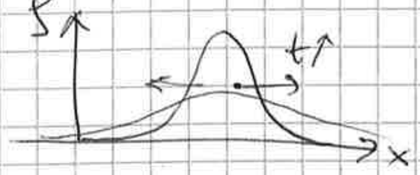
Anderson and Many-Body localization : Disorder + Quantum mechanics (+ interactions) (weak)

2. Diffusion:

Normally: • density inhomogeneity evens out over time:



$$\partial_t \rho = D \Delta \rho$$

• Energy diffuses and distributes over all degrees of freedom

$\Rightarrow$ thermalization / equilibration

$\Leftrightarrow$ 2nd law of thermodynamics : entropy increases and tends to its possible maximum

$\Longrightarrow$ defines the arrow of time!

$\Rightarrow$ In most systems, phase space is ergodically explored.

Basic underlying idea: Random walks of particles / energy packets : (kicked by interactions or random potential)

each step in time has no memory of past steps $\Rightarrow$

$\Longrightarrow$ Diffusion, conductivity, transport

BUT: Anderson (1958): "Quantum random walks" are different:

Energy is conserved (elastic collisions with disorder potential)

$\hookrightarrow$ memory $\Rightarrow$ can diffusion stop? $\Rightarrow$ $D = 0$?

YES! Due to quantum interference!

$\Rightarrow$ no diffusion / thermalization : equal energy surface is <u>not</u> explored uniformly as $t \to \infty$

no arrow of time visible after finite time!

Relevant for:

- spin diffusion
- light
- sound waves
- cold atoms : Bosons / fermions
- electrons : disordered metals / semiconductors

Requirement:

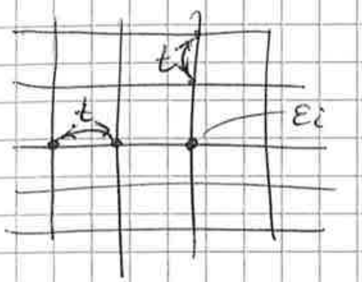
$\boxed{\text{sufficient disorder}}$

Simple model (Anderson)

Electrons hopping from site to site in a disordered system (e.g.: doped semiconductor)

$$H = \sum_i \epsilon_i \, c_i^+ c_i - t \sum_{\langle ij \rangle} \left(c_i^+ c_j + c_j^+ c_i \right)$$

on-site disorder (← impurities, amorphous structure)



e.g. $P(\epsilon_i) = \frac{1}{W}$ $\quad \epsilon_i \in \left[-\frac{W}{2}, \frac{W}{2} \right]$

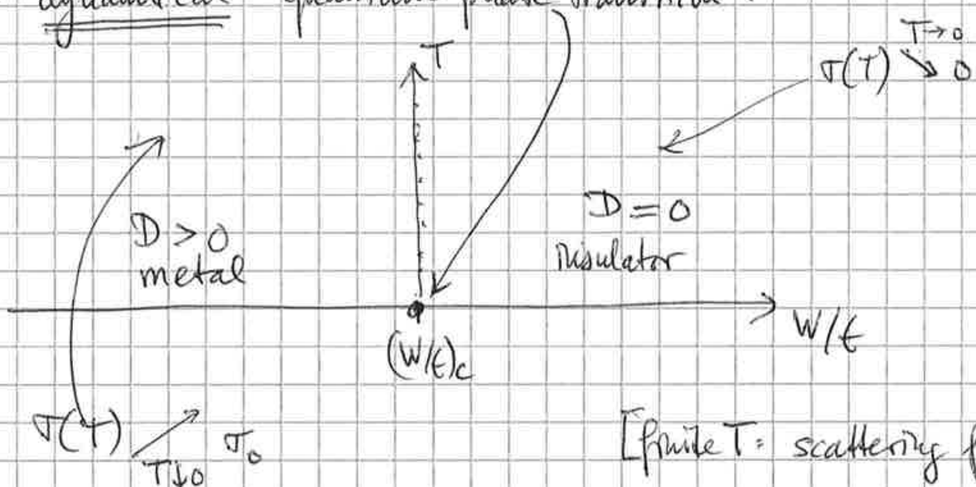
i.i.d. identically, idependently distributed

Central paramter: relative strength of disorder: $\frac{W}{t}$

General result: for $\frac{W}{t} \geq \frac{W_c}{t}$ (= function of dimension, lattice structure)

⟹ particles are localized!

⟹ dynamical quantum phase transition:



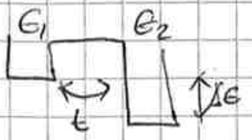
[finite T: scattering from phonons gives finite conductivity]

Characterization? Phenomenology? Nature of transition?

