Replica theory of the SK model

$J_{ij} = J_{ji}$ ← not independent

$$H_{SK} = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j = \underbrace{\sum_{i<j}}_{\text{count each link only once.}} J_{ij} s_i s_j \qquad \overline{J_{i<j}^2} = \frac{J^2}{N}$$

Naive approach: ($\Leftrightarrow$ replica symmetric approach):

$$h_{i,\mathrm{eff}} = \underbrace{\sum_{j(\neq i)} J_{ij} m_j}_{\text{random variable with mean}=0}$$

(looks like TAP without Onsager, but m_j should be interpreted as $m_j^{(i)} \leftarrow m_j$ in absence of 0)

and variance

$$\overline{\left(\sum_j J_{ij} m_j\right)^2} = \overline{\sum_{j,k\neq i} J_{ij} J_{ik} m_j m_k}$$

assume J, m uncorrelated:

$$\approx \sum_{j,k} \underbrace{\overline{J_{ij} J_{ik}}}_{\simeq \frac{J^2}{N}\delta_{jk}} \overline{m_j m_k} = \frac{J^2}{N}\sum_j m_j^2 = J^2 q$$

$$\boxed{\Longrightarrow P(h_{i,\mathrm{eff}}) = \frac{e^{-h^2/2J^2 q}}{\sqrt{2\pi J^2 q}}}$$

→ self-consistency:

$$q = \langle m_i^2\rangle = \langle \mathrm{th}(\beta h_{i,\mathrm{eff}})^2\rangle$$

$$\overset{z = h/\sqrt{J^2 q}}{=} \int \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \mathrm{th}^2\left(\beta (J^2 q)^{1/2} z\right)$$

"SK-equation" SK, 1975

Wrong! But why? (at low $T<T_c$ where $q\neq 0$)

$$\overset{q\ll 1}{\approx} \beta^2 J^2 q \int \frac{dz}{\sqrt{2\pi}} z^2 e^{-\frac{1}{2}z^2} = \beta^2 J^2 q$$

Correct results:
- $q=0$ for $T>T_c$ (high T)
- $(\beta J)_c = 1 \Rightarrow T_c = J$ ✓
- continuous transition with $q \sim \frac{T_c - T}{T_c} + \ldots$

Still: Wrong because:

1) J's and m's are correlated! $\Rightarrow$ h is not Gaussian distributed

2) Stability criterion: $(\beta J)^2\, \overline{(1-m_i^2)^2} \overset{!}{\leq} 1$ ($\Rightarrow$ recall pseudo-gap!) is violated by this "solution".

Good exercise: Add random fields and analyze stability criterion! $\Rightarrow$ Find still a transition

L100 Lev Landau

⇒ More sophisticated approach: replica theory (⇔ TAP which is very tricky)
(as for spherical p-spin); find the right saddle point
⇒ very interesting structure + physics!

$$\overline{Z^n}^{J_{ij}} = \overline{\sum_{\{S_i^a\}=\pm 1} \exp\left[-\sum_{i<j}\beta J_{ij}\left(\sum_{a=1}^n s_i^a s_j^a\right)\right]}^{J_{ij}}$$

$$\overset{\text{average over } J_{ij}}{=} \sum_{\{S_i^a\}=\pm 1} \exp\left[+\frac{(\beta J)^2}{2N}\sum_{i<j}\sum_{a,b=1}^n s_i^a s_j^a \, s_i^b s_j^b\right]$$

$$\sum_{i<j}\sum_{a,b} = \frac{1}{2}\left(\sum_{i,j} - \sum_{i\neq j}\right)\sum_{a,b}$$
$$= \frac{1}{2}\sum_{i,j}\left(2\sum_{a<b} + \sum_{a=b}\right) - \frac{N}{2}(i=j)\sum_{a,b}$$
$$= \frac{1}{2}n\sum_{i,j}(a=b) + \sum_{i,j}\sum_{a<b} - \frac{N}{2}n^2 \,(i=j)$$

$$= \sum_{\{S_i^a\}} \exp\left[\frac{(\beta J)^2}{2N}\left[\sum_{a<b=1}^n\left(\sum_i s_i^a s_i^b\right)^2 + \frac{1}{2}nN^2 - \frac{Nn^2}{2}\right]\right]$$

Hubbard-Stratonovich! $e^{x^2/2} = \int \frac{dy}{\sqrt{2\pi}} e^{-y^2/2} e^{yx}$ ← linear in x!

