag{126}$$

It turns out that the function $G(x)$ is proportional to the equal-time correlation function of a free bosonic field $\Phi(x, \tau)$ in the *two* dimensional space-(imaginary)time, which arises as a bosonized version of the Calogero-Sutherland model Eq.(51) of interacted fermions. More generally, a great number of models of interacted electrons in one dimension fall into the universality class of *Luttinger liquid*[9] which is characterized by a certain correlation functions at *large* separations in space and/or in time. All of them follow from the fact that the fermionic operator $\Psi(x, \tau)$ can be represented as

$$\Psi(x, \tau) = R \, e^{ik_F x} + L e^{-ik_F x}, \qquad R, \, L = \frac{1}{\sqrt{\pi}} \exp[\pm i \Phi_{R,L}(x, \tau)], \tag{127}$$

where $\Phi(x, \tau) = \frac{1}{2} \, [\Phi_R(x, \tau) + \Phi_L(x, \tau)]$ is the *free boson field* with the action:

$$S[\Phi] = \frac{1}{2\pi K} \int_0^{1/T} d\tau \int_{-\infty}^{+\infty} dx \, [(\partial_x \Phi)^2 + (\partial_\tau \Phi)^2] = \frac{L}{\pi K T} \sum_{k>0, \omega_n = 2\pi T n} |A_{k,\omega_n}|^2 \, (\omega_n^2 + k^2), \tag{128}$$

where

$$\Phi(x, \tau) = \sum_{k>0, \omega_n = 2\pi T \, n} \left\{ A_{k,\omega} e^{i(\omega_n \tau + kx)} + c.c \right\}, \tag{129}$$

$\omega_n = 2\pi T n$ (n are *all* integers) and $k = (2\pi/L)m > 0$ to avoid double-counting. The action Eq.(128) corresponds to:

$$\langle |A_{k,\omega}|^2 \rangle = \frac{\pi K T}{L} \frac{1}{k^2 + \omega_n^2}. \tag{130}$$

It is remarkable that *interaction* of fermions is encoded in only one single parameter K which is $K < 1$ for repulsion, $K = 1$ for the non-interacting fermions and $K > 1$ for attraction. In other words, with respect to long-distance properties the system of interacted fermions in one dimensions ($1 + 1$ space-time) is equivalent to a system of free bosons. The physical meaning of this result is that for systems of the Luttinger-liquid universality class all the multitude of effects of electron interaction reduces to dynamics and thermodynamics of the plasmon collective modes.

The density operator in this representation is given by:

$$\rho(x,\tau) - \rho_0 = \frac{1}{\pi}\,\partial_x \Phi(x,\tau) + A_1\,\cos[2k_F x + 2\Phi(x,\tau)] + A_2\,\cos[4k_F x + 4\Phi(x,\tau)] + ..., \tag{131}$$

where A_k are the structural constants which are determined from the details of the system at small distances.

The first term in Eq.(131) comes from the combination $[R^+R + L^+L]$ in $\Psi^\dagger\Psi$. It is analogous to the $\delta\rho \propto \nabla \mathbf{u}$ term in hydrodynamics, where $\mathbf{u}(x)$ is the mass displacement at a point x. The correct evaluation of this contribution requires the regularization $R^+R = R^+(x+a,\tau)R(x,\tau)$ (and the similar regularization for L), where $a = 1$ is the lattice constant which corresponds to $k_F = \pi$. Oscillating terms proportional to $e^{\pm 2ik_F x}$ arise from the cross combinations $R^+L + L^+R$ in $\Psi^\dagger\Psi$. Note that for interacting fermion system there is a vertex correction that involves momentum transfer far away from the Fermi points $\pm k_F$. As the result, the density operator expressed in terms of the field $\Psi(x,\tau)$ (which contains only momenta close to the Fermi points) is not simply equal to $\rho = \Psi^\dagger\Psi$ but may have higher order terms as well:

$$\rho(x,\tau) = \Psi^\dagger\Psi + c_1\,(\Psi^\dagger\Psi)^2 + ... \tag{132}$$

It is these higher order terms that generate combinations containing higher harmonics like $e^{\pm 4ik_F x}$.

Using Eq.(131) and Eq.(128) one can express the density-density correlation function through the free bosonic correlation function. To this end we use the identity valid for any Gaussian field theory:

$$\langle e^{in\Phi(x,\tau)}\, e^{-in\Phi(0,0)} \rangle = \exp\left\langle -\frac{n^2}{2}\,(\Phi(x,\tau) - \Phi(0,0))^2 \right\rangle_S.$$

Now observe that for $s \gg 1$ and $T \to 0$ we have in the thermodynamic limit $L \to \infty$:

$$\langle \Phi(s,0)\Phi(0,0)\rangle_S - \langle \Phi(0,0)\Phi(0,0)\rangle_S = -\frac{K}{2}\int_0^\pi dq\,\frac{1-\cos(qs)}{q} \approx \frac{K}{4}\,G(s),$$

where $G(s)$ is given by Eq.(126). Finally, for oscillating part of the density-density correlator we obtain:

$$\langle e^{in\Phi(s,\tau)}\, e^{-in\Phi(0,0)} \rangle = \exp\left\{ \frac{1}{4} K\, n^2\, G(s) \right\}. \tag{133}$$

The non-oscillating part is expressed through the second derivative of the correlation function of free boson field:

$$\frac{1}{\pi^2}\,\langle \partial_s \Phi(s,0)\partial_{s'}\Phi(s',0)\rangle_S = -\frac{K}{4\pi^2}\,\partial_s^2\,G(s). \tag{134}$$

Eqs.(133),(134) show that the phenomenology of the Luttinger liquid allows to relate the coefficient in front of the non-oscillating part of the density-density correlator with the coefficients in front of $G(s)$ in the exponent determining the amplitude of the oscillating terms. First of all we fix the interaction parameter K from the amplitude $1/(2\pi^2\beta)$ of the non-oscillating part. Eq.(134) suggests that:

$$K = \frac{2}{\beta}. \tag{135}$$

Then a comparison of Eq.(133) with the oscillating terms in Eqs.(123),(124),(125) shows that all coefficients κ in the amplitude of n-th harmonic $e^{\kappa\,G(s)}\,\cos(2\pi n\,x)$ are equal to $\kappa = n^2 K$ with K given by Eq.(135). This is exactly the parameter K that corresponds to the Calogero-Sutherland model Eq.(51). Thus we have demonstrated that the Wigner-Dyson level statistics at large level separations corresponds to the particle statistics of the Calogero-Sutherland model at zero temperature.