Two approaches to the localization transition:

- strong disorder: expansion in hopping t (locator expansion)
  - ↳ understand stability of trivial insulator ($t=0$)
  - ↳ useful to generalize to interacting case
- weak disorder: disorder ↑ ⟹ increase scattering ⟹ proliferation of weak localization (especially for low dimension)

Basic understanding: a two-well model:



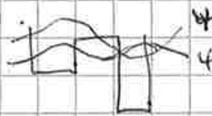
$$H = \epsilon_1 c_1^\dagger c_1 + \epsilon_2 c_2^\dagger c_2 - t(c_1^\dagger c_2 + c_2^\dagger c_1) \Leftrightarrow H = \begin{pmatrix} \epsilon_1 & -t \\ -t & \epsilon_2 \end{pmatrix}$$

$$\Delta\epsilon \equiv \epsilon_2 - \epsilon_1$$

$\Rightarrow$ Eigenfunctions: $\psi_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\quad \psi_2 = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$

where: $\dfrac{a_2}{a_1} = \dfrac{t}{\sqrt{t^2 + (\Delta\epsilon)^2}}$ and $E_{1,2} = \dfrac{\epsilon_1 + \epsilon_2}{2} \mp \sqrt{(\Delta\epsilon)^2 + t^2}$

Two limits: - weak disorder: $|\Delta\epsilon| \ll t \Rightarrow a_2 \simeq a_1 \Rightarrow \psi_{1/2}$ delocalize over both wells
("hybridization")



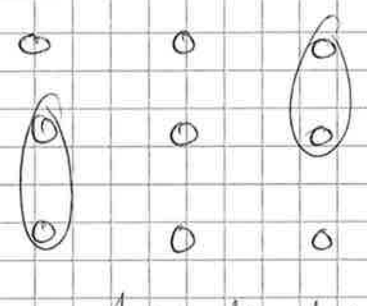
- strong disorder $|\Delta\epsilon| \gg t \Rightarrow a_2 \simeq \dfrac{t}{|\Delta\epsilon|} a_1 \ll a_1$

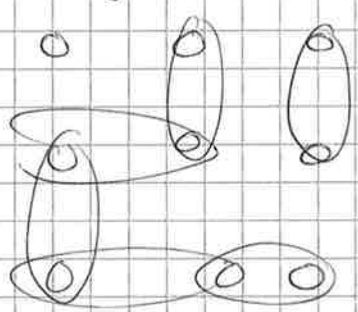
$\psi_{1,2}$ localized on one of the wells
no hybridization!

Hybridization / resonance only if $|\Delta\epsilon| \lesssim t$

Many ~~levels~~ wells (Anderson model) — qualitative picture of delocalization

Mark all resonances in the lattice: $|\Delta\epsilon|_{\text{nearest neighbor}} \lesssim t$





weak disorder $W \simeq t$

strong disorder $W \gg t$

$\Downarrow$

sparse resonances: do not percolate $\Rightarrow$ expect localized eigen-wavefunctions

$\Downarrow$

percolation of resonances $\Rightarrow$ delocalization of eigenfunctions

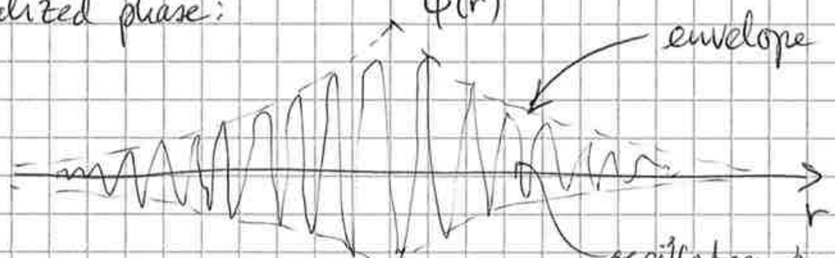
⚠ caution: This picture is too crude! Anderson localization and percolation are <u>not</u> in the same universality class! (What may be the main difference?)