⇒ $x = \sum s_i^a s_i^b$ ⇒ decouple sites!

$$= \sum_{\{S_i^a\}} e^{\frac{(\beta J)^2}{4} nN - O(n^2)} * \int \prod_{a<b=1}^n \left[dq^{ab}\left(\frac{\beta J N^{1/2}}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}N(\beta J)^2 q_{ab}^2} \exp\left((\beta J)^2 \sum_{a<b} q^{ab}\sum_i s_i^a s_i^b\right)\right]$$

$O(n) \underset{n\to 0}{\longrightarrow} 0$

all i are equivalent

$$= e^{\frac{(\beta J)^2}{4}nN} \int \prod_{a<b}\left[dq_{ab}\left(\frac{\beta J N^{1/2}}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}N(\beta J)^2 q_{ab}^2}\right]\left(\sum_{\{s^a\}=\pm 1} e^{(\beta J)^2\sum_{a<b} q^{ab} s^a s^b}\right)^N$$

$$\Rightarrow \overline{Z^n} = e^{nN\frac{(\beta J)^2}{4}} \int \prod_{a<b} dq_{ab}\left(\frac{\beta J N^{1/2}}{\sqrt{2\pi}}\right)\exp\left(-N\cdot G(\{q_{ab}\})\right)$$

with

$$G(q_{ab}) = \frac{1}{2}(\beta J)^2\sum_{a<b} q_{ab}^2 - \ln\left[\sum_{S^a}\exp\left((\beta J)^2\sum_{a<b} q_{ab} s^a s^b\right)\right]$$

Now: Saddle point! ($N\to\infty$! (before $n\to 0$ (?!)) ← works, even though not obvious.

Criterion: $\frac{\partial^2 G}{\partial q_{ab}\partial q_{cd}} \geq 0$ ⇒ viable saddle point.

Minimize G over viable saddle points.

Saddle point Eq.:

$$\frac{\partial G}{\partial q_{ab}} = 0 \iff$$

$$q_{ab} = \langle s^a s^b \rangle_{\text{eff}} = \frac{\sum_{\{s^a\}} s^a s^b \, e^{(\beta J)^2 \sum_{a<b} q_{ab} s^a s^b}}{\sum_{\{s^a\}} e^{(\beta J)^2 \sum_{a<b} q_{ab} s^a s^b}}$$

q_{ab} is the self-consistently calculated spin-spin overlap!

Plug into G: $$-\beta f = -\lim_{n\to 0} \frac{G(\{q^*_{ab}\})}{n} + \frac{(\beta J)^2}{4}$$

$$= \frac{(\beta J)^2}{4} - \frac{(\beta J)^2}{4} \lim_{n\to 0} \frac{1}{n} \sum_{a\neq b} q_{ab}^2 + \lim_{n\to 0} \frac{1}{n} \ln\left[\sum_{\{s_a\}} \exp (\beta J)^2 \sum_{a<b} q_{ab} s^a s^b\right]$$

Replica symmetry: $G(q_{ab}) = G(q_{\pi(a)\pi(b)}) \quad \pi \in S(n)$

Indeed: $\sum_{a<b} q_{ab}^2$ obvious by $\pi \in S(n)$

$\ln\left(\sum_s \exp(q_{ab} s_a s_b)\right)$: simply relabel s_a's.

$\Rightarrow$ with any q_{ab}, also $q_{\pi(a)\pi(b)}$ is a viable saddle point!

$\Rightarrow$ need to sum over equivalent saddle points, in case they are not invariant under $S(n)$ $\iff$ if replica symmetry is broken.

