2

GAUSSIAN ENSEMBLES. THE JOINT PROBABILITY DENSITY FUNCTION FOR THE MATRIX ELEMENTS

After examining the consequences of time-reversal invariance, we introduce Gaussian ensembles as a mathematical idealization. They are implied if we make the hypothesis of maximum statistical independence allowed under the symmetry constraints.

2.1 Preliminaries

In the mathematical model our systems are characterized by their Hamiltonians, which in turn are represented by Hermitian matrices. Let us look into the structure of these matrices. The low-lying energy levels (eigenvalues) are far apart and each may be described by a different set of quantum numbers. As we go to higher excitations, the levels draw closer, and because of their mutual interference most of the approximate quantum numbers lose their usefulness, for they are no longer exact. At still higher excitations the interference is so great that some quantum numbers may become entirely meaningless. However, there may be certain exact integrals of motion, such as total spin or parity, and the quantum numbers corresponding to them are conserved whatever the excitation may be. If the basis functions are chosen to be the eigenfunctions of these conserved quantities, all Hamiltonian matrices of the ensemble will reduce to the form of diagonal

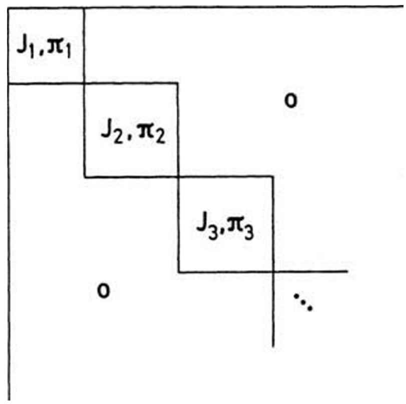

Figure 2.1. Block diagonal structure of a Hamiltonian matrix. Each diagonal block corresponds to a set of exact symmetries or to a set of exactly conserved quantum numbers. The matrix elements connecting any two diagonal blocks are zero, whereas those inside each diagonal block are random.

blocks. One block will correspond uniquely to each set of exact quantum numbers. The matrix elements lying outside these blocks will all be zero, and levels belonging to two different blocks will be statistically uncorrelated. As to the levels corresponding to the same block, the interactions are so complex that any regularity resulting from partial diagonalization will be washed out. (See Figure 2.1.)

We shall assume that such a basis has already been chosen and restrict our attention to one of the diagonal blocks, an $N \times N$ Hermitian matrix in which N is a large but fixed positive integer. Because nuclear spectra contain at least hundreds of levels with the same spin and parity, we are interested in (the limit of) very large N.

With these preliminaries, the matrix elements may be supposed to be random variables and allowed the maximum statistical independence permitted under symmetry requirements. To specify precisely the correlations among various matrix elements we need a careful analysis of the consequences of time-reversal invariance.

2.2 Time-Reversal Invariance

We begin by recapitulating the basic notions of time-reversal invariance. From physical considerations, the time-reversal operator is required to be antiunitary (Wigner, 1959) and can be expressed, as any other antiunitary operator, in the form

$$T = KC, \tag{2.2.1}$$

where K is a fixed unitary operator and the operator C takes the complex conjugate of the expression following it. Thus a state under time reversal transforms to

$$\psi^R = T\psi = K\psi^*, \tag{2.2.2}$$

ψ^* being the complex conjugate of ψ. From the condition

$$(\Phi, A\psi) = \left(\psi^R, A^R\Phi^R\right),$$

for all pairs of states ψ, Φ, and (2.2.2), we deduce that under time reversal an operator A transforms to

$$A^R = KA^TK^{-1}, \tag{2.2.3}$$

where A^T is the transpose of A. A is said to be self-dual if $A^R = A$. A physical system is invariant under time reversal if its Hamiltonian is self-dual, that is, if

$$H^R = H. \tag{2.2.4}$$

When the representation of the states is transformed by a unitary transformation, $\psi \to U\psi$, T transforms according to

$$T \to UTU^{-1} = UTU^{\dagger}, \tag{2.2.5}$$

or K transforms according to

$$K \to UKU^T. \tag{2.2.6}$$

Because operating twice with T should leave the physical system unchanged, we have