But: percolation argument gives a reasonable estimate for ↓

Quantitative (numerical) analysis: 3d cubic lattice: $\left(\frac{W}{t}\right)_c = 16.8$

Phenomenology:

localized phase:



envelope $\approx |\psi(r)| \sim \exp(-r/\xi)$

$\xi :=$ localization length

oscillatory amplitude

$W \to \infty$: $\quad \xi \to 0$ (localization on single sites)

$W \searrow W_c$: $\quad \xi \nearrow \infty$ as $(W - W_c)^{-\nu}$ $\leftarrow$ critical divergence

Dynamics from initial condition: Absence of diffusion and loss of arrow of time!

$$\psi(t=0, r) = \delta_{r,0} \Rightarrow \psi(t,R) = ?$$

Recall diffusion: $\frac{\partial \rho}{\partial t} = D \Delta \rho \Rightarrow \underset{\underset{|\psi(r,t)|^2}{|||}}{\rho(r,t)} \approx \frac{1}{(Dt)^{d/2}} e^{-r^2/2Dt} \Rightarrow \langle r^2 \rangle \propto Dt$

$\Leftrightarrow$ Here: Assume exponentially localized wavefunctions:

$$\psi(r,t) = \sum_\alpha \overbrace{\psi^\alpha(r=0)}^{c_\alpha} \cdot \psi^\alpha(r) e^{-iE_\alpha t} \qquad c_\alpha = \int \psi^\alpha(r)\psi(t=0) = \psi^\alpha(r=0)$$

eigenstates: $H\psi_\alpha = E_\alpha \psi_\alpha$

From $\int |\psi|^2 dr = 1$ and orthogonality of the ψ^α it follows that $\sum_\alpha |c_\alpha|^2 = 1$

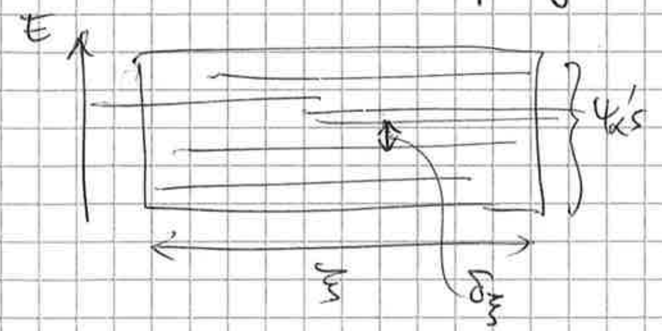
Note: only a finite number $(\sim \xi^d)$ of ψ_α have appreciable overlap with $r=0$ and thus finite $|c_\alpha|$.

$|\psi(t, r \gg \xi)| \lesssim e^{-2r/\xi}$ <u>always</u>, $\forall t$: either $\psi^\alpha(r)$ or $\psi^\alpha(r=0)$ are exponentially small!

$\Rightarrow$ there is no diffusion to large r $\Rightarrow D \equiv 0$!

What about short-time dynamics?

Consider the "level spacing in the localization volume"



$\nu :=$ average density of states $\left(\simeq \frac{1}{t} \text{ per site}\right) \stackrel{!}{=} \frac{1}{\delta_\xi \underbrace{\xi^d}}$

$\Rightarrow \delta_\xi = \frac{1}{\nu \cdot \underbrace{\xi^d}_{\#\text{ of sites in } \xi\text{-box}}}$

Dynamics is sensitive to localization only at times where discreteness of levels ($\Delta E \sim \delta_\xi$) can be resolved $\Leftrightarrow$ for $t > \tau_{\text{Thouless}} \simeq \frac{\hbar}{\delta_\xi}$!

Time evolution: Spreading during $t < \tau_{\text{Thouless}}$ up to the spatial extent ξ (anomalous diffusion!)

After that: no further expansion, quasiperiodic oscillations

$\Rightarrow$ dynamics forward and backward in time become indistinguishable!

2^{nd} law of thermodynamics has stalled before equilibrium is reached!

$\Leftrightarrow$ Bloch waves in periodic potentials: Difference: here there are exact resonances at large distances!

Dynamics
On the delocalized side?

Up to τ_{Th} no visible difference (as always at critical points!):

anomalous (slow) diffusion): $r \sim t^\alpha$ $\alpha \neq 1/2$ but $\alpha = 1/d < 1/2$ in $d > 2$

beyond τ_{Th}: crossover to normal diffusion!

Weak disorder approach

Limit of $\epsilon_i \to 0$ ($W=0$): no disorder $\Rightarrow$ $\psi_\alpha(r) = e^{i\vec{k}_\alpha \vec{r}}$: plane waves!