One can ask a question: how the Wigner-Dyson ensemble should be deformed in order to retain this analogy with the Calogero-Sutherland model also for finite temperatures $T \neq 0$. The answer is[10] that the proper deformation is given by the Gaussian non-invariant ensemble Eq.(5) at large values of the parameter B. The corresponding temperature of the Calogero-Sutherland model is[10]:

$$T = \frac{1}{4\beta B}. \tag{136}$$

The asymptotics of the correlation functions is given by Eq.(123)-(125) where one should substitute the deformed function $G(s)$ in a compactified space-time rolled into a cylinder of the circumference $1/T$ in the τ -direction:

$$G_T(s) = -\ln\left(2\pi^2 \frac{\sinh^2(\pi T s)}{(\pi T)^2}\right). \tag{137}$$

This function is proportional to the Green's function of the free bosonic field Eq.(128) at a finite temperature T and can be obtained from Eq.(130) by *summing* over Matsubara frequencies $\omega_n = 2\pi T\, n$ instead of integrating over ω.

It is seen from Eqs.(123)-(125) that breaking the basis invariance by introducing the finite bandwidth B and the corresponding temperature T of the bosonic system has an effect of making the amplitudes of the oscillating terms exponentially decaying for $s \gg 1/(\pi T) = (4\beta\, B)/\pi$.

XIV. FIELD THEORIES FOR RANDOM MATRIX ENSEMBLES

In this section we derive the field theory for an arbitrary Gaussian random matrix ensemble. This formalism, known as nonlinear super-symmetric sigma-model[11] has been first applied to the Wigner-Dyson random matrix ensemble. However, its real strength is in the possibility of extension to the non-invariant random matrix ensembles as well as to real disordered conductors with diffusive dynamics of particles.

We start by writing down the expression for retarded (G^R) or advanced G^A Green's function in terms of the functional integral over complex variables φ_n:

$$G^{R/A}_{nm} = \left([E_\pm - \mathbf{H}]^{-1}\right)_{nm} = \frac{\mp i}{Z} \int \mathcal{D}\varphi\, \mathcal{D}\varphi^*\ \varphi_n\, \varphi^*_m \exp\left[iS_\pm[\varphi]\right], \tag{138}$$

where

$$S_\pm[\varphi] = \pm \sum_{i,j} \varphi^*_i\ \left[E_\pm \delta_{ij} - H_{ij}\right]\ \varphi_j \tag{139}$$

and $E_\pm = E \pm (\omega/2 + i0)$. The sign $\pm$ stands for G^R or G^A, respectively. There would be no problem to average Eq.(138) over the Gaussian random entries H_{nm} if not the normalization constant (partition function) Z:

$$Z = \int \mathcal{D}\varphi\, \mathcal{D}\varphi^* \exp\left[iS_\pm[\varphi]\right]. \tag{140}$$

which also depends on H_{nm}. With the Z present one has a problem, the *problem of denominator*.

There are different ways of overcoming this problem, e.g. the *replica trick*. However, here we use another trick, the *super-symmetry* method[11]. In the core of this method is the calculus of anti-commuting (Grassmann) variables μ_m:

$$\mu_n\, \mu_m = -\mu_m\, \mu_n, \quad \mu_n^2 = 0. \tag{141}$$

One can define the Grassmann integral with the shortest table of integrals ever:

$$\int \mu_n\, d\mu_n = \frac{1}{\sqrt{\pi}}, \qquad \int d\mu_n = 0. \tag{142}$$

Since any function $f(\mu_n) = f(0) + f'(0)\,\mu_n$, the Grassmann integration is essentially a differentiation. Now let us compute the integral

$$\int \prod_i d\mu^*_i d\mu_i \exp\left[-\sum_n \mu^*_n\, a_n\, \mu_n\right] = \int \prod_i d\mu^*_i d\mu_i\, \mu^*_i \mu_i\, (-a_i) = \int \prod_i \mu^*_i d\mu^*_i\ \mu_i d\mu_i\, a_i = \prod_i \frac{a_i}{\pi}. \tag{143}$$

To accomplish this we expand the exponential function to leave only the term that contains a *complete* set $\mu_1^*\mu_1 ... \mu_n^*\mu_n$ of Grassmann variables and apply the table of integration Eq.(142).

The corresponding integral over the usual complex variables would give the following result:

$$\int \prod_i d\varphi_i^* d\varphi_i \exp\left[-\sum_n \varphi_n^*\, a_n\, \varphi_n\right] = \prod_i \frac{\pi}{a_i}. \tag{144}$$

We see a remarkable property: the product of the two integrals is equal to 1. This property remains true for any Gaussian integrals of commuting and anti-commuting variables. In particular,

$$Z^{-1} = \int \mathcal{D}\mu^*\, \mathcal{D}\mu \exp\left[iS_\pm[\mu]\right]. \tag{145}$$

Now the Green's functions can be represented without the denominator:

$$G_{nm}^{R/A} = \mp i \int \mathcal{D}\psi \quad \varphi_n\, \varphi_m^* \exp\left[iS_\pm[\psi]\right], \tag{146}$$

where

$$S_\pm[\psi] = S_\pm[\varphi] + S_\pm[\mu] = \pm \sum_{i,j} \psi_i^\dagger\, \left[E_\pm \delta_{ij} - H_{ij}\right]\, \psi_j. \tag{147}$$

Here we introduced the *super-vectors* ψ and $\psi^\dagger$:

$$\psi^\dagger = (\varphi^*, \mu^*), \qquad \psi = \begin{pmatrix} \varphi \\ \mu \end{pmatrix} \tag{148}$$

and the super-measure:

$$\mathcal{D}\psi = \mathcal{D}\varphi^*\, \mathcal{D}\varphi\, \mathcal{D}\mu^*\, \mathcal{D}\mu. \tag{149}$$

The action Eq.(147) and the integration measure Eq.(149) are *super-symmetric*, i.e. the commuting and anti-commuting variables enter in a fully symmetric way. The super-symmetry is however broken in the *pre-exponent* in Eq.(146), as it depends only on the commuting variables.

