So far : one pure state (light Gibbs state)

$\Rightarrow$ What to do with the cavity approach when there are <u>many</u> pure states?

$\Rightarrow$ cannot work with Gibbs state (not pure)

$\Rightarrow$ need to be careful: there are more states at higher free energy density $\Rightarrow$ needs to be taken into account when adding a site ($\rightarrow$ shift of free energy!)

Consider non-disordered system that undergoes a glass transition (otherwise: replace $\eta_{cav} \longrightarrow P(\eta_{cav})$)

For a given link $i \rightarrow j$ define:

$$\rho(\eta, F) := \sum_{\alpha = \text{pure states}} \delta(\eta - \eta^{(\alpha)}_{i\to j})\, \delta(F - F^{(\alpha)}) \overset{\text{expect:}}{=} Q(\eta) \exp(\beta m (F - F_{ref}))$$

Q is normalized!

for simplicity: think of Ising systems: $\eta_{i\to j} \Leftrightarrow h_{i\to j}$

where: Q depends on $f \equiv F/N$ and $m\beta = \Sigma'(f)$

Strategy:
- We show that this assumption is self-consistent
- We choose m as a parameter, which will select f and

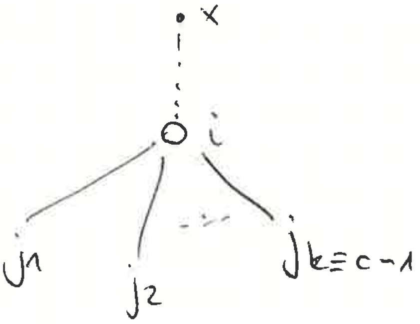
$$Q(\eta) = Q(\eta; f) = Q(\eta; m)$$

$\longrightarrow$ Tune which energy density you will study!

In the cavity:

$$\sum_\alpha \delta(\eta_1 - \eta^{(\alpha)}_{j_1\to i}) \cdots \delta(\eta_k - \eta^{(\alpha)}_{j_k\to i})\, \delta(F_{bef} - F^{(\alpha)}_{before}) = Q(\eta_1) \cdots Q(\eta_k) \cdot \exp(\beta m (F_{bef} - F'_{ref}))$$

Consider cavity recursion:



$$\eta_{i\to x} = \Phi[\eta_{j_1\to i}, \dots, \eta_{j_k\to i}]$$

as derived in a single pure state

In the system where site i is added we have:

last time's notation $z_{i\to x} = e^{-\beta \Delta F\{\eta\}}$

$$\rho(\eta_{i\to x}, F) = \sum_\alpha \delta(\eta_{i\to x} - \eta_{i\to x}^{(\alpha)})\, \delta(F - F^{(\alpha)})$$

$$= \sum_\alpha \delta(\eta_{i\to x} - \Phi(\{\eta^{(\alpha)}_{j_k\to i}\}))\, \delta\Big(F - \big(F^{(\alpha)}_{before} + \Delta F(\{\eta^{(\alpha)}_{j_k\to i}\})\big)\Big)$$

free energy before adding site i

free energy gain by adding site i in state α

plug a bunch of delta-functions:

$$= \sum_\alpha \int \prod_{\ell=1}^{k} d\eta_{j_\ell\to i}\, \delta(\eta_{j_\ell\to i} - \eta^{(\alpha)}_{j_\ell\to i}) \int dF_{bef}\, \delta(F_{bef} - F^{(\alpha)}_{before})$$

$$\times\, \underbrace{\delta(\eta_{i\to x} - \Phi(\{\eta_{j_\ell\to i}\}))\, \delta(F - (F_{bef} + \Delta F(\{\eta_{j_\ell\to i}\})))}$$

carry out $\sum$!

