

# 4

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## GAUSSIAN ENSEMBLES. LEVEL DENSITY

In this short chapter we reproduce a statistical mechanical argument of Wigner to “derive” the level density for the Gaussian ensembles. The joint probability density of the eigenvalues is written as the Boltzmann factor for a gas of charged particles interacting via a two dimensional Coulomb force. The equilibrium density of this Coulomb gas is such as to make the potential energy a minimum, and this density is identified with the level density of the corresponding Gaussian ensembles. That the eigenvalue density so deduced is correct, will be seen later when in Chapters 6 to 8 and 14 we will compute it for each of the four Gaussian ensembles for any finite  $N \times N$  matrices and take the limit as  $N \rightarrow \infty$ .

Another argument, again essentially due to Wigner, is given to show that the same level density, a “semi-circle”, holds for random Hermitian matrices with elements having an average value zero and a common mean square value.

### 4.1 The Partition Function

Consider a gas of  $N$  point charges with positions  $x_1, x_2, \dots, x_N$  free to move on the infinite straight line  $-\infty < x < \infty$ . Suppose that the potential energy of the gas is given by

$$W = \frac{1}{2} \sum_i x_i^2 - \sum_{i < j} \ln |x_i - x_j|. \quad (4.1.1)$$

The first term in  $W$  represents a harmonic potential which attracts each charge independently towards the point  $x = 0$ ; the second term represents an electrostatic repulsion between each pair of charges. The logarithmic function comes in if we assume the universe to be two-dimensional. Let this charged gas be in thermodynamical equilibrium at a temperature  $T$ , so that the probability density of the positions of the  $N$  charges is given by

$$P(x_1, \dots, x_N) = C \exp(-W/kT), \quad (4.1.2)$$

where  $k$  is the Boltzmann constant. We immediately recognize that (4.1.2) is identical to (3.3.8) provided  $\beta$  is related to the temperature by

$$\beta = (kT)^{-1}. \quad (4.1.3)$$

This system of point charges in thermodynamical equilibrium is called the Coulomb gas model, corresponding to the Gaussian ensembles.

Following Dyson (1962a, 1962b, 1962c), we can define various expressions that relate to our energy-level series in complete analogy with the classical notions of entropy, specific heat, and the like. These expressions, when computed from the observed experimental data and compared with the theoretical predictions, provide a nice method of checking the theory.

In classical mechanics the joint probability density in the velocity space is a product of exponentials

$$\prod_j \exp(-C_j v_j^2)$$

with constant  $C_j$ , and its contribution to the thermodynamic quantities of the model are easily calculated. We simply discard these trivial terms. The nontrivial contributions arise from the partition function

$$\psi_N(\beta) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\beta W} dx_1 \dots dx_N \quad (4.1.4)$$

and its derivatives with respect to  $\beta$ . Therefore it is important to have an analytical expression for  $\psi_N(\beta)$ . Fortunately, this can be deduced from an integral evaluated by Selberg (see Chapter 17).

**Theorem 4.1.1.** *For any positive integer  $N$  and real or complex  $\beta$  we have, identically,*

$$\psi_N(\beta) = (2\pi)^{N/2} \beta^{-N/2 - \beta N(N-1)/4} (\Gamma(1 + \beta/2))^{-N} \prod_{j=1}^N \Gamma(1 + \beta j/2). \quad (4.1.5)$$

Let us note the fact that the energy  $W$  given by (4.1.1) is bounded from below. More precisely,

$$W \geq W_0 = \frac{1}{4}N(N-1)(1+\ln 2) - \frac{1}{2} \sum_{j=1}^N j \ln j, \quad (4.1.6)$$

and this minimum is attained when the positions of the charges coincide with the zeros of the Hermite polynomial  $H_N(x)$  (cf. Appendix A.6).

Once the partition function is known, other thermodynamic quantities such as free energy, entropy, and specific heat can be calculated by elementary differentiation. Because all the known properties are identical to those of the circular ensembles, studied at length in Chapter 12 we do not insist on this point here.