Now when the problem of denominator is solved by the supersymmetry trick, the next step is to average over the Gaussian ensemble of H_{ij}. To this end we write:

$$\mp i \sum_{ij} \psi_i^\dagger \psi_j H_{ij} = \frac{\mp i}{2} \sum_{ij} \left(\psi_i^\dagger \psi_j H_{ij} + \psi_j^\dagger \psi_i H_{ji}\right)$$

Averaging of the r.h.s. is done independently for each pair of i, j using the identity:

$$r.h.s. - \frac{1}{A_{ij}} |H_{ij}|^2 = -\frac{1}{A_{ij}} \left(H_{ij} \pm \frac{i}{2} A_{ij}\, \psi_i^\dagger \psi_j\right) \left(H_{ji} \pm \frac{i}{2} A_{ij}\, \psi_j^\dagger \psi_i\right) - \frac{A_{ij}}{4}\, \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i.$$

From now on for simplicity we will consider the case $\beta = 2$. Then

$$\int dH_{ij} dH_{ij}^* \exp\left(r.h.s. - \frac{1}{A_{ij}} |H_{ij}|^2\right) = \int d\tilde{H}_{ij} d\tilde{H}_{ij}^* \exp\left(-\frac{1}{A_{ij}} |\tilde{H}_{ij}|^2\right) \exp\left(-\frac{A_{ij}}{4}\, \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i\right),$$

where

$$\tilde{H}_{ij} = H_{ij} \pm \frac{i}{2} \psi_i^\dagger \psi_j.$$

The simplicity of the case $\beta = 2$ is that $\tilde{H}_{ij}$ belongs to the same manifold of complex numbers as H_{ij}, so that one may replace in the integral over the entire manifold $\tilde{H}_{ij} \to H_{ij}$. Thus on the right hand side we obtain the normalization integral for the random matrix ensemble averaging. So we obtain for the disorder average:

$$\left\langle \exp\left(\mp i \sum_{ij} \psi_i^\dagger \psi_j H_{ij}\right) \right\rangle = \exp\left(-\frac{1}{4} \sum_{ij} A_{ij}\, \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i\right). \tag{150}$$

Now we define the super-matrix

$$\tilde{Q}_i = \psi_i \otimes \psi_i^\dagger = \begin{pmatrix} \varphi_i\varphi_i^* & \varphi_i\mu_i^* \\ \mu_i\varphi_i^* & \mu_i\mu_i^* \end{pmatrix} \equiv \begin{pmatrix} BB & BF \\ FB & FF \end{pmatrix} \tag{151}$$

and the super-trace:

$$\mathrm{STr}\begin{pmatrix} BB & BF \\ FB & FF \end{pmatrix} = BB - FF. \tag{152}$$

Then Eq.(150) can be conveniently rewritten as

$$\exp\left(-\frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{STr}[\tilde{Q}_i\tilde{Q}_j]\right).$$

Finally, the averaged Green's function can be represented as follows:

$$\langle G^{R/A}_{nm}\rangle = \mp i\int \mathcal{D}\psi \quad \varphi_n\,\varphi^*_m \exp\left[-F[\tilde{Q}]\right], \qquad F[\tilde{Q}] = \mp iE_\pm \sum_i \mathrm{STr}[\tilde{Q}_i] + \frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{STr}[\tilde{Q}_i\tilde{Q}_j], \quad \tilde{Q}_i = \psi_i\otimes\psi_i^\dagger \tag{153}$$

Thus we derived the *deterministic field theory* which is equivalent to the Gaussian random matrix ensemble and is suitable to compute the average Green's function. One can see that the matrix of variances

$$A_{ij} = \langle |H_{ij}|^2\rangle$$

plays a role of the coupling constant ("coupling matrix") in the corresponding action $F[\tilde{Q}]$.

The filed-theory representation Eq.(153) can be extended to consider the averaged product of $\langle G^R G^A\rangle$ which is necessary to be able to compute the *two-point* correlation functions. To this end, one introduce the double set of commuting and anti-commuting variables: one for G^R and another for G^A and then repeats with minor modifications all the above steps:

$$\left\langle G^R_{nm}\left(E+\frac{\omega}{2}\right)G^A_{n'm'}\left(E-\frac{\omega}{2}\right)\right\rangle = \int \mathcal{D}\Psi \quad \varphi^R_n\,\varphi^{R*}_m\,\varphi^A_{n'}\,\varphi^{A*}_{m'} \;\exp\left[-F[\bar{Q}]\right], \tag{154}$$

where the action $S[\bar{Q}]$ takes the form:

$$F[\bar{Q}] = -iE\sum_i \mathrm{STr}[\bar{Q}_i] - i\frac{\omega+i0}{2}\sum_i \mathrm{STr}[\Lambda\bar{Q}_i] + \frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{STr}[\bar{Q}_i\bar{Q}_j], \quad \bar{Q}_i = \Psi_i\otimes\bar{\Psi}_i, \tag{155}$$

with

$$\bar{\Psi} = \begin{pmatrix}\varphi^{R*} & \mu^{R*} & -\varphi^{A*} & -\mu^{A*}\end{pmatrix}, \quad \Psi = \begin{pmatrix}\varphi^R \\ \mu^R \\ \varphi^A \\ \mu^A\end{pmatrix}, \quad \Lambda = \Lambda_2\otimes\mathbf{1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{156}$$

There is another, in some sense *dual*, field-theory representation similar to Eq.(154). In contrast to Eq.(154) it involves the *inverse coupling matrix* $(A^{-1})_{ij}$ rather than the variance matrix A_{ij}. In order to obtain this representation one makes the *Hubbard-Stratonovich transformation*:

$$\exp\left\{-\frac{1}{4}\sum_{ij} A_{ij}\,\mathrm{STr}[(\Psi_i\otimes\bar{\Psi}_i)(\Psi_j\otimes\bar{\Psi}_j)]\right\} = \int \mathcal{D}P \exp\left\{-\sum_{ij} A^{-1}_{ij}\,\mathrm{STr}[P_iP_j] + i\sum_i \mathrm{STr}[(\Psi_i\otimes\bar{\Psi}_i)P_i]\right\}, \tag{157}$$

where P_i is a super-matrix field.

If one substitutes the last term in Eq.(155) for Eq.(157) the remaining integral over the Ψ fields is Gaussian which can be done using a generalization of Eqs.(143)-(144) for the case where the super-matrix K_i is not proportional to the unity matrix (for which $\mathrm{STr}[\ln \mathrm{K_i}] = 0$):

$$\int \mathcal{D}\Psi \;\exp\left\{\bar{\Psi}_i\,K_i\,\Psi_j\right\} = \exp\left\{-\sum_i \mathrm{STr}[\ln K_i]\right\}. \tag{158}$$

Now we are in a position to write down the full action of the dual representation:

$$F[P] = \sum_{ij} A_{ij}^{-1}\,\mathrm{STr}[P_i P_j] + \sum_i \mathrm{STr}[\ln(E - P_i + (\omega/2)\,\Lambda)]. \tag{159}$$

In order to appreciate the possibilities this representation is offering and also for further simplifications of Eq.(159) we compute the inverse coupling matrix A_{ij}^{-1} for the important case of the banded random matrix ensembles where the variance matrix A_{ij} is given by Eq.(4).