$$T^2 = \alpha \cdot 1, \quad |\alpha| = 1, \tag{2.2.7}$$

where 1 is the unit operator; or

$$T^2 = KCKC = KK^*CC = KK^* = \alpha \cdot 1. \tag{2.2.8}$$

But K is unitary:

$$K^*K^T = 1.$$

From these two equations we get

$$K = \alpha K^T = \alpha\left(\alpha K^T\right)^T = \alpha^2 K.$$

Therefore

$$\alpha^2 = 1 \quad \text{or} \quad \alpha = \pm 1, \tag{2.2.9}$$

so that the unitary matrix K is either symmetric or antisymmetric. In other words, either

$$KK^* = 1, \tag{2.2.10}$$

or

$$KK^* = -1. \tag{2.2.11}$$

These alternatives correspond, respectively, to an integral or a half-odd integral total angular momentum of the system measured in units of $\hbar$ (Wigner, 1959), for the total angular momentum operator $\mathbf{J} = (J_1, J_2, J_3)$ must transform as

$$J_\ell^R = -J_\ell, \quad \ell = 1, 2, 3. \tag{2.2.12}$$

For brevity we call the two possibilities the even-spin and odd-spin case, respectively.

2.3 Gaussian Orthogonal Ensemble

Suppose now that the even-spin case holds and (2.2.10) is valid. Then a unitary operator U will exist such that (cf. Appendix A.3)

$$K = UU^T. \tag{2.3.1}$$

By (2.2.6) a transformation $\psi \to U^{-1}\psi$ performed on the states ψ brings K to unity. Thus in the even-spin case the representation of states can always be chosen so that

$$K = 1. \tag{2.3.2}$$

After one such representation is found, further transformations $\psi \to R\psi$ are allowed only with R a real orthogonal matrix so that (2.3.2) remains valid. The consequence of (2.3.2) is that self-dual matrices are symmetric. In the even spin case every system invariant under time reversal will be associated with a real symmetric matrix H if the representation of states is suitably chosen. For even-spin systems with time-reversal invariance the Gaussian orthogonal ensemble E_{1G}, defined below, is therefore appropriate.

Definition 2.3.1. The Gaussian orthogonal ensemble E_{1G} is defined in the space T_{1G} of real symmetric matrices by two requirements:

(1) The ensemble is invariant under every transformation

$$H \to W^T H W \tag{2.3.3}$$

of T_{1G} into itself, where W is any real orthogonal matrix.
(2) The various elements H_{kj}, $k \leqslant j$, are statistically independent.

These requirements, expressed in the form of equations, read as follows:

(1) The probability $P(H)\,dH$ that a system of E_{1G} will belong to the volume element $dH = \prod_{k\leqslant j} dH_{kj}$ is invariant under real orthogonal transformations:

$$P(H')\,dH' = P(H)\,dH, \tag{2.3.4}$$

where

$$H' = W^T H W, \tag{2.3.5}$$

and

$$W^T W = W W^T = 1. \tag{2.3.6}$$

(2) This probability density function $P(H)$ is a product of functions, each of which depends on a single variable:

$$P(H) = \prod_{k\leqslant j} f_{kj}(H_{kj}). \tag{2.3.7}$$

Suppose, next, that we are dealing with a system invariant under space rotations. The spin may now be even or odd. The Hamiltonian matrix H which represents the system commutes with every component of $\mathbf{J}$. If we use the standard representation of the J matrices with J_1 and J_3 real and J_2 pure imaginary, (2.2.12) may be satisfied by the usual choice (Wigner, 1959)

$$K = e^{i\pi J_2} \tag{2.3.8}$$

for K. With this choice of K, H and K commute and H^R reduces to H^T. Thus a rotation-invariant system is represented by a real symmetric matrix H, and once again the ensemble E_{1G} is appropriate.

2.4 Gaussian Symplectic Ensemble

In this section we discuss a system to which E_{1G} does not apply, a system with odd-spin, invariant under time reversal, but having no rotational symmetry. In this case (2.2.11) holds, K cannot be diagonalized by any transformation of the form (2.2.6), and there is no integral of the motion by which the double-valuedness of the time-reversal operation can be trivially eliminated.

