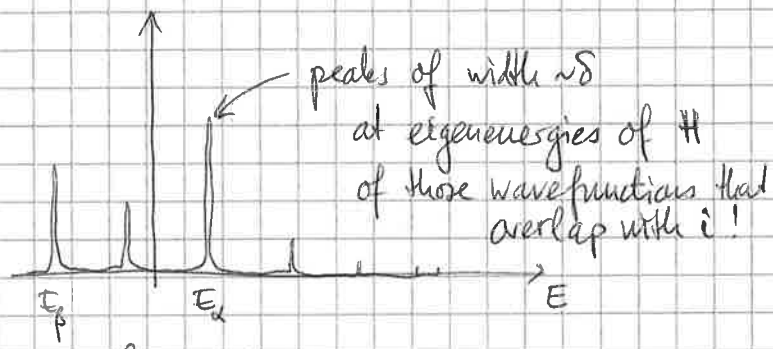
Anderson's approach to strong localization

Anderson Phys Rev 109, 1492, (1958)

"Locator expansion": Expand in small hopping; understand the physics of delocalization and decay.

Difference between localized and delocalized spectrum:

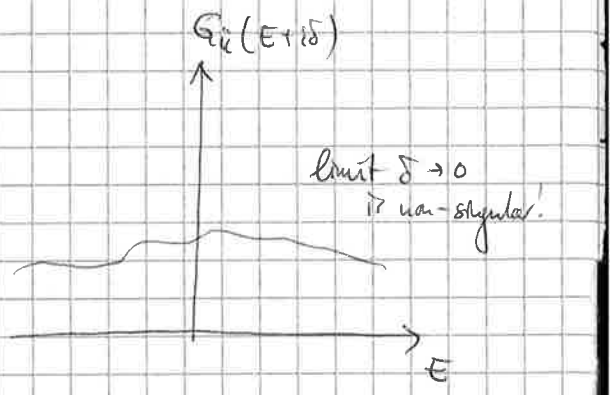
$G_{ii}(E+i\delta)$ ← Green's function (see below)



localized

"point spectrum"

$\Longleftrightarrow$



delocalized

"continuous spectrum"

$$H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{<ij>} t_{ij}\left(c_i^\dagger c_j + c_j^\dagger c_i\right)$$

Consider time evolution: (Schrödinger Eq.):

$$i\partial_t \psi_i = \epsilon_i \psi_i - \sum_{j\in\partial i} t_{ij}\psi_j \quad ; \quad \psi_i(0) = \delta_{ik}$$

Eigenfunctions $\epsilon_i \psi_i^\alpha - \sum_{j\in\partial i} t_{ij}\psi_j^\alpha = E^\alpha \psi_i^\alpha$

Now add a small decay rate δ (loss) to each site:

$$i\partial_t \psi_i = (\epsilon_i - i\delta)\psi_i - \sum_{j\in\partial i} t_{ij}\psi_j \equiv (H\psi)_i - [i\delta \mathbb{1}\psi]_i$$

$$\psi_i(0) = \delta_{ik}$$

Now analyze decay! In a finite volume V, for $\delta \to 0$, there is no decay possible!

But what happens in an infinite V?

Take $V\to\infty$ first, then $\delta \to 0$!

Q: Does some decay rate survive in the thermodynamic limit, even if $\delta \to 0$? (⇔ Does a particle escape to infinity, or does it remain localized, despite $V\to\infty$?)

$\Rightarrow$ Solution of evolution starting at site k:

$$\psi_k(t) = e^{-i(H-i\delta)t}|k\rangle$$

($|k\rangle$ ← initial state)

$$\psi_E := -i\int_0^\infty dt\, e^{iEt}\psi_k(t)$$: "part of the evolving wavefunction oscillating with frequency E"

$$\langle i|\psi_E\rangle = -i\langle i|\int_0^\infty dt\, e^{i(E-(H-i\delta))t}|k\rangle = \langle i|\frac{1}{E-(H-i\delta)}|k\rangle$$

$$\equiv G_{ik}(E+i\delta) = \underline{\text{Green's function}} \text{ (in energy and real space)}!$$

In the eigenbasis:

$$G_{ik}(E+i\delta) = \sum_\alpha \langle i|\psi_\alpha\rangle \frac{1}{E-(E_\alpha - i\delta)} \langle\psi_\alpha|k\rangle = \sum_\alpha \frac{(\psi_i^\alpha)^*\,\psi_k^\alpha}{E-E^\alpha+i\delta}$$

$\Rightarrow$ bunch of δ-functions, if ψ^α are localized!