Weak disorder $\Rightarrow$ elastic scattering $\vec{k} \to \vec{k}'$

Simple Fermi Golden rule scattering rate $\Rightarrow$ mean free time (between scattering) τ_{mf}

$$\hbar \tau_{mf}^{-1} \sim \frac{W^2}{t}$$

W^2 ← scattering potential $\sim W$

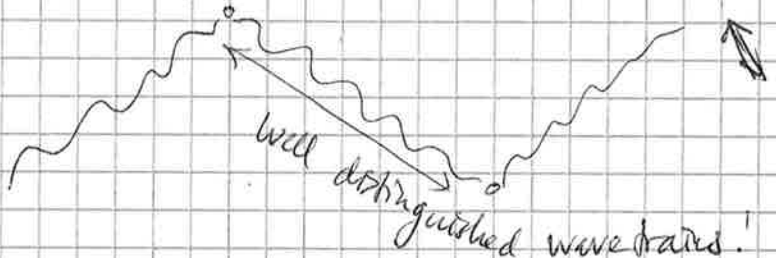
t ← only energy scale ($E_F \sim t$)

$\Rightarrow$ mean free path $l_{mf} = v_F \tau_{mf} \sim \left(\frac{t}{W}\right) a$ (lattice spacing a; $E_F \sim t \sim v_F \hbar k_F$, $k_F \sim \frac{1}{a}$ at half filling)

Fermi velocity $v_F \sim \frac{ta}{\hbar}$

Compare l_{mf} with $\lambda_F = \frac{1}{k_F}$ (Fermi wavelength):

As long as $l_{mf} \gg \lambda_F$ ($\Leftrightarrow$ $\boxed{k_F l_{mf} \gg 1}$) $\Rightarrow$ weak disorder effect.



$$\boxed{\begin{array}{l} E_F \tau_{mf} \gg 1 \\ \Leftrightarrow W \ll t \end{array}}$$

$\Rightarrow$ Disorder is perturbative $\Rightarrow$ diagrammatic perturbation theory as an expansion in $\frac{1}{k_F l_{(mf)}}$

Ioffe-Regel limit: $k_F l \to O(1)$: expect breakdown of plane waves and (possibly) of delocalized waves.

Measurement of $k_F l$:

Drude formula for conductivity:

$$\sigma_{Drude} = \frac{e \cdot n \cdot \tau_{mf}}{m} \sim e^2 k_F^d \frac{l_{mf}}{m v_F} \sim \frac{e^2 k_F^d\, l_{mf}}{\hbar\, k_F} \sim \underbrace{\frac{e^2}{\hbar}\, k_F^{d-2}}_{\text{natural unit of conductivity}} \overbrace{(k_F l_{mf})}^{\text{measures disorder}}$$

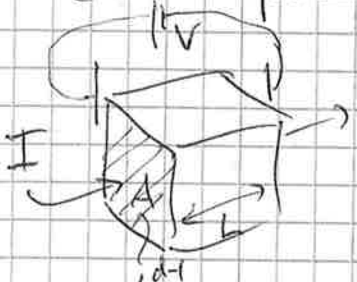
n: density: $\frac{e}{a^d} \sim e k_F^d$

Scaling theory of localization (Abrahams et al. 1979)

Goal: · understand role of dimension
· suggestion of continuous transition
· transition as a critical phenomenon!

Idea: look at conductivity σ as a function of scale L: do RG!
Further: Assume: σ is the only relevant parameter characterizing the system ($\Leftrightarrow$ "one parameter scaling")

$$[\sigma] = \left[\frac{e^2}{\hbar}\frac{1}{\text{length}^{d-2}}\right]$$



conductance
$$I = G \cdot V$$
$$G = \frac{A}{L}\sigma \quad \text{(Ohm's law)}$$
$$[G] = [e^2/\hbar] \ !$$

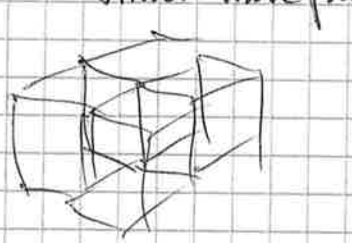
Dimensionless conductance: $G = G(L) = \sigma(L)\,\frac{L^{d-1}}{L}$

At the scale $L = \ell_{mf} \equiv \ell$: $G(\ell) = \frac{e^2}{\hbar}(k_F \ell)^{d-1}$

$\Rightarrow$ Define $g(L) \equiv \frac{G(L)}{e^2/\hbar} \Rightarrow g(\ell) \simeq (k_F\ell)^{d-1}$ "number of conducting channels"

Ioffe-Regel $\Leftrightarrow$ expect transition when $g \simeq \mathcal{O}(1)$! $\Rightarrow$ Flow of $g(L)$?