Every antisymmetric unitary operator can be reduced by a transformation (2.2.6) to the standard canonical form (cf. Appendix A.3)

$$Z = \begin{bmatrix} 0 & +1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & +1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \equiv \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \dot{+} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \dot{+} \cdots, \tag{2.4.1}$$

which consists of (2×2) blocks

$$\begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

along the leading diagonal; all other elements of Z are zero. We assume that the representation of states is chosen so that K is reduced to this form. The number of rows and columns of all matrices must now be even, for otherwise K would be singular in contradiction to (2.2.11). It is convenient to denote the order of the matrices by $2N$ instead of N. After one such representation is chosen, for which $K = Z$, further transformations $\psi \to B\psi$ are allowed, only with B a unitary $(2N \times 2N)$ matrix for which

$$Z = BZB^T. \tag{2.4.2}$$

Such matrices B form precisely the N-dimensional symplectic group (Weyl, 1946), usually denoted by $\mathrm{Sp}(N)$.

It is well known (Chevalley, 1946; Dieudonné, 1955) that the algebra of the symplectic group can be expressed most conveniently in terms of quaternions. We therefore introduce the standard quaternion notation for (2×2) matrices,

$$e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \tag{2.4.3}$$

with the usual multiplication table

$$e_1^2 = e_2^2 = e_3^2 = -1, \tag{2.4.4}$$

$$e_1e_2 = -e_2e_1 = e_3, \qquad e_2e_3 = -e_3e_2 = e_1, \qquad e_3e_1 = -e_1e_3 = e_2. \tag{2.4.5}$$

Note that in (2.4.3), as well as throughout the rest of this book, i is the ordinary imaginary unit and not a quaternion unit. The matrices e_1, e_2, and e_3, together with the (2×2) unit matrix

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

form a complete set, and any (2×2) matrix with complex elements can be expressed linearly in terms of them with complex coefficients:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)1 - \frac{i}{2}(a-d)e_1 + \frac{1}{2}(b-c)e_2 - \frac{i}{2}(b+c)e_3. \tag{2.4.6}$$

All the $(2N \times 2N)$ matrices will be considered as cut into N^2 blocks of (2×2) and each (2×2) block expressed in terms of quaternions. In general, a $(2N \times 2N)$ matrix with complex elements thus becomes an $(N \times N)$ matrix with complex quaternion elements. In particular the matrix Z is now

$$Z = e_2 I, \tag{2.4.7}$$

where I is the $(N \times N)$ unit matrix. It can be verified that the rules of matrix multiplication are not changed by this partitioning.

Let us add some definitions. We call a quaternion "real" if it is of the form

$$q = q^{(0)} + \mathbf{q} \cdot \mathbf{e} \equiv q^{(0)} + q^{(1)} e_1 + q^{(2)} e_2 + q^{(3)} e_3, \tag{2.4.8}$$

with real coefficients $q^{(0)}$, $q^{(1)}$, $q^{(2)}$, and $q^{(3)}$. Thus a real quaternion does not correspond to a (2×2) matrix with real elements. Any complex quaternion has a "conjugate quaternion"

$$\bar{q} = q^{(0)} - \mathbf{q} \cdot \mathbf{e}, \tag{2.4.9}$$

which is distinct from its "complex conjugate"

$$q^* = q^{(0)*} + \mathbf{q}^* \cdot \mathbf{e}. \tag{2.4.10}$$

A quaternion with $q^* = q$ is real; one with $q^* = -q$ is pure imaginary; and one with $\bar{q} = q$ is a scalar. By applying both types of conjugation together, we obtain the "Hermitian conjugate"

$$q^\dagger = \bar{q}^* = q^{(0)*} - \mathbf{q}^* \cdot \mathbf{e}. \tag{2.4.11}$$

A quaternion with $q^\dagger = q$ is Hermitian and corresponds to the ordinary notion of a (2×2) Hermitian matrix; one with $q^\dagger = -q$ is anti-Hermitian. The conjugate (Hermitian conjugate) of a product of quaternions is the product of their conjugates (Hermitian conjugates) taken in the reverse order:

$$\overline{q_1 q_2 \ldots q_n} = \bar{q}_n \ldots \bar{q}_2 \bar{q}_1, \tag{2.4.12}$$

$$(q_1 q_2 \ldots q_n)^\dagger = q_n^\dagger \ldots q_2^\dagger q_1^\dagger. \tag{2.4.13}$$