## 4.2 The Asymptotic Formula for the Level Density. Gaussian Ensembles

Since the expression (3.3.8) for  $P(x_1, \dots, x_N)$ , the probability that the eigenvalues will lie in unit intervals around  $x_1, x_2, \dots, x_N$ , is valid for all values of  $x_i$ , the density of levels

$$\sigma_N(x) = N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x, x_2, \dots, x_N) dx_2 \cdots dx_N \quad (4.2.1)$$

can be calculated for any  $N$  by actual integration (Mehta and Gaudin, 1960). The details of this tedious calculation are not given here, since an expression for  $\sigma_N(x)$ , derived by a different method, appears in Chapters 6, 7 and 8.

However, if one is interested in the limit of large  $N$ , as we certainly are, these complications can be avoided by assuming that the corresponding Coulomb gas is a classical fluid with a continuous macroscopic density. More precisely, this amounts to the following two assumptions:

- (1) The potential energy  $W$  given by (4.1.1) can be approximated by the functional

$$W(\sigma) = \frac{1}{2} \int_{-\infty}^{\infty} dx x^2 \sigma(x) - \frac{1}{2} \int_{-\infty}^{\infty} dx dy \sigma(x) \sigma(y) \ln |x - y|. \quad (4.2.2)$$

- (2) The level density  $\sigma(x)$  will be such as to minimize the expression (4.2.2), consistent with the requirements

$$\int_{-\infty}^{\infty} dx \sigma(x) = N, \quad (4.2.3)$$

and

$$\sigma(x) \geq 0. \quad (4.2.4)$$

The first integral in (4.2.2) reproduces the first sum in (4.1.1) accurately in the limit of large  $N$ . The same is not true of the second integral, for it neglects the two-level correlations, which may be expected to extend over a few neighboring levels; however, because the total number of levels is large their effect may be expected to be small. The factor  $1/2$  in the second term of (4.2.2) comes from the condition  $i < j$  in (4.1.1).

The problem of finding the stationary points of the functional  $W(\sigma)$ , (4.2.2), with the restriction (4.2.3) leads us to the integral equation

$$-\frac{1}{2}x^2 + \int_{-\infty}^{\infty} dy \sigma(y) \ln|x-y| = C, \quad (4.2.5)$$

where  $C$  is a Lagrange constant. Actually (4.2.5) has to hold only for those values of  $x$  for which  $\sigma(x) > 0$ . One cannot add a negative increment to  $\sigma(x)$  where  $\sigma(x) = 0$ , and therefore the functional differentiation is not valid; hence (4.2.5) cannot be derived for such values of  $x$ . It is not difficult to solve (4.2.5) (Mushkelishvili, 1953). This will not be done here, but the solution will be given and then verified.

Differentiation of (4.2.5) with respect to  $x$  eliminates  $C$ . Before carrying it out, we must replace the integral with

$$\lim_{\varepsilon \rightarrow 0} \left( \int_{-\infty}^{x-\varepsilon} dy + \int_{x+\varepsilon}^{\infty} dy \right) \sigma(y) \ln|x-y|. \quad (4.2.6)$$

When (4.2.6) is differentiated with respect to  $x$ , the terms arising from the differentiation of the limits drop out and only the derivative of  $\ln|x-y|$  remains. The integral becomes a principal value integral and (4.2.5) becomes

$$P \int_{-\infty}^{\infty} \frac{\sigma(y)}{x-y} dy = x. \quad (4.2.7)$$

Conversely, if (4.2.7) is satisfied by some  $\sigma(y)$  and this  $\sigma$  is an even function, then it will satisfy (4.2.5) also. We try

$$\sigma(y) = \begin{cases} C(A^2 - y^2)^{1/2}, & |y| < A, \\ 0, & |y| > A. \end{cases} \quad (4.2.8)$$

Elementary integration gives

$$\begin{aligned} \int \frac{(A^2 - y^2)^{1/2}}{x-y} dy &= x \sin^{-1}\left(\frac{y}{A}\right) - (A^2 - y^2)^{1/2} \\ &+ (A^2 - x^2)^{1/2} \ln \left( \frac{A(x-y) - x(A^2 - y^2)^{1/2} - y(A^2 - x^2)^{1/2}}{A(x-y) - x(A^2 - y^2)^{1/2} + y(A^2 - x^2)^{1/2}} \right). \end{aligned} \quad (4.2.9)$$

Taking the principal value of (4.2.9) between the limits  $(-A, A)$ , we find that only the first term gives a nonzero contribution, which is  $\pi x$ . Hence (4.2.7) gives

$$C = 1/\pi, \quad (4.2.10)$$

and (4.2.3) gives

$$\frac{1}{\pi} \frac{\pi}{2} A^2 = N. \quad (4.2.11)$$

Thus

$$\sigma(x) = \begin{cases} \frac{1}{\pi} (2N - x^2)^{1/2}, & |x| < (2N)^{1/2}, \\ 0, & |x| > (2N)^{1/2}. \end{cases} \quad (4.2.12)$$

This is the so-called “semicircle law” first derived by Wigner.