<u>Recipe by Anderson</u>: Green's function is very ill-behaved, spiky

$\Rightarrow$ Study instead the local self-energy: It captures the possibility of decay!

$$[G_{ii}(E+i\delta)]^{-1} =: E+i\delta-\epsilon_i - S_i(E+i\delta)$$

($S_i = 0$ if $t_{ij}=0$! $\Rightarrow$ S_i is correction due to hopping.)

<u>Why S_i?</u> $\Rightarrow$ Look at <u>behavior of $S_i(E+i\delta)$</u> in the loc/deloc regime:

(E^α ← one of the eigenvalues)

(i) <u>loc</u>: $G_{ii}(E=E^\alpha+i\delta) = \frac{|\psi_i^\alpha|^2}{i\delta}$ + small terms from other $\beta\neq\alpha$ with $|E^\beta-E^\alpha| \gg \delta$

$$\Rightarrow (G_{ii})^{-1} \approx \frac{i\delta}{|\psi_i^\alpha|^2} \overset{!}{=} E^\alpha_{+i\delta} - \epsilon_i - S_i(E^\alpha+i\delta)$$

$$\Rightarrow \mathrm{Im}\, S_i(E^\alpha+i\delta) \propto \delta \xrightarrow[\delta\to 0]{} 0 \quad \text{"no decay"}$$

<u>No singularity in $S(E)$!</u> $\mathrm{Re}(S_i(E^\alpha+i\delta)) \simeq E^\alpha-\epsilon_i$

$\Leftrightarrow$ eigenenergies E^α are solutions of $S_i(E^\alpha) = E^\alpha-\epsilon_i$!

(ii) delocalized: $\sum_{\alpha | E^\alpha \in [\epsilon, \epsilon+d\epsilon]} |\psi_i^\alpha|^2 \delta_{\gamma\gamma} := $ local density of states

$$G_{ii}(E+i\delta) \simeq \int d\epsilon \frac{\rho_i(\epsilon)}{E-\epsilon+i\delta} = -i\pi \rho_i(E) + \int d\epsilon\, \rho_i(\epsilon) \mathcal{P}\frac{1}{E-\epsilon}$$

$$\Rightarrow \mathrm{Im}\left[(G_{ii})^{-1}\right] \neq 0 \Rightarrow \mathrm{Im}\left(S_i(E+i\delta)\right) \neq 0$$

even when $\delta \to 0$!

Conclusion: $\mathrm{Im}(S_i(E+i\delta\to 0)) \begin{cases} = 0 & \text{localized} \\ > 0 & \text{delocalized} \end{cases}$

$\Rightarrow \mathrm{Im}(S) \neq 0 \Leftrightarrow$ spontaneous symmetry breaking.

Finite response to infinitesimal perturbation δ!

Note: 1) Delocalized phase is the symmetry-broken phase

2) $\delta \gtrless 0$ distinguishes $\begin{cases} \text{retarded} \\ \text{advanced} \end{cases}$ propagation

If $\delta \to 0$ does not break symmetry $\Rightarrow$ no arrow of time

If $\mathrm{Im}\,\delta \gtrless 0$ for $\delta \to 0 \Rightarrow$ spontaneous breaking of "time reversal symmetry"

$\Rightarrow$ decay to infinity singles out the forward direction!

Perturbative analysis of self-energy : Hopping expansion

Write $H = \underbrace{H_0}_{\sum_i \epsilon_i c_i^\dagger c_i} - \underbrace{T}_{\sum_{\langle i,j\rangle} t_{ij}(c_i^\dagger c_j + c_j^\dagger c_i)}$

$$G_i(E) = \frac{1}{E - H_0 + T} = \frac{1}{G_0^{-1} + T}$$

$\Rightarrow G = G_0 - G_0 T G_0 + G_0 T G_0 T G_0 - \ldots$ where $G_0 = \frac{1}{E-H_0}$; $[G_0]_{ij} = \frac{\delta_{ij}}{E-\epsilon_i}$

$G_{ii} = ?$ $\rightarrow$ Dyson equation:

$$G_{ii} \overset{!}{=} G_{0,ii} + \Big[\underbrace{-G_{0,ii} \underbrace{T_{ii}}_{=0}}_{} + G_{0,ii} T_{ij} \underbrace{G_{0,jj} T_{ji}}_{\text{no internal index} = i!} - \underbrace{[G_0 T G_0 T G_0 T]_{ii}}_{} \Big] G_{ii}$$

$\Big\} = G_{0,ii} S_i(E)$

Check: $G_{ii}\left(1 - S_i(E)\, G_{0,ii}\right) = G_{0,iii}$

$$\Leftrightarrow \frac{1}{G_{ii}} = \frac{1}{G_{0,ii}} - S_i(E) = E - \epsilon_i - S_i(E) \quad \checkmark$$

$$\Rightarrow S_i(E) = \Big[\underbrace{TG_0T - TG_0TG_0T + TG_0TG_0TG_0T - \dots}_{\text{with no return to } i!}\Big]_{ii}$$

perturbative series for self-energy; expansion in hopping matrix T.