The matrix element of a matrix inverse with respect to a matrix $A_{ij} = A(i-j)$ is given by:

$$A_{ij}^{-1} = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, G(k)\, e^{ik(i-j)}, \quad G(k) = \left[\sum_{m=-\infty}^{+\infty} \tilde{A}(k + 2\pi\, m)\right]^{-1}, \quad \tilde{A}(k) = \int_{-\infty}^{+\infty} dr\, A(r)\, e^{-ikr}. \tag{160}$$

Note that the summation over the reciprocal lattice vector $2\pi m$ is important.

For the case of the exponential $A(r) = e^{-|r|/B}$, we have:

$$A(k) = \frac{2B}{1 + B^2 k^2}, \quad \sum_{m=-\infty}^{+\infty} \frac{2B}{1 + B^2(k + 2\pi m)^2} = \frac{\sinh\left(\frac{1}{B}\right)}{\cosh\left(\frac{1}{B}\right) - \cos k}.$$

One can see that for large $B \gg 1$ the inverse coupling matrix A_{ij}^{-1} is extremely simple:

$$G(k) \approx B\,(1 - \cos k), \quad A_{ij}^{-1} \approx \frac{B}{2}\,(2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}), \quad A_0^{-1} = \sum_j A_{ij}^{-1} \approx \frac{1}{2B}. \tag{161}$$

Eq.(161) shows that locally it is just the lattice second derivative. However, the pre-factor in front of it is large $\sim B$ and it becomes infinite for the Wigner-Dyson ensemble. This means a large cost of variation of P_i in the space. Let us consider $P_i = P_0$ being independent of i to the first approximation. Then the action becomes:

$$N^{-1}\,F[P_0] = A_0^{-1}\,\mathrm{STr} P_0^2 + \mathrm{STr}\ln(E - P_0). \tag{162}$$

Here we also neglected a term proportional to ω which is legitimate as long as $\omega \ll \sqrt{B}$. Minimizing the action with respect to P_0 we obtain a saddle-point equation:

$$P_0(E - P_0) = A_0/2. \tag{163}$$

The solution to this saddle-point equation is *degenerate*:

$$P_0 = \frac{1}{2}\,\left(E + iQ\,\sqrt{2A_0 - E^2}\right), \tag{164}$$

where Q is a super-matrix obeying the constraint

$$Q^2 = 1. \tag{165}$$

Another constraint comes from the requirement of super-symmetry:

$$\mathrm{STr} Q = 0. \tag{166}$$

Indeed, a super-matrix Q obeying the constraint Eq.(165) after the diagonalization must contain only ± 1 on the diagonal. The super-symmetry (the symmetry between commuting and anti-commuting variables) implies that it must have the same diagonal elements corresponding to fermionic and bosonic variables of the given type (R or A). But then necessarily $\mathrm{STr} Q = 0$.

Now take into account (as the first order expansion in ω) the term in Eq.(159)proportional to the energy difference ω and allow for slow variations of the field $Q = Q_i$ in space which do not violate the saddle-point condition Eq.(165). Then we obtain plugging Eq.(164) into Eq.(159):

$$F[Q] = -\frac{1}{4}\,(\pi\rho\, A_0)^2 \sum_{ij} A_{ij}^{-1}\,\mathrm{STr}[Q_i Q_j] - i\frac{\pi\rho\omega}{2} \sum_i \mathrm{STr}[\Lambda Q_i], \tag{167}$$

where

$$\rho = \rho_0(E) = \frac{\sqrt{2A_0 - E^2}}{\pi A_0}, \quad A_0 = \sum_i A_{ij}. \tag{168}$$

This is the celebrated action of the *nonlinear σ model*[4,11].

For the particular case of banded random matrices with A_{ij} given by Eq.(4) we obtain:

$$F[Q] = -D \sum_i \mathrm{STr}[(Q_i - Q_{i+1})^2] - i\frac{\pi\rho\omega}{2} \sum_i \mathrm{STr}[\Lambda Q_i], \tag{169}$$

where $D = \frac{1}{2}B\left(B - \frac{E^2}{4}\right) \sim B^2$. The continuous limit of this model is the diffusive nonlinear σ-model:

$$F[Q] = -D \int dx\ \mathrm{STr}[(\nabla Q)^2] - i\frac{\pi\rho\omega}{2} \int dx\ \mathrm{STr}[\Lambda Q(x)], \tag{170}$$

which was originally derived by Efetov[11] to describe the crossover from the diffusive dynamics to the Anderson localization in the quasi-one dimensional multi-channel disordered wire. This demonstrates the isomorphism of the problem of quasi-one dimensional localization and the problem of banded random matrices[4].

Closing this chapter we note that the derivation of Eq.(167) from the exact Eq.(159) requires the *saddle-point approximation* Eq.(163). Thus Eq.(167) is justified only if the energy cost of space variations of Q_i is high. This happens when the variance matrix A_{ij} has a form of a banded matrix which is approximately constant at $|i-j| < B$, where $B \gg 1$. For the Wigner-Dyson ensemble $A_{ij} = 1$ and the bandwidth is maximum possible $B \sim N$ (in particular, $A_0 = N$). In the limit $N \to \infty$ all spacially varying configurations of the field Q (*non-zero modes*) are strictly forbidden. Neglecting them we obtain the *zero-mode* nonlinear sigma-model[11] which describes the statistics of energy levels in fully chaotic quantum systems of confined geometry (*quantum dots*):

$$F_{WD}[Q] = -i\frac{\pi\, s}{2}\ \mathrm{STr}[\Lambda Q], \quad s = \frac{\omega}{\Delta}. \tag{171}$$

XV. HOW TO COMPUTE OBSERVABLES: SEMI-CIRCLE LAW FROM THE SOLUTION TO A QUADRATIC EQUATION

Let us demonstrate how to compute observable quantities within the field theory using the simplest example of the mean density of states. It is given by

$$\rho(E) = (-2\pi i)^{-1} \left(\langle G^R_{nn}(E)\rangle - \langle G^A_{nn}(E)\rangle\right) = \frac{1}{2\pi} \int \mathcal{D}\Psi\ (\varphi^{R*}_n \varphi^R_n + \varphi^{A*}_n \varphi^A_n)\ e^{-F[\bar{Q}]}. \tag{172}$$