Actually the two-level correlation function can be calculated (cf. Sections 6.2, 7.2 and 8.2) and the above intuitive arguments put to test. Instead, we shall derive an exact expression for the level-density valid for any  $N$ . The limit  $N \rightarrow \infty$  can then be taken (cf. Appendix A.9) to obtain the “semicircle law”.

We have noted in Section 4.1 that without any approximation whatever the energy  $W$  attains its minimum value when the points  $x_1, x_2, \dots, x_N$  are the zeros of the  $N$ th order Hermite polynomial. The postulate of classical statistical mechanics then implies that in the limit of very large  $N$  the level density is the same as the density of zeros of the  $N$ th order Hermite polynomial. This later problem has been investigated by many authors, and we may conveniently refer to the relevant mathematical literature (Szegő, 1959).

### 4.3 The Asymptotic Formula for the Level Density. Other Ensembles

Numerical evidence shows, as we said in Chapter 1, that the local statistical properties of the Gaussian ensembles are shared by a much wider class of matrices. In particular even the eigenvalue density, which is a global property, follows the “semi-circle law”. Wigner (1955) first considered bordered matrices, i.e., real symmetric matrices  $H$  with elements

$$H_{jk} = \begin{cases} \pm h, & \text{if } |j - k| \leq m, \\ 0, & \text{if } |j - k| > m. \end{cases} \quad (4.3.1)$$

Except for the symmetry of  $H$ , the signs of  $H_{jk}$  are random. He then calculates the moments of the level density and derives an integral equation for it. The calculations are long. The final result is that in the limit of  $h^2/m \rightarrow 0$  and the order of the matrices is

infinite, the eigenvalue density is a “semi-circle”. Here we present still another heuristic argument in its support, again essentially due to Wigner.

Consider a matrix  $H$  with elements  $H_{ij}$  all having an average value zero and a mean square value  $V^2$ . Let the order  $N$  be large enough so that the density of its eigenvalues may be taken to be a continuous function. Let this function be  $\sigma(\varepsilon, V^2)$ , so that the number of eigenvalues lying between  $\varepsilon$  and  $\varepsilon + \delta\varepsilon$  is given by  $\sigma(\varepsilon, V^2) d\varepsilon$ . If we change the matrix elements by small quantities  $\delta H_{ij}$  such that the  $\delta H_{ij}$  themselves all have the average value zero and a mean square value  $v^2$ , the change in a particular eigenvalue at  $\varepsilon_i$  can be calculated by the second order perturbation theory

$$Z(\varepsilon, V^2) = \delta H_{ii} + \sum_{j \neq i} \frac{|\delta H_{ij}|^2}{\varepsilon_i - \varepsilon_j} + \dots \quad (4.3.2)$$

The  $\delta H_{ii}$  do not produce, on the average, any change in  $\varepsilon_i$ . The eigenvalues  $\varepsilon_j$  which lie nearest to  $\varepsilon_i$  give the largest contribution to (4.3.2) with an absolute value  $v^2/\bar{s}$  where  $\bar{s}$  is the mean spacing at  $\varepsilon_i$ . But as there are eigenvalues on both sides of  $\varepsilon_i$ , the two contributions arising from the two nearest eigenvalues nearly cancel out, leaving quantities of a higher order in  $v^2$ . The sum in (4.3.2) can therefore be approximated by

$$Z(\varepsilon, V^2) \approx v^2 \int \frac{\sigma(\varepsilon', V^2)}{\varepsilon - \varepsilon'} d\varepsilon', \quad (4.3.3)$$

where the integral in (4.3.3) is a principal value integral and

$$V^2 = \langle |H_{ij}|^2 \rangle, \quad v^2 = \langle |\delta H_{ij}|^2 \rangle. \quad (4.3.4)$$