Interpretation of g:
Construct wave functions in cube $(2L)^d$ from those in cubes (L^d)



1) effective hopping between cubes:
$$\Leftrightarrow t_{eff} \simeq \hbar/\tau_{diff}(L) \qquad \tau_{Diff} \sim \frac{L^2}{D} = \tau_{Thouless}$$

recall: Einstein: $D = \sigma/e^2\nu$

2) effective disorder: level spacing between eigenfunctions:
$$W_{eff} = \delta_L = \frac{1}{\nu L^d}$$

$\Rightarrow$ effective disorder parameter: $\boxed{W_{eff}/t_{eff}} = \frac{1}{\hbar}\frac{L^2}{D}\cdot\frac{1}{\nu L^d} = \frac{e^2}{\hbar}\frac{1}{\sigma L^{d-2}} = \boxed{\frac{1}{g(L)}}$!
(for a rescaled Anderson problem, with "sites" being the wavefunctions at scale L)

Scaling with one parameter (assumed) $\Rightarrow$ (8)

$$g(2L) \overset{!}{=} \Phi(g(L)) \quad \leftarrow \text{some } L\text{-independent function!}$$

$$\Leftrightarrow \frac{g(2L)-g(L)}{g(L)} = \frac{\Phi(g(L))}{g(L)} - 1 \approx \underbrace{\frac{\Delta g}{g}}_{\Delta(\ln g)} \cdot \underbrace{\frac{L}{\Delta L (= 2L - L)}}_{\substack{=1 \\ \Delta(\log L)}} \approx \underline{\frac{d \ln g}{d \ln L}}$$

is only a function of g!

$$\Rightarrow \beta \equiv \frac{d \ln g}{d \ln L} \overset{!}{=} \beta(g) \leftarrow \text{only fct. of } g, \ \underline{\text{not of } L \text{ as well}}!$$

beta function (flow of conductance with scale) $\Rightarrow \boxed{\beta(g) = ?}$

<u>Two limits:</u>

$g \ll 1$: Strong disorder $\Rightarrow$ expect all wavefunctions localized $\Rightarrow g(L) \approx g_0 \, e^{-L/\xi}$

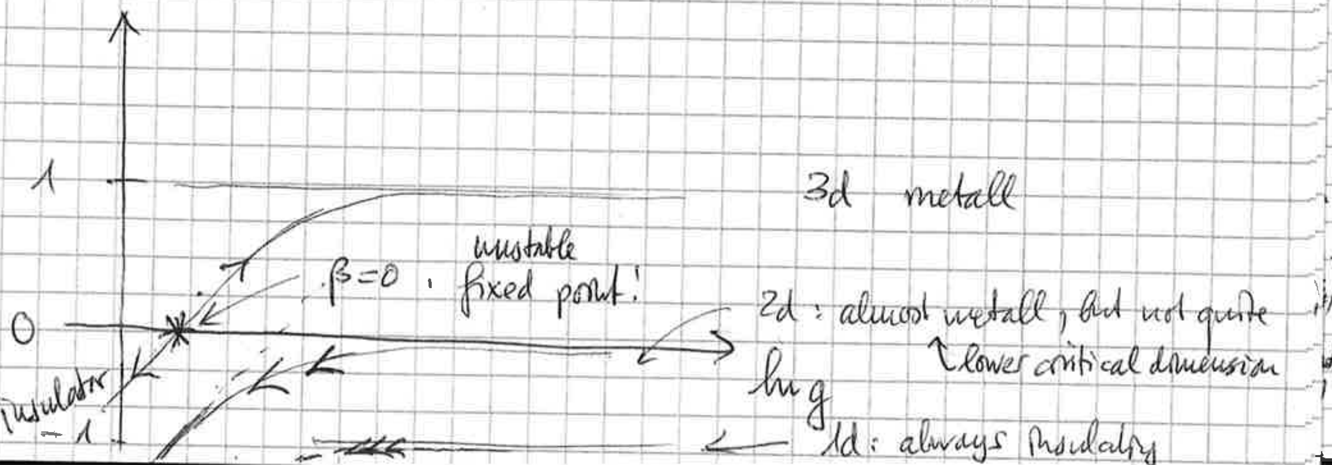
$$\Rightarrow \beta(g) \approx \frac{d(-L/\xi)}{d \ln L} = -L/\xi = \underline{\ln(g/g_0)} \xrightarrow{g \to 0} -\infty$$

$g \gg 1$: Disorder is perturbative $\Rightarrow$ compute σ via Kubo formula:

$$\sigma = \frac{dj}{dE} = \frac{-1}{i\omega} \frac{dj_\omega}{dA_\omega} = \frac{-1}{i\omega} \langle [j_\omega, j_{-\omega}] \rangle$$ in perturbation theory of disorder potential!

$$\Rightarrow \dots \Rightarrow \quad \beta(g) = d - 2 - \frac{\#}{g} \leftarrow \text{loop corrections} \quad \text{(see below)}$$

simple Ohm's law: $g \approx (\sigma_{Drude} / \frac{e^2}{h}) \, L^{(d-2)}$ (Drude)