One can check that the pre-exponent in Eq.(172) can be represented as

$$(\varphi^{R*}_n \varphi^R_n + \varphi^{A*}_n \varphi^A_n) = \mathrm{STr}[\Pi\, \Psi_n \otimes \bar{\Psi}_n], \tag{173}$$

where

$$\Pi = \Pi^R - \Pi^A, \qquad \Pi^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi^A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{174}$$

Now we introduce an infinitesimal *source field* h_n and add to the action Eq.(155) a term

$$\delta F[\bar{Q}, h] = -i \sum_i h_i\, \mathrm{STr}[\Pi\, \bar{Q}_i].$$

One can easily check that the density of states is given by a differentiation of the partition function with respect to the field h:

$$\rho(E) = \frac{1}{2\pi i}\ \frac{\partial}{\partial h_n} \int \mathcal{D}\Psi\ e^{-F[\bar{Q},h]}\Big|_{h\to 0}, \qquad F[\bar{Q}, h] = F[\bar{Q}] + \delta F[\bar{Q}, h]. \tag{175}$$

Note that the additional term in the action proportional to h enters exactly like the term proportional to ω, so that in the final action of the sigma-model Eq.(167) one can simply substitute

$$\frac{\omega}{2}\,\Lambda \to \frac{\omega}{2}\,\Lambda + h_n\,\Pi. \tag{176}$$

Then Eq.(175) results in:

$$\rho(E) = \rho_0(E)\,\frac{1}{2}\int \mathcal{D}Q\ \mathrm{STr}[\Pi Q_n]\ e^{-F[Q]}, \tag{177}$$

where $\rho_0(E)$ is given by Eq.(168) and $F[Q]$ is given by Eq.(167) at $\omega = 0$. We see that the quantity $\rho_0(E)$ which appear in Eq.(167) from the solution of the quadratic saddle-point equation Eq.(163) is not accidentally of the form of a semi-circle as the mean density of states is proportional to it. In the case of the Wigner-Dyson ensemble the functional $F[Q]$ at $\omega = 0$ is simply zero and the integral in Eq.(177) is a constant independent of E. Thus we conclude that the semicircle law appears in this formalism from the solution of a quadratic saddle-point equation.

Other statistics such as the two-point correlation functions can also be easily computed using the formalism of the nonlinear sigma model, however not so simply as the semi-circle law. For this one needs a proper parametrization of the matrix Q which resolves the constraints Eqs.(165),(166).

XVI. SYMMETRY OF SUPER-MATRICES $\bar{Q}$ AND Q.

Let us return back to the derivation of the functional representation in terms of $\bar{Q}$. It appears[12] that by a change of variables:

$$\begin{aligned}
\varphi^{R/A} &= \pm i\sqrt{\lambda_{1/2}}\ e^{\pm i\varphi/2 + i\Omega}\,(1 - \frac{1}{2}\chi^*_{R/A}\chi_{R/A}) \\
\mu^{R/A} &= \pm i\sqrt{\lambda_{1/2}}\ e^{\pm i\varphi/2 + i\Omega}\,\chi_{R/A},
\end{aligned} \tag{178}$$

where $\lambda_{1/2} \geq 0$, $0 \leq \varphi \leq 2\pi$, $0 \leq \Omega \leq \pi$ and $\chi_{R/A}, \chi^*_{R/A}$ are the new anti-commuting variables, one can represent $\bar{Q} = \Psi \otimes \bar{\Psi}$ in the following form:

$$\bar{Q} = U\,\bar{\Sigma}\,U^{-1} = \begin{pmatrix} u_R & 0 \\ 0 & u_A \end{pmatrix} \begin{pmatrix} \bar{\Sigma}_{RR} & \bar{\Sigma}_{RA} \\ \bar{\Sigma}_{AR} & \bar{\Sigma}_{AA} \end{pmatrix} \begin{pmatrix} u_R^{-1} & 0 \\ 0 & u_A^{-1} \end{pmatrix}. \tag{179}$$

The beauty of this form is that the commuting and anti-commuting variables are separated by factorization. Namely, the outer matrices U, U^{-1} containing 2×2 matrices $u_{R/A}$ and $u^{-1}_{R/A}$

$$u_{R/A} = \begin{pmatrix} 1 - \frac{1}{2}\chi^*_{R/A}\chi_{R/A} & -\chi^*_{R/A} \\ \chi_{R/A} & 1 + \frac{1}{2}\chi^*_{R/A}\chi_{R/A} \end{pmatrix}_{BF}, \quad u^{-1}_{R/A} = \begin{pmatrix} 1 - \frac{1}{2}\chi^*_{R/A}\chi_{R/A} & \chi^*_{R/A} \\ -\chi_{R/A} & 1 + \frac{1}{2}\chi^*_{R/A}\chi_{R/A} \end{pmatrix}_{BF} \tag{180}$$

are made of the anti-commuting variables. The inner matrix $\bar{\Sigma}$

$$\bar{\Sigma} = \begin{pmatrix} \bar{\Sigma}_{RR} & \bar{\Sigma}_{RA} \\ \bar{\Sigma}_{AR} & \bar{\Sigma}_{AA} \end{pmatrix}_{RA} \equiv \begin{pmatrix} \bar{\Sigma}_{BB} & 0 \\ 0 & \bar{\Sigma}_{FF} \end{pmatrix}_{BF},$$

which is diagonal in the FB space, contains only commuting variables with only BB sector non-zero:

$$\bar{\Sigma}_{BB} = \begin{pmatrix} \lambda_1 & \sqrt{\lambda_1\lambda_2}\,e^{i\varphi} \\ -\sqrt{\lambda_1\lambda_2}\,e^{-i\varphi} & -\lambda_2 \end{pmatrix}_{RA}, \quad \bar{\Sigma}_{FF} = 0. \tag{181}$$

One can show that the factorized form Eq.(179) is common to both the field $\bar{Q}$ and the *dual field* Q, with matrices U, U^{-1} being exactly the same. However the structure of the inner matrices $\bar{\Sigma}$ and Σ are different. Efetov has shown[11] that the constraints Eqs.(165),(166) give rise to the following structure of Σ:

$$\Sigma_{BB} = \begin{pmatrix} \lambda & \sqrt{\lambda^2 - 1}\,e^{i\varphi} \\ -\sqrt{\lambda^2 - 1}\,e^{-i\varphi} & -\lambda \end{pmatrix}_{RA}, \tag{182}$$

$$\Sigma_{FF} = \begin{pmatrix} \lambda_F & \sqrt{1 - \lambda_F^2}\,e^{i\varphi_F} \\ \sqrt{1 - \lambda_F^2}\,e^{-i\varphi_F} & -\lambda_F \end{pmatrix}_{RA}, \tag{183}$$

where $\lambda \geq 1$, $-1 \leq \lambda_F \leq 1$, and $\varphi, \varphi_F \in (0, 2\pi)$.