$$= \int \Big[\prod_{\ell=1}^{k} \big(d\eta_{j_\ell\to i}\, Q(\eta_{j_\ell\to i})\big)\Big] \times \int dF_{bef}\, \exp\big(\beta m (F_{bef} - F'_{ref})\big) \times (\,''\,)$$

do $\int dF_{bef}$

$$= \int [\quad '' \quad]\, \delta(\eta_{i\to x} - \Phi(\{\eta\})) \cdot \exp\big(\beta m (F - \Delta F(\{\eta\}) - F'_{ref})\big)$$

$$= \underbrace{\left\{ \int [\quad '' \quad]\, \delta(\eta_{i\to x} - \Phi(\{\eta\}))\, \frac{e^{-\beta m \Delta F(\{\eta\})}}{\langle \exp(-\beta m \Delta F)\rangle} \right\}}_{Q^{new}(\eta_{i\to x})} \cdot \exp\big(\beta m (F - F^{new}_{ref})\big)$$

where: $Q^{new}(\eta_{i\to x})$ is normalized:

$$\Leftrightarrow \quad \langle \exp(-\beta m \Delta F)\rangle \stackrel{!}{=} \int \left[\prod_{\ell=1}^{k} d\eta_{j_\ell \to i}\, Q(\eta_{j_\ell \to i})\right] e^{-\beta m \Delta F(\{\eta\})}$$

and the exponential distribution of free energies has the new reference value: F_{ref}^{new}

with: $e^{-\beta m F_{ref}^{new}} = e^{-\beta m F_{ref}'} \cdot \langle \exp(-\beta m \Delta F)\rangle$

$$\Leftrightarrow \quad \boxed{F_{ref}^{new} = F_{ref}' - \frac{1}{\beta m} \ln \langle \exp(-\beta m \Delta F)\rangle}$$

1 step-cavity approach: $\Leftrightarrow$

Find fixed point of $Q(\eta)$:

$$Q^{new}(\eta_{i\to x}) \stackrel{!}{=} Q^*(\eta_{i\to x}) = \int \left[\prod_{\ell=1}^{k} d\eta\, Q^*(\eta)\right] \delta(\eta_{i\to x} - \Phi(\{\eta\})) \times \frac{e^{-\beta m \Delta F\{\eta\}}}{\langle \exp(-\beta m \Delta F)\rangle}$$

"Reweighting" is very important!

How to solve this? $\Rightarrow$ Population dynamics!

<u>Note</u>: The fixed point Q^* depends on m!

The larger m, the more heavily one biases η's that result from a cavity iteration that lowers the free energy, ($\Delta F < 0$)

$\Rightarrow m \uparrow \Rightarrow$ select lower free energy states!

What free energy density is selected? How to calculate $\Phi(m)$ ($\Leftrightarrow \Sigma(f)$)

Recall: $Z_{tree} = \prod_{i\ sites} z_i \Big/ \prod_{\langle ij\rangle\ links} z_{ij}$

Same for m clones: $Z^{(m)} = \prod_{i\ sites} \underbrace{e^{-\beta\Delta f_i \cdot m}}_{z_i^m} \Big/ \prod_{\langle ij\rangle} e^{-\beta\Delta f_{\langle ij\rangle}\cdot m}$

$$\Rightarrow \Phi(m)(-\beta N) = \log Z^{(m)} = \sum_{sites} \log \langle e^{-\beta\Delta f_{site} m}\rangle_{Q(\eta;m)} - \sum_{links} \log \langle e^{-\beta\Delta f_{link}\cdot m}\rangle_{Q(\eta;m)}$$

$$\Rightarrow -\beta\Phi(m) = \log\langle e^{-\beta m\Delta f_{site}}\rangle_{Q(\eta;m)} - \frac{c}{2}\log\langle e^{-\beta m \Delta f_{link}}\rangle_{Q(\eta,m)}$$

$\Downarrow$

Obtain $\Sigma(f)$ → ground state / → threshold ⇒ complexity.

even for continuous RSB may be a reasonable approx.

⇒ Describe glassy phases qualitatively at the 1step RSB level!

• Interesting application in random satisfiability problems / glass problems:
How to actually find solutions / low energy states??

• Disordered systems (⇔ non-equivalent sites) ⇒ not just $Q(\eta)$ but $Q(\eta_{i\to j})$

but $\mathcal{R}[Q(\eta)] = \sum_{links} \delta(Q - Q_{\langle ij\rangle})$

distribution over (sites)/links

distribution over pure states.

This is needed if one wants to analyze thermodynamics. (Rather heavy!)

But: If one is interested in a given sample / realization:

⇒ Calculate $Q_{i\to j}$ from $Q_{j_k\to i}$ ⇒ "survey propagation"

η: belief.