Physics: 1d: localization sets in very fast:

small L ($\ll l$): $g(L) \approx \frac{e^2}{\hbar} \frac{l}{L} \gg \frac{e^2}{\hbar}$

then: $E_{Th} = \frac{1}{\tau_{Th}} \approx \frac{\hbar D}{L^2}$ decreases faster than level spacing: $\delta_L \approx \frac{1}{\nu L}$

$\Rightarrow$ Disorder becomes more and more relevant

$\Rightarrow$ at $g(L) \approx 1 \Leftrightarrow L \approx l \Rightarrow$ crossover to exponential localization $g \sim \exp(-L/\xi)$

$$\Rightarrow \boxed{\xi_{loc} \approx \underbrace{l_{mf}}_{\sim \frac{1}{W^2}}} \quad (1d)$$

$\sim \frac{1}{W^2}$ in weak disorder!

More general insight: In $d = 1,2$: quantum corrections to classical diffusion / random walks are important.

Classical diffusion: random walk with step size l_{mf}:

Probabilities for different paths add up!

$$P_{A\to B} = \sum_{\gamma: A\to B} P_\gamma$$

$\Leftrightarrow$ Quantum diffusion:

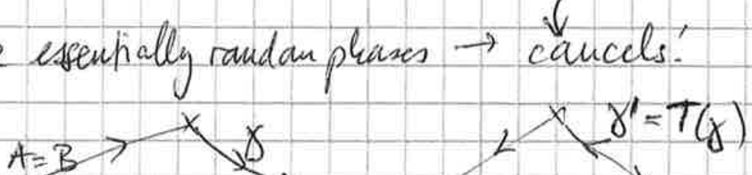
Amplitudes add!

transition matrix element

$$P_{A\to B} = |M_{A\to B}|^2 = \Big|\sum_{\gamma: A\to B} A_\gamma\Big|^2 = \sum_\gamma |A_\gamma|^2 + \sum_{\gamma\neq\gamma'} A^*_\gamma A_{\gamma'}$$

For $A \neq B$ A_γ and $A_{\gamma'}$ have essentially random phases $\rightarrow$ cancels!

BUT: $A = B$ is special:



Time-reversed paths have same scattering amplitude!

$|A_\gamma + A_{\gamma'}|^2 = 4|A_\gamma|^2$ not only $2|A_\gamma|^2$!

$\Rightarrow$ enhanced probability to return to the origin due to constructive interference of time reversed paths!
($\Leftrightarrow$ "weak localization corrections")

Interesting mesoscopic quantum interference effect!

Role of dimension?

In dimensions d a random walk returns on average $N_{ret}(t)$ times:

$$N_{ret}(t) = \int_0^t dt' \, \rho(r=0,t') = \int_0^t dt' \frac{1}{(Dt')^{d/2}} = \begin{cases} t^{1/2} & d=1 \\ \ln t & d=2 \\ \text{finite} & d>2 \end{cases}$$

$$\rho = \frac{1}{(Dt)^{d/2}} e^{-r^2/2Dt}$$

$\Rightarrow$ In $d \leq 2$, random walkers will surely come back (not enough phase space to escape!)
$\Rightarrow$ Quantum interference becomes important.

Perturbative calculation: yields:

$$g = L^{d-2}\left(\sigma_{Drude} - \int^{L^2/D} \frac{D\,dt}{(Dt)^{d/2}}\right)$$

$$\Rightarrow \beta(g) = \frac{d\ln g}{d\ln L} = \ldots = d-2 - \frac{c_d}{g}$$

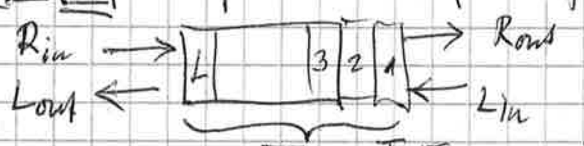
number depends on d

$$\equiv \varepsilon - \frac{a}{g} \quad \text{for } d-2 = \varepsilon \; (\ll 1)$$

(ε - expansion)

$\Rightarrow \boxed{g^* = \frac{a}{\varepsilon}}$ critical point $\Rightarrow$ critical exponents

Remarks: $\boxed{1d}$: simple exercise of transfer matrices:



$$T_{tot} = T_L T_{L-1} \cdots T_1$$

transfer matrices

$$\begin{pmatrix} R_{i+1}^{[in]} \\ L_{i+1}^{[out]} \end{pmatrix} = (T_i) \begin{pmatrix} R_i^{[out]} \\ L_i^{[in]} \end{pmatrix}$$

$$T_i = \begin{pmatrix} 1/\sqrt{t} & -i\sqrt{r/t} \\ i\sqrt{r/t} & 1/\sqrt{t} \end{pmatrix}_i$$

In terms of scattering: $S^{in \to out}$:

$$\begin{pmatrix} R_i \\ L_{i+1} \end{pmatrix} = S_i^{in-out} \begin{pmatrix} R_{i+1} \\ L_i \end{pmatrix} \doteq \begin{pmatrix} \sqrt{t} & +i\sqrt{r} \\ i\sqrt{r} & \sqrt{t} \end{pmatrix} \begin{pmatrix} R_{i+1} \\ L_i \end{pmatrix}_{in}$$

transmission + reflection: $t + r = 1$

transfer matrices T_i: random, but identically distributed.