The ensemble averages being indicated by  $\langle \rangle$ . Let us calculate the change in the number of eigenvalues lying in an interval  $(\varepsilon, \varepsilon + \delta\varepsilon)$ . This can be done in two ways; one gives, as is obvious from the way of writing,

$$\sigma(\varepsilon, V^2)Z(\varepsilon, V^2) - \sigma(\varepsilon + \delta\varepsilon, V^2)Z(\varepsilon + \delta\varepsilon, V^2) \approx -\frac{\partial(\sigma Z)}{\partial \varepsilon} \delta\varepsilon, \quad (4.3.5)$$

while the other gives in a similar way

$$v^2 \frac{\partial \sigma}{\partial V^2}. \quad (4.3.6)$$

If all the matrix elements  $H_{ij}$  are multiplied by a constant  $c$ , the values  $\varepsilon_i$  are also multiplied by  $c$ , while  $V^2$  is multiplied by  $c^2$ . Hence,

$$\sigma(c\varepsilon, c^2 V^2) c d\varepsilon = \sigma(\varepsilon, V^2) d\varepsilon. \quad (4.3.7)$$

Setting  $cV = 1$  the last equation gives

$$\sigma(\varepsilon, V^2) = \frac{1}{V} \sigma(\varepsilon/V, 1),$$

which could have been inferred by dimensional arguments. Putting

$$Z(\varepsilon, V^2) = \frac{v^2}{V} Z_1(\varepsilon/V), \quad \sigma(\varepsilon, V^2) = \frac{1}{V} \sigma_1(\varepsilon/V), \quad (4.3.8)$$

in (4.3.3), (4.3.5) and (4.3.6), we obtain

$$\frac{\partial(Z_1 \sigma_1)}{\partial x} = \frac{1}{2} \frac{\partial(x \sigma_1)}{\partial x}, \quad x = \varepsilon/V, \quad (4.3.9)$$

$$Z_1(x) = P \int \frac{\sigma_1(x')}{x - x'} dx'. \quad (4.3.10)$$

When  $x = 0$ , by symmetry requirement  $Z_1 = 0$ ; therefore (4.3.9) gives, on integration

$$Z_1(x) = x/2. \quad (4.3.11)$$

Finally we have the boundary condition

$$\int \sigma(\varepsilon, V^2) d\varepsilon = \int \sigma_1(x) dx = N. \quad (4.3.12)$$

Equations (4.3.10), (4.3.11) and (4.3.12) together are equivalent to the integral equation (4.2.7) together with (4.2.3). The solution, as there, is the semi-circle law (4.2.12):

$$\sigma(\varepsilon, V^2) = \begin{cases} \frac{1}{2\pi V^2} (2NV^2 - \varepsilon^2)^{1/2}, & \varepsilon^2 < 2NV^2, \\ 0, & \varepsilon^2 > 2NV^2. \end{cases} \quad (4.3.13)$$

Olson and Uppulury (1972) and later Wigner extended these considerations to include a still wider class of matrices to have the “semi-circle law” as their eigenvalue density.

For the two level correlation function or the spacing distribution no such argument has yet been found.

### Summary of Chapter 4

As a consequence of Selberg’s integral one has the partition function

$$\psi_N(\beta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\beta W} dx_1 \cdots dx_N \quad (4.1.4)$$

$$= (2\pi)^{N/2} \beta^{-N/2 - \beta N(N-1)/4} (\Gamma(1 + \beta/2))^{-N} \prod_{j=1}^N \Gamma(1 + \beta j/2), \quad (4.1.5)$$

where

$$W = \frac{1}{2} \sum_{j=1}^N x_j^2 - \sum_{1 \leq j < k \leq N} \ln |x_j - x_k|. \quad (4.1.1)$$

For a large class of random matrices the asymptotic density of eigenvalues is the “semi-circle”

$$\sigma(x) = \begin{cases} \frac{1}{\pi} (2N - x^2)^{1/2}, & |x| < (2N)^{1/2}, \\ 0, & |x| > (2N)^{1/2}. \end{cases} \quad (4.2.12)$$