⇔ $Q(\eta)$: survey of beliefs (in diff. states)

V

- Use m to favor states at low energy ($\Leftrightarrow$ satisfiable configurations in 3-SAT)

Idea: Run survey propagation:

- select randomly a link $i \to j$
- update it, by computing $Q_{i\to j}$ from $Q_{j_k \to i}$, $j_k \neq j$
- repeat until convergence is reached.

Now: Identify sites with very biased surveys (construct $Q_i(y_i)$ from all $Q_{j\to i}(y_{j\to i})$!)

$\Leftrightarrow$ these sites have the same strong belief (= preference for a given σ_i^* in most states.

$\Rightarrow$ It is a good idea to fix the variable σ_i into this σ_i^* and consider it non-dynamic

$\Rightarrow$ this reduces the size of the problem, but keeps most "solutions" or low energy configurations

↳ Iterate for the smaller problem, freeze "biased" variables successively $\Rightarrow$ find eventually a low energy state!

$\Rightarrow$ Very successful algorithm for hard satisfiability problems!

Remarks :

- In principle one could go ~~to~~ 2step RSB :
  One would have to consider clusters of pure states
  and then describe the statistics of clusters.
  This becomes nearly impossible to implement.
  ↳ no simple way how to implement continuous RSB (like for SK) on a random graph.

- But for infinite connectivity, $c \to \infty$: SK solution is good.
  ↳ large c expansions ? .... ⇒ interesting open problems

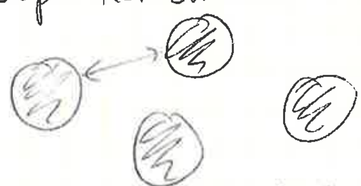
Summary of classical glass physics

- two universality classes

$p = 2$ spins (e.g. SK model) | $p > 2$ spins

- continuous transition : ⇔ sudden break-up of paramagnetic state into clusters (states)
  ...
  ⇔ $\langle s_i \rangle$ freezes suddenly to finite value ⇒ $q_{EA} > 0$ jumps!
  Later: freezing transition

- states organize in bigger clusters and superclusters ⇔ States are well separated
  "1step" RSB structure :
  → hierarchy of states in ultrametric tree



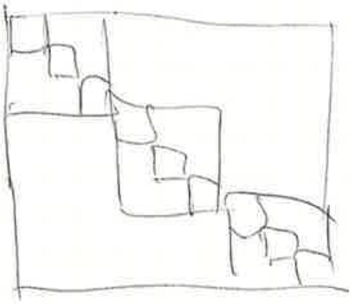
- permanent criticality, avalanches
  'deeper lying states : truly stable traps with ~~finite~~ large barriers

-

– solution :

mean field :

full replica symmetry breaking :



⟺ continuous

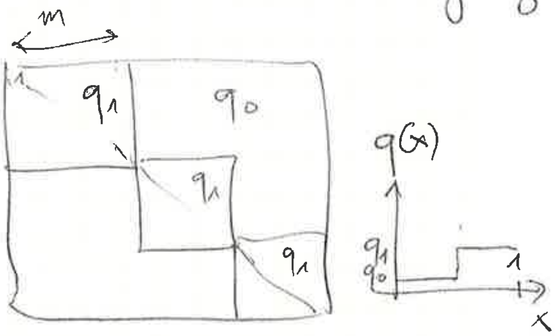
$q(x) \Longleftrightarrow P(q) = \frac{dx}{dq}$

essentially only one marginally stable solution

⟹ only one relevant free energy density [presumably]

⟺ whole range of free energy densities with stable states.

– 1step RSB // cavity approach with reweighting



m ⟷ # of clones

⟷ legendre conjugate to free energy density

· varying m ⟺ select different states

$\phi(m) = \Phi(m)/m$

extremize over m ⟹ ground state (lowest free energy)

So far : everything classical !

Interesting phase space structure

⟶ how is the glass landscape probed / sensed by quantum fluctuations ?