To make practical calculations possible we also give (without derivation) the expressions for the Jacobians of the transformation from original variables to the variables of the above paramerization. They are

$$J[\bar{Q}] = \frac{\pi}{4\,\lambda_1\lambda_2}, \tag{184}$$

and

$$J[Q] = \frac{1}{8(\lambda - \lambda_F)^2}, \tag{185}$$

for the theories with coupling matrices A_{ij}^{-1} and A_{ij}, respectively.

The matrices $\bar{\Sigma}_{BB}$ of the structure Eq.(181) as well as the matrices Σ_{BB} of the structure Eq.(182) can be diagonalized by the *pseudo-unitary* rotation R:

$$\bar{\Sigma}_{BB} = R\,\bar{D}\,R^{-1}, \quad \Sigma_{BB} = R\,D\,R^{-1}, \quad R^{-1} = \Lambda_2\,R^{\dagger}\,\Lambda_2, \tag{186}$$

where

$$\bar{D} = \begin{pmatrix} |\lambda_1 - \lambda_2|\;\theta(\lambda_1 - \lambda_2) & 0 \\ 0 & -|\lambda_1 - \lambda_2|\;\theta(\lambda_2 - \lambda_1) \end{pmatrix}, \quad D = \Lambda_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{187}$$

It is clear that the rotation matrix R can be multiplied by a diagonal matrix

$$\begin{pmatrix} e^{i\Phi_R} & 0 \\ 0 & e^{i\Phi_A} \end{pmatrix} \in U(1) \otimes U(1)$$

without violating the condition of pseudo-unitarity and without changing the matrix Σ_{BB} or $\bar{\Sigma}_{BB}$. To eliminate the redundant degrees of freedom (which lead to the divergency of the functional integrals) the group of pseudo-unitary matrices $U(1,1)$ should be factorized as $R\,(U(1) \otimes U(1))$, where R being a factor-group $\frac{U(1,1)}{U(1)\otimes U(1)}$.

On top of that the diagonal matrix $\bar{D}$ has a free parameter

$$\lambda_1 - \lambda_2 \in \mathbf{R}.$$

The complete symmetry of the manifold of matrices $\bar{\Sigma}_{BB}$ and thus the complete symmetry of $\bar{Q}$ is:

$$\bar{Q} \in \frac{\mathbf{U(1,1)}}{\mathbf{U(1)} \otimes \mathbf{U(1)}} \otimes \mathbf{R}.$$

In contrast to that the symmetry of matrices Σ_{BB} is simply $\Sigma_{BB} \in \frac{U(1,1)}{U(1)\otimes U(1)}$. Its counterpart Σ_{FF} has the symmetry $\Sigma_{FF} \in \frac{U(2)}{U(1)\otimes U(1)}$ as it can be diagonalized by the *unitary* rotation matrix R_F. The complete symmetry of the Q field in the Efetov's nonlinear sigma-model is therefore:

$$Q \in \frac{\mathbf{U(1,1)}}{\mathbf{U(1)} \otimes \mathbf{U(1)}} \otimes \frac{\mathbf{U(2)}}{\mathbf{U(1)} \otimes \mathbf{U(1)}}.$$

Note by passing that the number of independent variables in $\bar{Q}$ and Q is different. While both have 4 anti-commuting variables, the number of commuting variables is $2 + 2 = 4$ for the filed Q and $2 + 1 = 3$ or the field $\bar{Q}$.

Thus we see that the duality transformation and the saddle-point approximation not only invert the coupling matrix A_{ij} but also change the symmetry of the *target space*. Such type of duality is encountered in the string theory and is called $T - duality$.

XVII. EIGENFUNCTION STATISTICS

In this section we show how to compute eigenfunction statistics using the field theory formalism. As usual, the starting point is to express the physical quantity of interest in terms of the Green's functions. To this end we study the product:

$$K_{l,m} = [G^R_{nn}]^l\,[G^A_{nn}]^m = \left(\sum_i \frac{|\Psi_i(n)|^2}{E - E_i + i\delta}\right)^l \left(\sum_{i'} \frac{|\Psi_{i'}(n)|^2}{E - E_{i'} - i\delta}\right)^m, \tag{188}$$

where we used the representation of Green's functions in terms of exact eigenfunction $\Psi_i(n)$ and exact eigenvalues E_n of a random matrix Hamiltonian. Let us multiply Eq.(188) by an infinitesimal δ^{l+m-1} average over realizations of the random matrix ensemble and do the limit $\delta \to 0$. This trick singles out only one state of the double sum, the one that is accidentally at the energy E:

$$\lim_{\delta\to 0} \delta^{l+m-1}\, K_{l,m} = \lim_{\delta\to 0} \left\langle \sum_i \frac{|\Psi_i(n)|^{2(l+m)}\ \delta^{l+m-1}}{(E-E_i+i\delta)^l (E-E_i-i\delta)^m}. \right\rangle \tag{189}$$

The smallness of the interval $|E-E_i| \sim \delta$ is the reason why the power of δ is $l+m-1$ and not $l+m$. Indeed, let the joint probability distribution function for $\Psi_i(n)$ and E_i be $P(\Psi_i, E_i)$. It is a smooth function of E_n which does not change at a scale $\delta \to 0$. Then averaging in Eq.(189) can be performed as follows:

$$\sum_i \int d\Psi_i dE_i\, P(\Psi_i, E_i)\, \frac{|\Psi_i|^{2(l+m)}}{(E-E_i+i\delta)^l (E-E_i-i\delta)^m} \approx \sum_i \int d\Psi_i\, P(\Psi_i, E)\, |\Psi_i|^{2(l+m)}\, C_{l,m}, \tag{190}$$

where

$$C_{l,m} = \int_{-\infty}^{+\infty} \frac{dE_i}{(E-E_i+i\delta)^l (E-E_i-i\delta)^m} = (2\delta)^{1-(l+m)}\ 2\pi i^{m-l} \frac{(l+m-2)!}{(l-1)!(m-1)!}. \tag{191}$$