Physical content: Look at 2nd order in T:

$$S_i(E+i\delta) = \sum_{j\neq i} t_{ij} \frac{1}{E - \epsilon_j + i\delta} t_{ji} + O(t^3)$$

$$\xrightarrow{\delta\to 0} \sum_{j\neq i} \frac{t_{ij}^2}{E-\epsilon_j} - i\pi \underbrace{\sum_{j\neq i} \delta(E-\epsilon_j) t_{ij}^2}_{\text{essentially always zero with probability 1!}} - i\delta \sum_{j\neq i} \frac{t_{ij}^2}{(E-\epsilon_j)^2}$$

2nd order perturbation of 0th order eigenenergy ϵ_i

Looks like Fermi Golden rule decay rate. BUT: Here decay is not possible!
Final states are discrete: exact energy conservation cannot be satisfied.
Average over ϵ_j should not be taken! $\Leftrightarrow$ Study Prob[Im(S)], not $\langle \mathrm{Im}(S)\rangle$!

Important insight: At any finite order of perturbation theory $\Rightarrow \mathrm{Im}(S_i(E+i\delta)) \propto \delta$!

since t_{ij} and $G_{0,ii} = \frac{1}{E-\epsilon_i+i\delta}$ are real apart from part of δ!

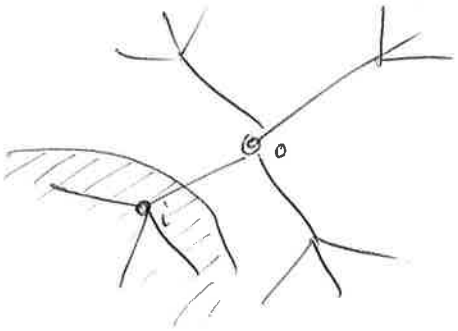
$\Rightarrow$ Im $S_i \neq 0$ can only occur if perturbation theory diverges!

$\Rightarrow$ Proof of localization by proving the convergence of the expansion, (Anderson)
[Mathematical proofs (e.g. Fröhlich+Spencer '84): judicious organization of resummation
Holdhanor + Aizenmann of perturbation series.]

Delocalization $\Leftrightarrow$ spontaneous emergence of Im $S_i < 0$ (decay rate) due to infinitesimal coupling to a bath (δ-decay rate)

Exactly solvable case: Localization on the Cayley tree (Bethe lattice)

Abou-Chacra, Thouless, Anderson '73



hierarchical, tree-like lattice **without** loops

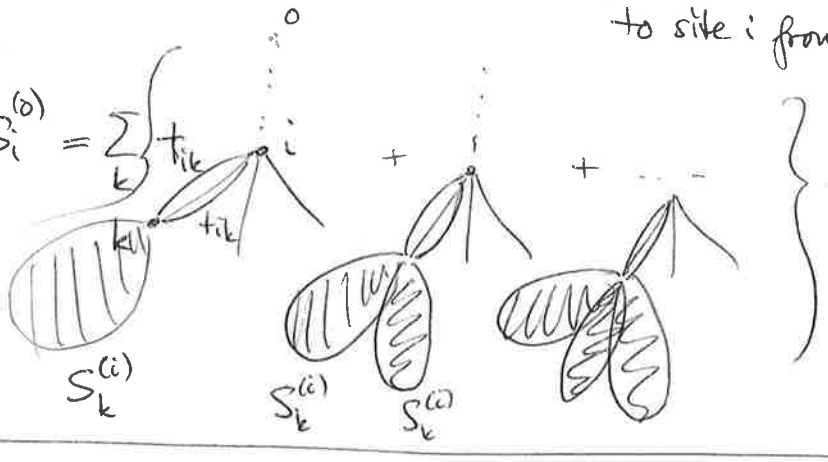
$\Rightarrow$ allows to resum S_i conveniently!