Now consider a general $(2N \times 2N)$ matrix A which is to be written as an $(N \times N)$ matrix Q with quaternion elements q_{kj}; $k, j = 1, 2, \ldots, N$. The standard matrix operations on A are then reflected in Q in the following way:

Transposition

$$\left(Q^T\right)_{kj} = -e_2 \bar{q}_{jk} e_2. \tag{2.4.14}$$

Hermitian conjugation

$$\left(Q^\dagger\right)_{kj} = q_{jk}^\dagger. \tag{2.4.15}$$

Time reversal

$$\left(Q^R\right)_{kj} = e_2 \left(Q^T\right)_{kj} e_2^{-1} = \bar{q}_{jk}. \tag{2.4.16}$$

The matrix Q^R is called the "dual" of Q. A "self-dual" matrix is one with $Q^R = Q$. That is if $q_{jk} = \left[\begin{smallmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{smallmatrix}\right]$, then $Q = [q_{jk}]$ is self-dual if

$$a_{jk} = d_{kj}, \qquad b_{jk} = -b_{kj} \quad \text{and} \quad c_{jk} = -c_{kj}. \tag{2.4.17}$$

The usefulness of quaternion algebra is a consequence of the simplicity of (2.4.15) and (2.4.16). In particular, it is noteworthy that the time-reversal operator K does not appear explicitly in (2.4.16) as it did in (2.2.3). By (2.4.15) and (2.4.16) the condition

$$Q^R = Q^\dagger \tag{2.4.18}$$

is necessary and sufficient for the elements of Q to be real quaternions. When (2.4.18) holds, we call Q "quaternion real".

A unitary matrix B that satisfies (2.4.2) is automatically quaternion real. In fact, it satisfies the conditions

$$B^R = B^\dagger = B^{-1}, \tag{2.4.19}$$

which define the symplectic group. The matrices H which represent the energy operators of physical systems are Hermitian as well as self-dual:

$$H^R = H, \qquad H^\dagger = H, \tag{2.4.20}$$

hence are also quaternion real. From (2.4.15) and (2.4.16) we see that the quaternion elements of a self-dual Hermitian matrix must satisfy

$$q_{jk}^\dagger = \bar{q}_{jk} = q_{kj}, \tag{2.4.21}$$

or $q_{jk}^{(0)}$ must form a real symmetric matrix, whereas $q_{jk}^{(1)}$, $q_{jk}^{(2)}$, and $q_{jk}^{(3)}$ must form real antisymmetric matrices. Thus the number of real independent parameters that define a $(2N \times 2N)$ self-dual Hermitian matrix is

$$\frac{1}{2}N(N+1) + \frac{1}{2}N(N-1) \cdot 3 = N(2N-1).$$

From this notational excursion, let us come back to the point. Systems having odd-spin, invariance under time-reversal, but no rotational symmetry, must be represented by self-dual, Hermitian Hamiltonians. Therefore the Gaussian symplectic ensemble, as defined below, should be appropriate for their description.

Definition 2.4.1. The Gaussian symplectic ensemble E_{4G} is defined in the space T_{4G} of self-dual Hermitian matrices by the following properties:

(1) The ensemble is invariant under every automorphism

$$H \to W^R H W \tag{2.4.22}$$

of T_{4G} into itself, where W is any symplectic matrix.
(2) Various linearly independent components of H are also statistically independent.