Let us define also the moment of the $|\Psi_i(n)|^2$ at an energy E:

$$\langle |\Psi_i|^{2p} \rangle_E = \frac{1}{\rho(E)} \left\langle \sum_i |\Psi_i|^{2p}\ \delta(E-E_i) \right\rangle \equiv \frac{1}{\rho(E)} \sum_i \int d\Psi_i \int dE_i\ P(\Psi_i, E_i)\ |\Psi_i|^{2p}\ \delta(E-E_i). \tag{192}$$

Comparing Eqs.(192),(190) we arrive at:

$$\langle |\Psi_i|^{2(l+m)} \rangle_E = \frac{1}{2\pi\rho(E)}\, i^{l-m}\ \frac{(l-1)!(m-1)!}{(l+m-2)!} \lim_{\delta\to+0} \left\{ (2\delta)^{l+m-1} \langle [G^R_{nn}(E+i\delta)]^l [G^A_{nn}(E-i\delta)]^m \rangle \right\}. \tag{193}$$

This is the expression of the eigenfunction moments in terms of the retarded and advanced Green's functions we were looking for. One can see that any non-trivial moment $m+l>1$ requires a non-trivial limiting procedure.

The next standard step is to represent the average of the Green's functions in terms of the functional integral. It begins with the standard representation similar to Eq.(172):

$$[G^R_{nn}(E+i\delta)]^l [G^A_{nn}(E-i\delta)]^m = \frac{i^{m-l}}{l!m!} \int \mathcal{D}\Psi\ (\varphi^{R*}_n \varphi^R_n)^l\, (\varphi^{A*}_n \varphi^A_n)^m\ e^{-F[\bar{Q}]}. \tag{194}$$

Then the analogy with Eq.(172) would suggest that we write $\varphi^{R*}_n \varphi^R_n = \mathrm{STr}[\Pi^R \bar{Q}]$, raise the $h\mathrm{STr}[\Pi^R\bar{Q}]$ into the exponent with the help of the l-times differentiation with respect to the background field h and then switch to the super-matrix filed Q as in Eq.(177). However, in trying to do these "standard" steps we make a mistake. The reason is that the field $\bar{Q}_n$ is not slow-varying with n and the background field h_n should also contain fast space variations. This is what makes a difference compared to the case of the constant in space symmetry breaking field $\frac{1}{2}\omega\Lambda$ in Eq.(176).

One possible remedy[11] is to single out the bi-linear combinations of φ^*_n and φ_n which do not contain fast space variations. We show how to do this for the product $\varphi^{R*}_n \varphi^R_n \varphi^{A*}_n \varphi^A_n$. As the result of averaging over random matrix ensemble should not depend on n (*translational invariance on the average*) one can do the sum over n and then divide the result by N. Switching to the Fourier-components $\varphi(p)$we can represent this sum as

$$\sum_{p_1,p_2,q} \varphi^{R*}(p_1)\varphi^R(-p_1+q)\varphi^{A*}(p_2)\varphi^A(-p_2-q) = \sum_{p_1,p_2,q} \varphi^{R*}(p_1)\varphi^R(-p_2+q)\varphi^{A*}(p_2)\varphi^A(-p_1-q).$$

Two sums in the above expression is a mere re-labeling of momenta, all what is really needed is that the sum of all momenta is zero. However, this re-labeling becomes a non-trivial operation if one assumes that the momentum q is small. In assuming so we select a definite *domain of summation* such that the corresponding bi-linear combination of φ is slow varying in space. Then *one single sum* can be presented as

$$\sum_{p_i} \varphi^{R*}(p_1)\varphi^R(p_2)\varphi^{A*}(p_3)\varphi^A(p_4)\ \delta(p_1+p_2+p_3+p_4) = \tag{195}$$

$$\sum_{p_1,p_2,q\ll 1} \varphi^{R*}(p_1)\varphi^R(-p_1+q)\varphi^{A*}(p_2)\varphi^A(-p_2-q) + \sum_{p_1,p_2,q\ll 1} \varphi^{R*}(p_1)\varphi^R(-p_2+q)\varphi^{A*}(p_2)\varphi^A(-p_1-q) + \text{remainder}.$$

In the first term of Eq.(195) the bi-linear combinations $\varphi^{R*}(p_1)\varphi^{R}(-p_1+q)$ and $\varphi^{A*}(p_2)\varphi^{A}(-p_2-q)$ are slow, while in the second term slow are the combinations $\varphi^{R*}(p_1)\varphi^{A}(-p_1-q)$ and $\varphi^{A*}(p_2)\varphi^{R}(-p_2+q)$. In the remainder we collect all terms where there is no bi-linear slow combinations. The meaning of the above procedure of singling out the slow bi-linear combinations is that only such combinations lead to the divergent functional integral in the limit when $E_+ - E_- = 2i\delta$ tends to zero. The average of the remainder is not singular and can be neglected.

Performing this procedure in Eq.(194) one obtains $(l+m)!$ possibilities to break the product $(\varphi_n^{R*}\varphi_n^{R})^l\,(\varphi_n^{A*}\varphi_n^{A})^m$ into the product of slow bi-linear combinations. All of them appear to make the same contribution to Eq.(194). Thus one can consider only one such term, do all the standard manipulations with the source fields as we explained above for the case of the mean density of states and multiply the result by $q!$. The final result for the simplest choice $m=1$ is:

$$\langle|\Psi_n|^{2k}\rangle_E = -\frac{k}{2}\,\lim_{\delta\to 0}\left\{(2\pi\rho\delta)^{k-1}\,\int \mathcal{D}Q\,(\mathrm{STr}[\Pi^R Q_n])^{k-1}\,\mathrm{STr}[\Pi^A Q_n]\,e^{-F[Q]}\right\}. \tag{196}$$

One can see that it is $k!$ times larger than the one obtained by the "naive" manipulations with the background field. We spent some time to go into detail of this subtlety in order to show that sometimes "exact" manipulations with the background fields are dangerous if the fast varying components of the fields are treated improperly or simply omitted.