Replica symmetry $\iff$ invariance under $S(n)$

$\iff q_{a\neq b} \equiv q$ ($q_{aa} \equiv 1$ can be defined for convenience, but was not necessary)

Physical content of saddle point q_{ab}?

"Overlap matrix": how does it connect to the distribution of physical overlaps: $P(q) := \sum_{\alpha,\beta} w_\alpha w_\beta \, \delta(q - q_{\alpha\beta})$?

Note: $P(q)$ in general depends on couplings J_{ij} (not self-averaging!) But we can ask about its disorder average: $\overline{P(q)}$, which can be computed via the replica approach.

Idea: compute the sum over spin correlators squared, both directly ~~and~~ using pure states, and via replicas $\Rightarrow$ Relation between $\overline{P(q)}$ and q_{ab}!:

1-point function: $$q^{(1)} := \frac{1}{N}\sum_i \overline{\langle S_i\rangle^2} = \frac{1}{N}\sum_i \sum_{\alpha\beta} \overline{w_\alpha w_\beta \langle S_i\rangle_\alpha \langle S_i\rangle_\beta}$$
$$= \sum_{\alpha\beta} \overline{w_\alpha w_\beta q_{\alpha\beta}} = \int dq\, \overline{\sum_{\alpha\beta} w_\alpha w_\beta \delta(q-q_{\alpha\beta})}\, q = \int dq\, \overline{P(q)}\, q$$

k-point function:
$$q^{(k)} := \frac{1}{N^k}\sum_{i_1\dots i_k} \overline{\langle S_{i_1}\dots S_{i_k}\rangle^2} = \frac{1}{N^k}\sum_{i_1\dots i_k}\sum_{\alpha\beta} \overline{w_\alpha w_\beta \langle S_{i_1}\dots S_{i_k}\rangle_\alpha \langle S_{i_1}\dots S_{i_k}\rangle_\beta}$$
$$= \frac{1}{N^k}\sum_{i_1\dots i_k}\sum_{\alpha\beta} \overline{w_\alpha w_\beta \langle S_{i_1}\rangle_\alpha \cdots \langle S_{i_k}\rangle_\alpha \langle S_{i_1}\rangle_\beta \cdots \langle S_{i_k}\rangle_\beta} = \sum_{\alpha\beta} \overline{w_\alpha w_\beta q_{\alpha\beta}^k}$$
$$= \int dq\, \overline{P(q)}\, q^k$$

$\Rightarrow q^{(k)}$ is just the k'th moment of $\overline{P(q)}$!

Calculating the same quantities with replica?

E.g.: $$q^{(1)} = \frac{1}{N}\sum_i \overline{\langle S_i\rangle^2} = \overline{\frac{1}{Z^2}\sum_{\{S_i^{a=1}, S_i^{a=2}\}} \frac{1}{N}\sum_i S_i^1 S_i^2\, e^{-\beta H(S^1) - \beta H(S^2)}}$$
$$= \lim_{n\to 0} \overline{\sum_{S^a} \frac{1}{N}\sum_i S_i^1 S_i^2\, e^{-\beta \sum_{a=1}^n H(S^a)}}$$

added: $n-2$ extra replicas $\Rightarrow Z^{n-2} \xrightarrow[n\to 0]{} Z^{-2}$

Now: disorder average, decouple, take saddle point
$$\Rightarrow q^{(1)} = \int \prod_{a<b} dq_{ab}\, e^{-NG(q_{ab})}\, q_{12} \approx q_{12}^{SP} e^{-NG(q_{ab}^{SP})} \xrightarrow[G\sim n\to 0]{} q_{12}^{SP}$$

But: If q_{ab} breaks replica symmetry q_{12} will differ from q_{1a} with general a. $\longrightarrow$??