Idea: define $S_i^{(0)}$:= self-energy in the absence of hopping t_{i0}: "cavity method for site i".

$$S_i^{(0)}(E) = \left(T^{(0)} G_0 T^{(0)} - T^{(0)} G_0 T^{(0)} G_0 T^{(0)} + \cdots\right)$$

↖ no internal index $i \Leftrightarrow$ never come back to site i from subtree!

$\Leftrightarrow$



$$S_i^{(0)} = \sum_{k \in \mathrm{div}\{0\}} t_{ik} \left(G_{0,kk} + G_{0,kk} S_k^{(i)} G_{0,kk} + G_{0,kk} S_k^{(i)} G_{0,kk} S_k^{(i)} G_{0,kk} + \cdots\right) t_{ki}$$

$$= \sum_k t_{ik} \frac{1}{E - \epsilon_k + i\delta - S_k^{(i)}(E+i\delta)} t_{ki} \qquad (*)$$

$\Rightarrow$ closed recursion for $S_i^{(0)}(E+i\delta)$ as a function of $S_k^{(i)}(E+i\delta)$!

Write $S_i^{(0)} = E_i - i\Delta_i$ real + imaginary parts

$(*) \Rightarrow$

$$E_i = \sum_{k \in \mathrm{div}\{0\}} \frac{t_{ik}^2 (E - \epsilon_k - E_k)}{(E - \epsilon_k - E_k)^2 + (\delta + \Delta_k)^2}$$

$$\Delta_i = \sum_k \frac{t_{ik}^2 (\Delta_k + \delta)}{(E - \epsilon_k - E_k)^2 + (\delta + \Delta_k)^2}$$

Anderson: Check for stability of localized regime!

Q: Is $\Delta_i \sim \delta \to 0$ as $\delta \to 0$

Or: $\Delta_i(\delta \to 0) \to$ finite?

Linear stability analysis of the recursion relation!

Instability $\Leftrightarrow$ For $\delta = 0$, small Δ_k's : Δ's grow indefinitely under iteration $\Rightarrow$ non-linearity set in and stop growth

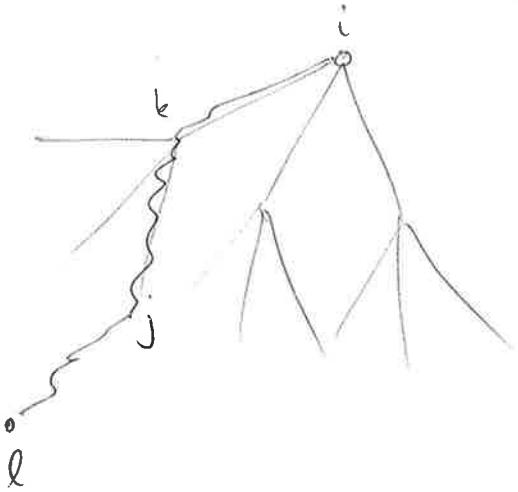
$\Longleftrightarrow$ delocalization!

Thus study:

$$\Delta_i = \sum_k \frac{t_{ik}^2 \Delta_k}{(E - \epsilon_k - E_k)^2} \qquad E_i = \sum_k \frac{t_{ik}^2}{E - \epsilon_k - E_k}$$

$$= \sum_{k,j \in \partial k \setminus \{i\}} \left(\frac{t_{ik}}{E - \epsilon_k - E_k}\right)^2 \left(\frac{t_{kj}}{E - \epsilon_j - E_j}\right)^2 \Delta_j \cdots$$

↑ iterate further



$$= \sum_{\text{leaves } \ell} \left(\prod_{s=1}^{L} \left(\frac{t_{k_{s-1} k_s}}{E - \epsilon_{k_s} - E_{k_s}} \right)^2 \right) \Delta_{\text{leaf}}$$

k_s: $p: i \to \ell = \{i, k_1, k_2, \ldots, k_{L-1}, \ell\}$, with $k_0 = i$, $k_L = \ell$

Assume $\Delta_{\text{leaf}} = \text{const.}$

$\Longleftrightarrow$ Compute $\Delta_i = \Delta_{\text{leaf}} \cdot \underbrace{\left(\sum_\ell \underbrace{e^{-\beta E_{\text{path } i \to \ell}}}_{\equiv \prod_{s=1}^{L} \left(\frac{t_{k_{s-1} k_s}}{E - \epsilon_{k_s} - E_{k_s}} \right)^2} \right)}_{Z_L}$