This is the result of a saddle-point approximation used in the derivation of the nonlinear sigma-model. No such danger appear for the dual representation which did not involve any approximation:

$$\langle|\Psi_n|^{2k}\rangle_E = -\frac{1}{2\,(k-1)!}\,\lim_{\delta\to 0}\left\{(2\pi\rho\delta)^{k-1}\,\int \mathcal{D}\bar{Q}\,(\mathrm{STr}[\Pi^R \bar{Q}_n])^{k-1}\,\mathrm{STr}[\Pi^A \bar{Q}_n]\,e^{-F[\bar{Q}]}\right\}. \tag{197}$$

One can do one more step without specifying the functionals $F[Q]$ and $F[\bar{Q}]$ using the fact that the structure of dependence of the Q and $\bar{Q}$ fields on the anti-commuting variables Eqs.(179),(180) is the same. To this end we define[13]the *generating functions* $Y[Q]$ as the functional integral of $e^{-F[Q]}$ done over all the super-matrices Q_i, except the one at a space point n:

$$Y[Q_n] = \int_{Q_i, i\neq n} \mathcal{D}Q\, e^{-F[Q]}. \tag{198}$$

If the generating function is known the eigenfunction moments are given by the integral over one single super-matrix Q_n.

One can show quite generally that this function does not depend on the anti-commuting variables. Then the integration of the anti-commuting variables is very simple as it involves only the pre-exponent in Eqs.(196),(197). As the result of this integration the additional factor $(k-1)$ appears in these equations. However, the main thing is to understand how it comes that the infinitesimal factor δ^{k-1} is compensated by the integral over the super-matrix Q_n. There is only one scenario of for this to happen in the framework of the nonlinear sigma-model: this is to absorb δ into the variable $\lambda \to \delta\lambda$ which can take arbitrary large values. For this the pre-exponent in Eq.(196) must be proportional to λ^{k-2} in the limit of large λ and also the generating function $Y(Q_n) = Y(u)$ must be a function of one single variable $u = 2\pi\rho\delta\lambda$. One can show that this is indeed the case:

$$\langle|\Psi_n|^{2k}\rangle_E = \frac{k(k-1)}{N}\,\int_0^\infty du\, u^{k-2}\, Y(u). \tag{199}$$

This is a remarkable formula, as it implies that the distribution function of $|\Psi|^2$ for any unitary ensemble $\beta = 2$ is:

$$\mathcal{P}(|\Psi|^2) = N^{-1}\,\frac{\partial^2}{\partial u^2}\,Y(u)\Big|_{u=|\Psi|^2}. \tag{200}$$

For the dual theory Eq.(197), the generating function $\bar{Y}[\bar{Q}_n]$ defined similar to Eq.(198) may depend on the *two* variables. This is because there are not one but two *non-compact* variables λ_1 and λ_2 that may take arbitrary large values. Introducing new variables

$$s = (\lambda_1 + \lambda_2), \qquad r = (\lambda_1 - \lambda_2),$$

one obtains:

$$\langle|\Psi_n|^{2k}\rangle_E = \frac{1}{4\pi\rho\, N}\,\frac{1}{(k-2)!}\,\int_0^\infty ds\,\int_{-\infty}^{+\infty} dr\;\, s^{k-2}\,\bar{Y}(s,r). \tag{201}$$

Let us apply Eq.(200) to the simplest case of the eigenfunction statistics in the Wigner-Dyson random matrix theory. In this case the variables of the super-matrix Q_i are locked to their values at $i = n$. Thus there is no integration in Eq.(198) whatsoever and we obtain:

$$Y[Q] = \exp\left[-\pi N\rho\delta\ \mathrm{STr}[\Lambda Q_n]\right] \to \exp(-2\pi\rho\delta\lambda\ N). \tag{202}$$

Then Eq.(200) immediately gives for $\beta = 2$ Wigner-Dyson RME the Gaussian eigenfunction distribution:

$$\mathcal{P}(|\Psi|^2) = N\ e^{-N|\Psi|^2}. \tag{203}$$

Note that the Gaussian form of the distribution function is not a consequence of the Gaussian distribution of the entries of $\mathbf{H}$ but rather a consequence of the *central limit theorem* at any distribution of independently fluctuating entries which variance matrix A_{ij} does not depend on $i-j$. One can show that for the orthogonal Gaussian ensemble $\beta = 1$ it deviates from the Gaussian:

$$\mathcal{P}(|\Psi|^2) = \sqrt{\frac{N}{2\pi\,|\Psi|^2}}\ e^{-N|\Psi|^2/2}. \tag{204}$$

The simplest non-trivial application[12] of Eq.(201) for the problem that cannot be treated by the nonlinear sigma-model is calculating the eigenfunction distribution function for the one-dimensional Anderson model. This model is described by the random matrix Hamiltonian

$$\mathbf{H}_{ii} = \varepsilon_i, \quad \mathbf{H}_{i,i\pm1} = 1,$$

where ε_i is the Gaussian random variable with the variance $w \ll 1$ (weak disorder case). Outside the center of the band $E = 0$ the result for the eigenfunction distribution function is amazingly simple:

$$\mathcal{P}(|\Psi|^2) = \frac{V_{\mathrm{loc}}}{N}\ \frac{e^{-V_{\mathrm{loc}}|\Psi|^2}}{|\Psi|^2}, \tag{205}$$

where V_{loc} is the localization radius. This distribution is not normalizable and should be cut a small values of $|\Psi|^2$. However, there is another way of normalizing it. This is the requirement that $\langle|\Psi_n|^2\rangle = N^{-1}$. The first moment of the distribution is perfectly well defined and gives the above pre-factor.

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[1] M.L.Mehta, *Random Matrices*, Academic Press (1991).
[2] N.Rosenzweig and C.E.Porter, Phys.Rev. **120**, 1698 (1960).
[3] H.Kunz and B.Shapiro, Phys.Rev. E **58**, 400 (1998).
[4] Y.V.Fyodorov and A.D.Mirlin, Phys.Rev.Lett., **67**, 2405 (1991).
[5] B.Sutherland in: *Lecture notes in Physics*, **242**, Springer Verlag, Berlin (1985).
[6] N.I.Muskhelishvili, *Singular integral equations*, Groningen, Noordhoff (1953).
[7] G.Szego, *Orthogonal polynomials*, American Mathemetical Society, New York, (1959).
[8] L.D.Landau and E.M.Lifshitz, *Quantum Mechanics*.
[9] A.O.Gogolin, A.M.Tsvelik and A.A.Nersesyan, *The bosonization approach to strongly correlated systems*, Cambridge University Press, Cambridge, 1998.
[10] V.E.Kravtsov and A.M.Tsvelik, Phys.Rev.B, **62**, 9888 (2000).
[11] K.B.Efetov, *Supersymmetry in disorder and chaos*, Cambridge University Press, Cambridge, 1997.
[12] A.Ossipov and V.E.Kravtsov, Phys. Rev.B, **73**, 033105 (2006).
[13] A.D.Mirlin, Phys. Reports, **326**,259 (2000).