These requirements put in the form of equations read as follows:

(1) The probability $P(H)\,dH$ that a system E_{4G} will belong to the volume element

$$dH = \prod_{k \leqslant j} dH_{kj}^{(0)} \prod_{\lambda=1}^{3} \prod_{k<j} dH_{kj}^{(\lambda)} \tag{2.4.23}$$

is invariant under symplectic transformations; that is,

$$P(H')\,dH' = P(H)\,dH, \tag{2.4.24}$$

if

$$H' = W^R H W, \tag{2.4.25}$$

where

$$W^R W = 1 \quad \text{or} \quad W Z W^T = Z. \tag{2.4.26}$$

(2) The probability density function $P(H)$ is a product of functions each of which depends on a single variable:

$$P(H) = \prod_{k \leqslant j} f_{kj}^{(0)}\left(H_{kj}^{(0)}\right) \prod_{\lambda=1}^{3} \prod_{k<j} f_{kj}^{(\lambda)}\left(H_{kj}^{(\lambda)}\right). \tag{2.4.27}$$

2.5 Gaussian Unitary Ensemble

Mathematically a much simpler ensemble is the Gaussian unitary ensemble E_{2G} which applies to systems without invariance under time reversal. Such systems are easily created in principle by putting an ordinary atom or nucleus, for example, into an externally generated magnetic field. The external field is not affected by the time-reversal operation. However, for the unitary ensemble to be applicable, the splitting of levels by the magnetic field must be at least as large as the average level spacing in the absence of the magnetic field. The magnetic field must, in fact, be so strong that it will completely "mix up" the level structure that would exist in zero field; for otherwise our random hypothesis cannot be justified. This state of affairs could never occur in nuclear physics. In atomic or molecular physics a practical application of the unitary ensemble may perhaps be possible.

A system without time-reversal invariance has a Hamiltonian that may be an arbitrary Hermitian matrix not restricted to be real or self-dual. Thus we are led to the following definition.

Definition 2.5.1. The Gaussian unitary ensemble E_{2G} is defined in the space T_{2G} of Hermitian matrices by the following properties:

(1) The probability $P(H)\,dH$ that a system of E_{2G} will belong to the volume element

$$dH = \prod_{k \leqslant j} dH_{kj}^{(0)} \prod_{k<j} dH_{kj}^{(1)}, \tag{2.5.1}$$

where $H_{kj}^{(0)}$ and $H_{kj}^{(1)}$ are real and imaginary parts of H_{kj}, is invariant under every automorphism

$$H \to U^{-1} H U \tag{2.5.2}$$

of T_{2G} into itself, where U is any unitary matrix.

(2) Various linearly independent components of H are also statistically independent.

In mathematical language these requirements are

(1)

$$P(H')\, dH' = P(H)\, dH, \tag{2.5.3}$$

if

$$H' = U^{-1} H U, \tag{2.5.4}$$

where U is any unitary matrix.

(2) $P(H)$ is a product of functions, each of which depends on a single variable:

$$P(H) = \prod_{k \leqslant j} f_{kj}^{(0)}\big(H_{kj}^{(0)}\big) \prod_{k<j} f_{kj}^{(1)}\big(H_{kj}^{(1)}\big). \tag{2.5.5}$$

2.6 Joint Probability Density Function for the Matrix Elements

We now come to the question of the extent to which we are still free to specify the joint probability density function $P(H)$. It will be seen that the two postulates of invariance and statistical independence elaborated above fix uniquely the functional form of $P(H)$.

The postulate of invariance restricts $P(H)$ to depend only on a finite number of traces of the powers of H. We state this fact as a lemma (Weyl, 1946).

Lemma 2.6.1. *All the invariants of an $(N \times N)$ matrix H under nonsingular similarity transformations A,*

$$H \to H' = A H A^{-1},$$

can be expressed in terms of the traces of the first N powers of H.

Actually the trace of the jth power of H is the sum of the jth powers of its eigenvalues λ_k, $k = 1, 2, \ldots, N$, of H,

$$\operatorname{tr} H^j = \sum_{k=1}^{N} \lambda_k^j \equiv p_j, \quad \text{say},$$

and it is a well-known fact that any symmetric function of the λ_k can be expressed in terms of the first N of the p_j; see, for example, Macdonald (1979) or Mehta (1989MT).

The postulate of statistical independence excludes everything except the traces of the first two powers, and these, too, may occur only in an exponential. To see this we will need the following lemma.

Lemma 2.6.2. *If three continuous and differentiable functions $f_k(x)$, $k = 1, 2, 3$, satisfy the equation*

$$f_1(xy) = f_2(x) + f_3(y), \tag{2.6.1}$$

then they are necessarily of the form $a \ln x + b_k$ ($k = 1, 2, 3$), with $b_1 = b_2 + b_3$.