$\det T_i = 1 \Rightarrow \det T_{tot} = 1$

$\rightarrow$ spectrum of $T_{tot} = (\lambda, 1/\lambda)$ $\qquad \lambda[T] = ?$

For length N $(T = T_N \cdots T_1) \Rightarrow \lambda_N \sim \exp(N/\xi)$ "Lyapunov exponent"

$\Rightarrow$ All ~~typical~~ entries of T typically scale as λ_N!

In particular $T_{(11)} = \frac{1}{\sqrt{t_{tot}}} \sim \lambda_N$

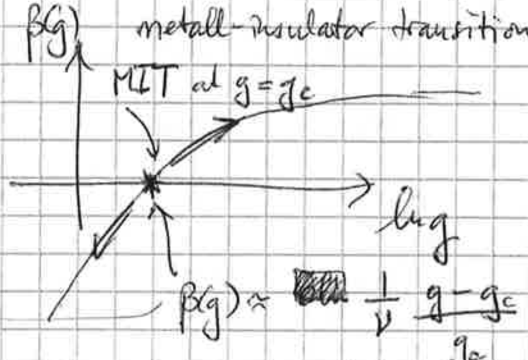
$\Rightarrow$ total transmission through the block $\approx t_{tot} \sim \frac{1}{\lambda_N^2} \sim \exp(-2N/\xi)$

$\Rightarrow$ A long block reflects everything, transmission is zero!
"complete backscattering"

$\Leftrightarrow$ all eigenfunctions are exponentially localized

NB: many exact techniques to compute ξ as function of energy, statistics of ψ_α's etc exist.

• $\boxed{3d}$: Integrate the flow!



$\Rightarrow \frac{d\ln g}{d\ln L} = \beta(g)$

$\Leftrightarrow \underbrace{\frac{dg}{g}\cdot\frac{g_c}{g-g_c}}_{dg\left(\frac{1}{g-g_c}-\frac{1}{g}\right)} = \frac{1}{\nu}\, d\ln L \quad \xrightarrow{\text{Integrate}} \quad \frac{g(L)-g_c}{g(L)}\cdot\frac{g_0}{g_0-g_c} = \left(\frac{L}{\ell}\right)^{1/\nu}$

define: $g_0 \equiv g(L=\ell)$ conductance at l_{mf}

For $L > L^*$:
- either localization $(g_0 < g_c)$ $\xi \simeq L^*$
- Ohmic behavior $g \sim \left(\frac{L}{L^*}\right)$

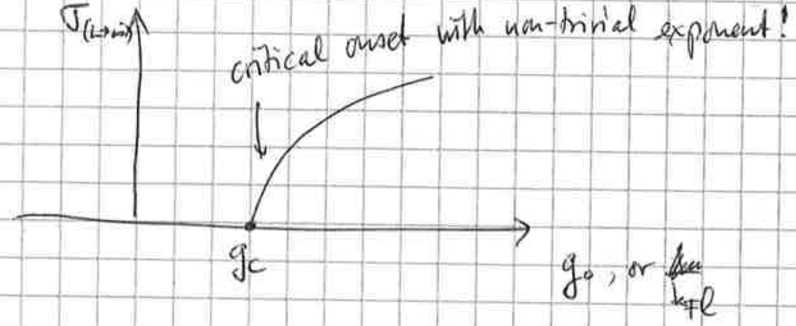
Crossover out of critical regime: when $\frac{g(L^*)-g_c}{g(L^*)} \approx O(1)$

$\boxed{\nu_{3D} \approx 1.45}$ with no special symmetries $\qquad \Leftrightarrow L^* \sim \ell\cdot\left(\frac{g_0-g_c}{g_c}\right)^{-\nu} \Rightarrow$ critical divergence of localization or correlation length $L^* \sim \xi$!