$\Rightarrow$ Directed polymer problem! [Simplify: $\epsilon_k + E_k = \text{Re}(S_k) + \xi_k$ assume uncorrelated. Neglect shift by $\text{Re}(S_k)$]

slight correlation of denominators

now: $t_{ij} \equiv t$

Recall solution:

$$Z_L = \exp(-L\phi) \quad \text{where} \quad \phi = \max_{0 \le m \le 1} \phi(m)$$

$$\text{with} \quad e^{-m\phi(m)} \equiv K \cdot \underbrace{\left(\frac{t^2}{(E - \epsilon)^2}\right)^{\epsilon m}}_{t = c - 1}$$

Instability $\Leftrightarrow Z_L \gg 1 \Leftrightarrow \phi < 0$. Localization transition $\Leftrightarrow \phi = 0$

Note: • Delocalisation happens at specific value $\phi = 0$ of the "free energy" of the effective directed polymer, not at a thermodynamic transition of the polymer.

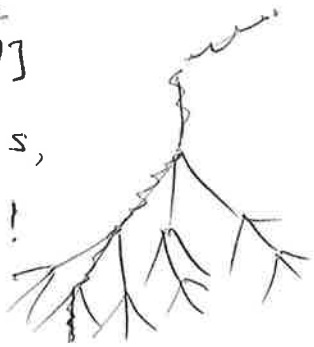
• Indeed, $\overline{\left(\frac{t^2}{(E-\epsilon)^2}\right)^m}^{\epsilon}$ is only defined for $m < \frac{1}{2}$ (otherwise it is divergent!)

$\Rightarrow m = 1$ is not within the physical domain

$\Rightarrow$ the polymer is always in its frozen phase

[because $1/\epsilon_i^2$ has a heavy tailed distribution!]

$\Leftrightarrow$ Anderson delocalisation happens along optimal paths, those that dominate the partition function Z_L!



Critical wavefunctions are extremely sparse, nearly 1d objects $\Leftrightarrow$ see multi-fractality below!

Transition point

Approximate: $p(\epsilon) = \frac{1}{W}$ for $\epsilon \in [-\frac{W}{2}, \frac{W}{2}]$ (neglect effects of self-energies)

Take energy $E = 0$ for simplicity:

$$\Rightarrow \phi(m) = -\frac{1}{m} \ln\left[K \int_{-W/2}^{W/2} \frac{d\epsilon}{W} \left(\frac{t^2}{\epsilon^2}\right)^m\right] = -\frac{1}{m} \ln\left[\frac{K}{1-2m} \left(\frac{2t}{W}\right)^{2m}\right]$$

1) $m = m^*$ such that $\left.\frac{\partial \phi}{\partial m}\right|_{m=m^*} = 0$ (maximum!)

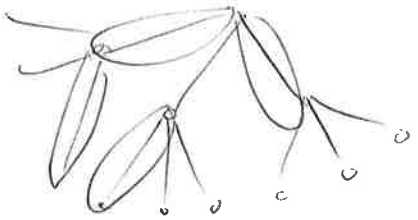
2) Delocalisation $\Leftrightarrow \phi = 0 \Leftrightarrow \frac{K}{1-2m^*} \left(\frac{2t}{W}\right)^{2m^*} = 1 \qquad (\star)$

1): using 2): $\left.\frac{\partial}{\partial m}\left\{\ln\left[\frac{K}{1-2m}\left(\frac{2t}{W}\right)^{2m}\right]\right\}\right|_{m=m^*} = 0$

$$\Leftrightarrow \ln\left(\frac{2t}{W}\right) + \frac{1}{1-2m^*} = 0$$

Injecting $(\star)$: $\Rightarrow \frac{W}{2t} / \ln(W/2t) = eK \Rightarrow \boxed{\left(\frac{W}{t}\right)_c \approx \underbrace{2eK}_{\text{fairly large!}} \cdot \ln(2eK)}$

- Note: simple nearest neighbor resonance condition



resonance $\Leftrightarrow$ $t_{ij} \geq \text{const.} \times |\epsilon_i - \epsilon_j|$

Would result in a density $g \sim \frac{t}{W}$ of resonant bonds.

They percolate when $g \cdot K = 1$ (always find a band to go further!)

$\Rightarrow$ This would wrongly predict $\left(\frac{W}{t}\right)_c \propto \frac{1}{g_c} \sim K$

$\Leftrightarrow$ correct result: $\left(\frac{W}{t}\right)_c \sim K \ln K$ at large K

Reason: Small denominators $\frac{1}{\epsilon}$ are very important:
resonances at further distance matter!

Delocalization is not just simple percolation!

Energy dependence: $E \neq 0$