Proof. Differentiating (2.6.1) with respect to x, we have

$$f_1'(xy) = \frac{1}{y} f_2'(x),$$

which, on integration with respect to y, gives

$$\frac{1}{x} f_1(xy) = f_2'(x) \ln y + \frac{1}{x} g(x), \tag{2.6.2}$$

where $g(x)$ is still arbitrary. Substituting $f_1(xy)$ from (2.6.2) into (2.6.1),

$$x f_2'(x) \ln y + g(x) - f_2(x) = f_3(y). \tag{2.6.3}$$

Therefore the left-hand side of (2.6.3) must be independent of x; this is possible only if

$$x f_2'(x) = a \quad \text{and} \quad g(x) - f_2(x) = b_3,$$

that is, only if

$$f_2(x) = a \ln x + b_2 = g(x) - b_3,$$

where a, b_2 and b_3 are arbitrary constants.

Now (2.6.3) gives

$$f_3(y) = a \ln y + b_3,$$

and finally (2.6.1) gives

$$f_1(xy) = a \ln(xy) + (b_2 + b_3).$$

□

Let us now examine the consequences of the statistical independence of the various components of H. Consider the particular transformation

$$H = U^{-1} H' U, \tag{2.6.4}$$

where

$$U = \begin{bmatrix} \cos\theta & \sin\theta & 0 & \dots & 0 \\ -\sin\theta & \cos\theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \tag{2.6.5}$$

or, in quaternion notation (provided N is even),

$$U = \begin{bmatrix} \cos\theta + e_2 \sin\theta & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}. \tag{2.6.6}$$

This U is, at the same time, orthogonal, symplectic, and unitary.

Differentiation of (2.6.4) with respect to θ gives

$$\frac{\partial H}{\partial\theta} = \frac{\partial U^T}{\partial\theta} H' U + U^T H' \frac{\partial U}{\partial\theta} = \frac{\partial U^T}{\partial\theta} U H + H U^T \frac{\partial U}{\partial\theta}, \tag{2.6.7}$$

and by substituting for U, U^T, $\partial U/\partial\theta$ and $\partial U^T/\partial\theta$ from (2.6.5) or (2.6.6) we get

$$\frac{\partial H}{\partial\theta} = AH + HA^T, \tag{2.6.8}$$

where

$$A = \frac{\partial U^T}{\partial\theta} U = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.6.9}$$

or, in quaternion notation, A is diagonal.

$$A = \begin{bmatrix} -e_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{2.6.10}$$

If the probability density function

$$P(H) = \prod_{(\alpha)} \prod_{j \leqslant k} f_{kj}^{(\alpha)}\big(H_{kj}^{(\alpha)}\big) \tag{2.6.11}$$

is invariant under the transformation U, its derivative with respect to θ must vanish; that is

$$\sum \frac{1}{f_{kj}^{(\alpha)}} \frac{\partial f_{kj}^{(\alpha)}}{\partial H_{kj}^{(\alpha)}} \frac{\partial H_{kj}^{(\alpha)}}{\partial \theta} = 0. \tag{2.6.12}$$

Let us write this equation explicitly, say, for the unitary case. Equations (2.6.8) and (2.6.12) give

$$\begin{aligned}
&\left[\left(-\frac{1}{f_{11}^{(0)}} \frac{\partial f_{11}^{(0)}}{\partial H_{11}^{(0)}} + \frac{1}{f_{22}^{(0)}} \frac{\partial f_{22}^{(0)}}{\partial H_{22}^{(0)}}\right)\big(2H_{12}^{(0)}\big) + \frac{1}{f_{12}^{(0)}} \frac{\partial f_{12}^{(0)}}{\partial H_{12}^{(0)}}\big(H_{11}^{(0)} - H_{22}^{(0)}\big)\right] \\
&\quad + \sum_{k=3}^{N}\left(-\frac{1}{f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} H_{2k}^{(0)} + \frac{1}{f_{2k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{2k}^{(0)}} H_{1k}^{(0)}\right) \\
&\quad + \sum_{k=3}^{N}\left(-\frac{1}{f_{1k}^{(1)}} \frac{\partial f_{1k}^{(1)}}{\partial H_{1k}^{(1)}} H_{2k}^{(1)} + \frac{1}{f_{2k}^{(1)}} \frac{\partial f_{2k}^{(1)}}{\partial H_{2k}^{(1)}} H_{1k}^{(1)}\right) = 0.
\end{aligned} \tag{2.6.13}$$