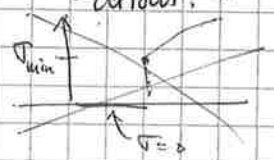
Conductivity: on the metallic side, $g_0 > g_c$

$$\Rightarrow \quad g(L) \simeq \frac{L}{L_*}\, g(L_*) \approx \frac{L}{L^*}$$

$$\Leftrightarrow \quad \sigma = \frac{e^2}{\hbar}\frac{g(L)}{L^{d-2}} = \frac{e^2}{\hbar L^*} \simeq \frac{e^2}{\hbar l_{mf}} (g_0 - g_c)^{\nu}$$



($\Leftrightarrow$ earlier wrong expect-ations:



$\sigma(T=0)$ serves as an "order parameter"

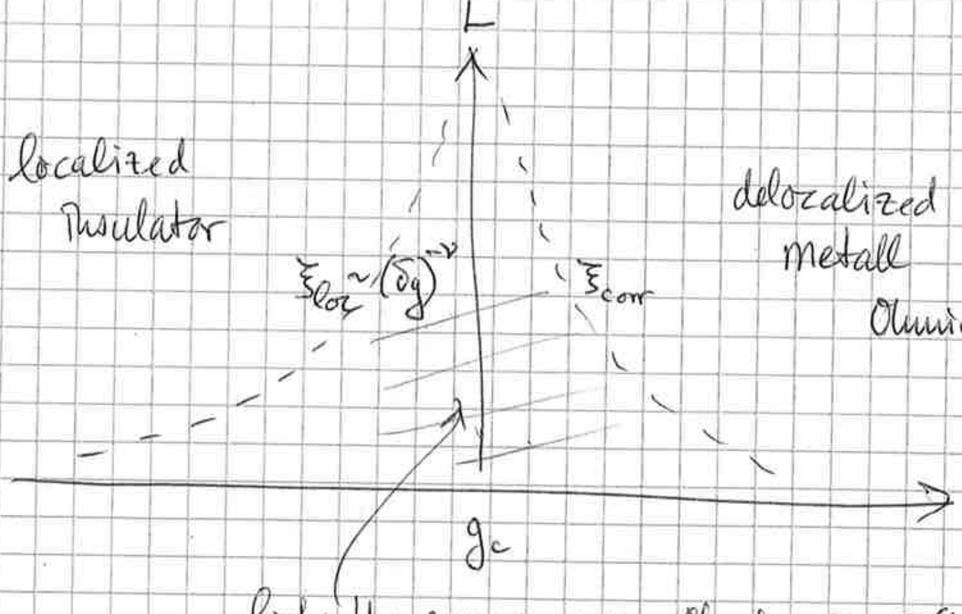
But: no symmetry is broken in a thermodynamic sense!

Anderson transition is <u>continuous</u>, even though not thermodynamic, but only <u>dynamic</u>

Note: $\sigma(T=0) \neq 0 \Leftrightarrow$ metall

The metall is the "symmetry broken", "ordered" phase!

<u>Quantum critical fan:</u>



looks the same: non-Ohmic, $\sigma = \sigma(L) \approx g_c L^{2-d}$

$$\frac{d(R^2)}{dt} = D(R) \sim R^{2-d} \Longrightarrow \text{anomalous diffusion } R \sim t^{1/d}$$

Exercise: marginal case of $\underline{d=2}$ $(\epsilon = 0)$

Exact perturbative calculation yields:

$$\beta(g) = \underbrace{(d-2)}_{=0} - \frac{1}{\pi^2}\frac{1}{g} = \frac{d \ln g}{d \ln L}$$

$$g_0 = \frac{1}{2\pi} k_F l$$

Integrate the flow until L^* where $|g(L^*) - g_0| \approx g_0$

to find $L^* = l \cdot \exp\left(\frac{\pi}{2} k_F l\right)$

$\Rightarrow$ 2d disordered systems are <u>always localized</u>

but: for weak disorder $(k_F l \gg 1)$

localization sets in at exponentially large length/time scales

<u>Probing weak localisation</u>:

logarithmic dependence.

$$\frac{\sigma_{2D}}{e^2/\hbar} = \frac{\sigma_{Drude}}{e^2/\hbar}\left(= \frac{1}{2\pi} k_F l\right) - \frac{1}{\pi^2}\ln(L/l)$$

The $\log(L)$ correction comes from accumulating quantum interference.

and requires
- coherence: no collision of the electron with other particles (electrons or phonons)
- time reversal symmetry (time reversed paths interfere positively only when TRS is preserved!)

replace L by:

$$\Rightarrow L \to L_{eff} = \min(L_\phi, L_B, L_{sample\ size})$$

L_ϕ: inelastic scattering length: $\rightarrow$ probes interactions with phonons/electrons

L_B: strength of magnetic field that destroys interference

$$\pi L_B^2 \cdot B = \phi_0 = \text{flux quantum!}$$