VIII

Quantum glasses:

fluctuations / non-commutativity :

do not commute

- transverse fields (Ising) : $H = -\sum_{ij} s_i^z J_{ij} s_i^z - \Gamma \sum_i s_i^x$

  (also p-spin version with $\Gamma \sum_i s_i^x$)

- quantum rotors with M components $\quad H = \frac{g}{2M} \sum_i L_i^2 + \frac{M}{\sqrt{N}} \sum_{ij} J_{ij} \vec{n}_i \cdot \vec{n}_j$

  $[n_i^\mu, n_j^\nu] = 0$

  $$[L_i^{\mu\nu}, n_j^\sigma] = i\delta_{ij} (\delta_{\mu\sigma} n_j^\nu - \delta_{\nu\sigma} n_j^\mu) \qquad \mu\nu = 1, \dots, M$$

  $L^{\mu\nu}$: generator of rotation of vector n in the $\mu\nu$-plane

  $\Rightarrow$ acts like a transverse field ; Ising is like $M \to 1$

  Interactions commute ! Fluctuations come from local operator

$\Leftrightarrow$ non-commuting interactions

$$H = -\sum_{ij} \vec{S}_i \cdot \vec{S}_j \cdot J_{ij} \qquad \vec{S} : \text{generators of } SU(2)$$

$\hookrightarrow$ or of SU(M)

e.g. Heisenberg interactions

$$\Rightarrow [S_i^x S_j^x, S_i^y S_k^y] \neq 0 \quad !$$

$\Rightarrow$ fluctuations due to noncommuting interactions !

$\Rightarrow$ $\boxed{\text{Very different quantum phase transitions !}}$

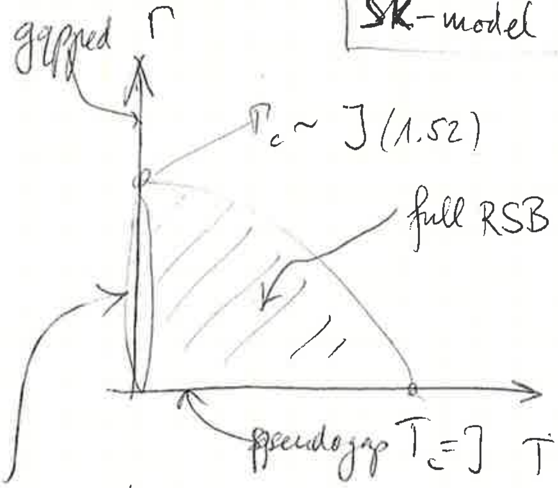
Mean field theory ? $\quad N \to \infty$: fully connected

extra simplification: $M \to \infty$ (rotors or SU(M))

(Sachdev, Ye, Read)

$\Rightarrow$ solvable !

transverse field
SK-model, similar for rotor models



everywhere gapless : critical phase ⟺ marginal stability
↳ collective spin waves of arbitrarily low frequency!
↳ nature: Soft directions of $\frac{\partial^2 G}{\partial m_i \partial m_j}$: oscillations along those directions yield low frequency modes.

Note: this is spectacular: there is no continuous symmetry ⇒ no Goldstone modes. Yet the glass is gapless!

⟺ Heisenberg (su(2)) fully connected

Bray + Moore 1980: proof that T_c still exists, but only $\sim 0{,}14\, J$

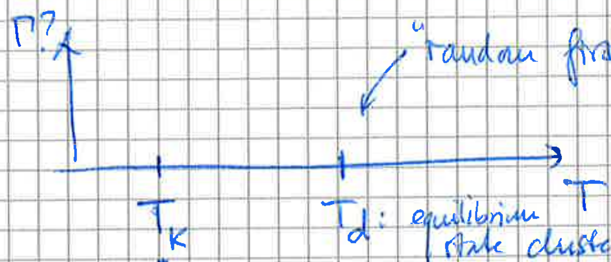
Current Interest: Sachdev + Ye: Decompose $\vec{S}^{\alpha\beta} \in su(M)$ as $f_\alpha^+ f_\beta$ or $b_\alpha^+ b_\beta$ (bosonic or fermionic partons) with $\alpha\beta = 1, \dots, M$; $\sum_\alpha b_\alpha^+ b_\alpha = \frac{M}{2}$

$M \to \infty$ becomes solvable! ⇒ gapless spectrum with no quasi-particles at high $T > T_c$! (Self-energies/scattering rates $\sim \omega^{1/2}$ as $\omega \to 0$
$\gg \omega$ ⇒ no sharp quasi-particles)

Extremely fast thermalization! "Fastest scramblers"
very strong scattering
⇒ unusually high spectral weight at low frequencies

What about 1step-RSB systems in transverse fields?