The braces at the left-hand side of this equation depend on mutually exclusive sets of variables and their sum is zero. Therefore each must be a constant; for example,

$$-\frac{H_{2k}^{(0)}}{f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} + \frac{H_{1k}^{(0)}}{f_{2k}^{(0)}} \frac{\partial f_{2k}^{(0)}}{\partial H_{2k}^{(0)}} = C_k^{(0)}. \tag{2.6.14}$$

On dividing both side of (2.6.14) by $H_{1k}^{(0)} H_{2k}^{(0)}$ and applying the Lemma 2.6.2, we conclude that the constant $C_k^{(0)}$ must be zero, that is,

$$\frac{1}{H_{1k}^{(0)}} \frac{1}{f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} = \frac{1}{H_{2k}^{(0)}} \frac{1}{f_{2k}^{(0)}} \frac{\partial f_{2k}^{(0)}}{\partial H_{2k}^{(0)}} = \text{constant} = -2a, \quad \text{say}, \tag{2.6.15}$$

which on integration gives

$$f_{1k}^{(0)}\big(H_{1k}^{(0)}\big) = \exp\big[-a\big(H_{1k}^{(0)}\big)^2\big]. \tag{2.6.16}$$

In the other two cases we also derive a similar equation. Now because the off-diagonal elements come on as squares in the exponential and all invariants are expressible in terms of the traces of powers of H, the function $P(H)$ is an exponential that contains traces of at most the second power of H.

Because $P(H)$ is required to be invariant under more general transformations than we have here considered, one might think that the form of $P(H)$ is further restricted. This, however, is not so, for

$$\begin{aligned} P(H) &= \exp\bigl(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\bigr) \\ &= e^c \prod_j \exp\bigl(bH_{jj}^{(0)}\bigr) \prod_{k\leqslant j} \exp\bigl[-a\bigl(H_{kj}^{(0)}\bigr)^2\bigr] \prod_\lambda \prod_{k<j} \exp\bigl[-a\bigl(H_{kj}^{(\lambda)}\bigr)^2\bigr] \end{aligned} \tag{2.6.17}$$

is already a product of functions, each of which depends on a separate variable. Moreover, because we require $P(H)$ to be normalizable and real, a must be real and positive and b and c must be real.

Therefore we have proved the following theorem (Porter and Rosenzweig, 1960a).

Theorem 2.6.3. *In all the above three cases the form of $P(H)$ is automatically restricted to*

$$P(H) = \exp\bigl(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\bigr), \tag{2.6.18}$$

where a is real and positive and b and c are real.

In the foregoing discussion we have emphasized the postulate of statistical independence of various components of H even at the risk of frequent repetitions. This statistical independence is important in restricting $P(H)$ to the simple form (2.6.18), and hence makes the subsequent analytical work tractable. However, it lacks a clear physical motivation and therefore looks somewhat artificial.

The main objection to the assumption of statistical independence, leading to (2.6.18), is that all values of $H_{kj}^{(\lambda)}$ are not equally weighted and therefore do not correspond to all "interactions" being "equally probable". By a formal change Dyson (1962a, I) has defined his "circular ensembles", which are esthetically more satisfactory and equally easy to work with. We shall come to them in Chapters 9 to 11. They give equivalent results as we will see in Chapter 11. On the other hand, Rosenzweig (1963), has emphasized the "fixed strength" ensemble briefly considered in Chapter 27. Others (Leff, 1963; Fox and Kahn, 1964) have arbitrarily tried the so-called "generalized" ensembles related to classical orthogonal polynomials other than the Hermite polynomials. We will study them in Chapter 19.

If we keep only the first requirement that $P(H)$ is invariant under $H \to UHU^{-1}$, then $P(H)$ may be any function of the traces of powers of H. People have studied in particular the case when $P(H) \propto \exp(-\operatorname{tr} V(H))$, where $V(x)$ is a polynomial, preferably of even degree with the coefficient of the highest power positive. This case is briefly mentioned in Section 19.4.

2.7 Gaussian Ensemble of Hermitian Matrices With Unequal Real and Imaginary Parts

The ensembles so far considered were characterized by two requirements: (i) the probability $P(H)\,dH$ that a system belongs to the volume element dH is such that $P(H)$ is invariant under $H \to U^{-1}HU$, where U is any matrix which is either real orthogonal, symplectic or unitary according to the symmetry of the system; and (ii) various linearly independent components of H are also statistically independent.

If for our system the time reversal invariance is only weakly violated, then the appropriate ensemble will be almost an orthogonal or symplectic ensemble slightly mixed with the unitary ensemble. Keeping the hypothesis (ii) that various linearly independent parts of H are also statistically independent, we should now take

$$P(H) \propto \exp\bigl(-\operatorname{tr}\bigl(H_1^2/c_1 + H_2^2/c_2\bigr)\bigr), \tag{2.7.1}$$

where $H = H_1 + H_2$, H_1 and H_2 are Hermitian, H_1 is symmetric (self-dual) and H_2 is anti-symmetric (anti-self-dual). If $c_2 = 0$, then $H_2 = 0$ with probability 1, and we have the orthogonal (symplectic) ensemble; if $c_2 = c_1$, then we have the unitary ensemble. For a small violation of the time reversal invariance, $c_2 \ll c_1$. Since it does not increase the mathematical difficulties and the analytical solution is as elegant, we will treat in Chapter 14 the general case where c_1 and c_2 are arbitrary real numbers.

Note that under real orthogonal transformations the traces of powers of H_1 and H_2 (i.e. of real and imaginary parts of H) are invariant and so is the probability density $P(H)$ of Eq. (2.7.1). However, under unitary transformations the real and imaginary parts of H mix up and the above $P(H)$ is no longer invariant unless $c_1 = c_2$.

2.8 Anti-Symmetric Hermitian Matrices

Though physically not relevant, the mathematical analysis of a Gaussian ensemble of anti-symmetric (or that of anti-self-dual quaternion) Hermitian matrices is equally elegant. As above, the probability will be taken as

$$P(H)\,dH, \quad dH = \prod_{j<k} dH_{jk},$$

and

$$P(H) \propto \exp\bigl(-a \operatorname{tr} H^2\bigr).$$

Summary of Chapter 2

The probability density $P(H)$ of a random matrix H is proportional to $\exp(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c)$ with certain constants a, b, c in the following three cases:

(1) If H is a Hermitian symmetric random matrix, its elements H_{jk} with $j \geqslant k$ are statistically independent, and $P(H)$ is invariant under all real orthogonal transformations of H. The resulting ensemble is named as Gaussian orthogonal.

(2) If H is a Hermitian random matrix, its diagonal elements H_{jj} and the real and imaginary parts of its off-diagonal elements H_{jk} for $j > k$ are statistically independent, and $P(H)$ is invariant under all unitary transformations of H. The resulting ensemble is named as Gaussian unitary.

(3) If H is a Hermitian self-dual random matrix, its diagonal elements H_{jj} and the four quaternionic components of its off-diagonal elements H_{jk} with $j > k$ are statistically independent, and $P(H)$ is invariant under all symplectic transformations of H. The resulting ensemble is named as Gaussian symplectic.

Moreover,

(4) For a Hermitian anti-symmetric random matrix H, it is not unreasonable to take the elements H_{jk} with $j > k$ as Gaussian variables with the same variance.

(5) Similarly, for a Hermitian random matrix H, with $P(H)$ not invariant under unitary transformations of H, it is not unreasonable to take its symmetric and anti-symmetric parts to have the probability densities prescribed under cases (1) and (4) above.

Invariance of $P(H)$ under orthogonal, unitary or symplectic transformations of H is required by physical considerations and depend on whether the system described by the Hamiltonian H has or does not have certain symmetries like time reversal or rotational symmetry. The statistical independence of the various real parameters entering H is assumed for simplicity.