Recall:

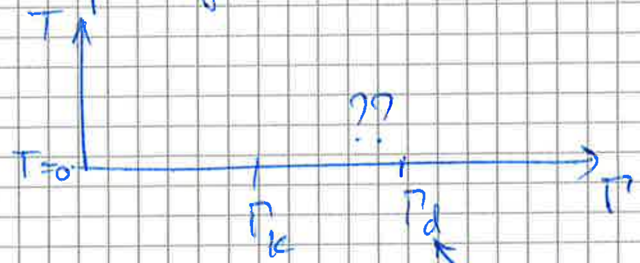


"random first order transition": onset of in-state correlations is jump-like

T_d: equilibrium state clusterizes

T_K: equilibrium consists in one (the lowest) state → freezing.

What to expect from a transverse field?



This is impossible! At $T \to 0$ the entropy of many states cannot compensate for loss of entropy in the liquid state:

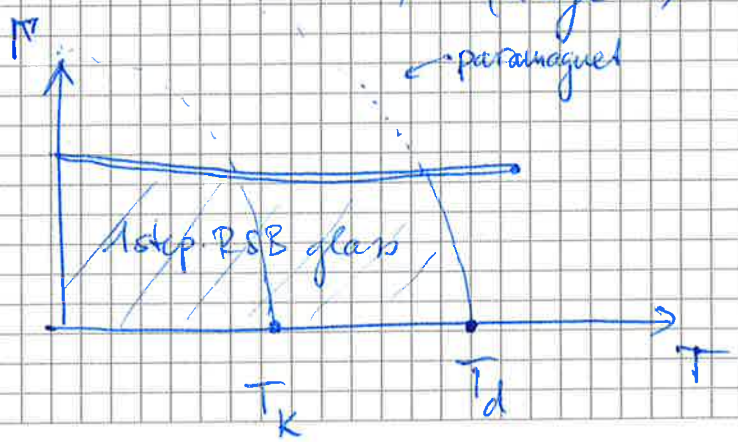
Because there is no entropy at $T \to 0$!

And: $\langle s_i \rangle = m_i$ cannot turn on smoothly: energy gain cannot compensate loss in quantum fluctuations!

⇒ The only possibility is a first order transition:

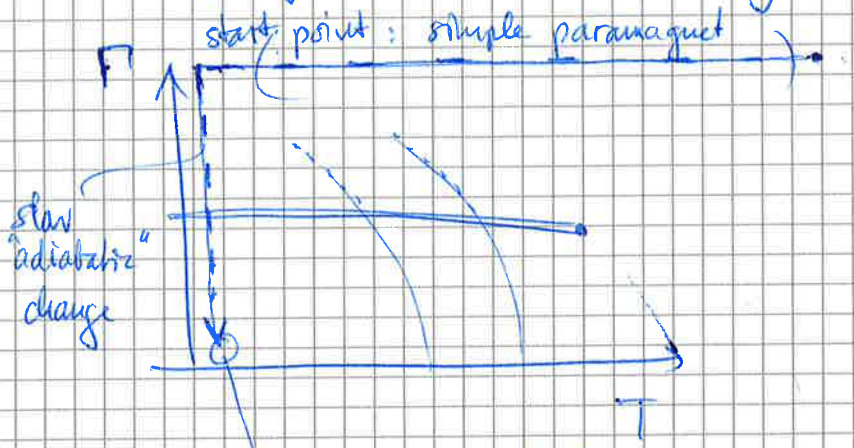
PM (large Γ) ⟶ Glass state (small Γ)

paramagnet



What happens as one explores this phase diagram with a path $(\Gamma, T)(t)$?

Idea: While it may be hard to find the GS of the p-spin glass (or 3-SAT) by adiabatic computation:



low energy state??

⇒ Would contradict rigorous theorems that one cannot find (in non-exponential time) states deeper down from the threshold than a certain margin.

$\boxed{\text{What is the catch?}}$

Mean field: Paramagnet : $\langle s_i^z \rangle = 0$ ($\langle s_i^x \rangle \neq 0$ along Γ)

⟷ Spin glass : $\langle s_i^z \rangle \simeq \pm 1$

extremely different!

↳ tunneling from low energy paramagnet to equal energy spin glass state: requires macroscopically many spins to change state → tunneling rate is $\exp(\#\cdot N)$, no matter what state you want to end in!

⇒ "Nucleation" of the glass state (or any) happens only on exp. long time scales