In fact: one needs to sum/average over all equivalent saddle points.

$$\Rightarrow \quad q^{(1)} = \lim_{n\to 0} \frac{\sum_{\pi \in S(n)} q_{\pi(1)\pi(2)}}{\sum_{\pi \in S(n)} 1} = \lim_{n\to 0} \frac{\sum_{a\neq b} q_{ab}}{n(n-1)} \quad \longleftarrow \text{different choices for } \pi(1)\,\pi(2)$$

Similarly: $$q^{(k)} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} (q_{ab})^k \overset{!}{=} \int dq\, \overline{P(q)}\, q^k$$

Or by Taylor expansion: $$\int dq\, \overline{P(q)}\, f(q) = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} f(q_{ab})$$

$$\Leftrightarrow \quad \overline{P(q)} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \delta(q - q_{ab})$$
$$= (\text{fraction of elements in } q_{ab} = q')\cdot \delta(q-q')$$

$\Longrightarrow$ Explicit relation between probability of overlaps and saddle point q_{ab}. Recall for RS: $q_{a\neq b} = q_0$

$$\Rightarrow \overline{P(q)} = \lim \frac{n(n-1)}{n(n-1)}\, \delta(q-q_0) = \delta(q-q_0)$$

$\Rightarrow$ single pure state

Recall 2: 1step ansatz:

$$q_{ab} = \begin{pmatrix} \overbrace{1 \quad q_1 \quad q_1 \quad q_1}^{m} & q_0 & \cdots & q_0 \\ & 1 & & & & \\ & & 1 & & & q_0 \\ & & & 1 & & \\ & & & & 1 \quad q_1 & \\ & q_0 & & & q_1 \quad 1 & \\ & & & & & 1 \quad q_1 \\ & & & & & q_1 \quad 1 \end{pmatrix} \Rightarrow \overline{P(q)} = \lim_{n\to 0} \left[\frac{(m-1)\delta(q-q_1) + (n-m)\delta(q-q_0)}{(n-1)} \right]$$

$$= (1-m)\,\delta(q-q_1) + m\,\delta(q-q_0)$$

How about the SK model?

What is the right structure of the Saddle point of $G(q_{ab})$?

RS ansatz : $q_{a\neq b} = q_0$! Find $q_0 \neq 0$ at $T = T_c = J$. ✓

but : $\frac{\partial^2 G}{\partial q_{ab} \partial q_{cd}}$ is not positive ! $\Rightarrow$ not a stable S.P. !

(the "replicon" eigenvalue $\Leftrightarrow \lambda_{rep} = 1 - J^2 \overline{\chi_{ii}^2} \overset{!}{\geq} 0$

$\Leftrightarrow$ (TAP validity)

is violated ! $\lambda_{rep} < 0$.

$\Rightarrow$ Replica symmetry must be broken ! Also $S(T=0) < 0$ ↯

1 step : Computation of $G(q_{ab})$: Decouple groups of m replica by Hubbard - Stratonovich, and all replica by another one.

Result : i) energy $U(T \to 0)$ comes much close to simulation

ii) $S(T=0) = -0.01$ ($\Leftrightarrow -0.16$ for RS)

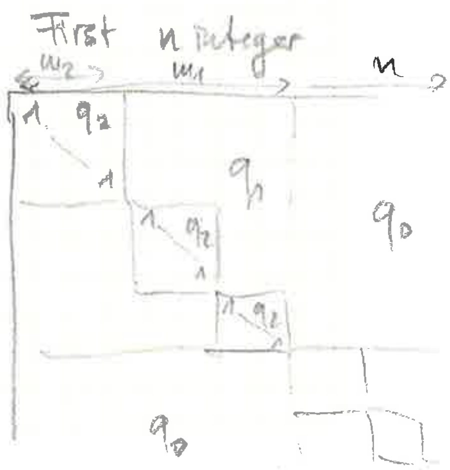
$\to$ much better

iii) replicon (worst eigenvalue of $\partial^2 G$) is less negative

↳ Further blocking structure of q_{ab} ?!

2-steps : • n/m_1 big groups of size m_1, $n > m_1$

• m_1/m_2 small groups of size m_2 inside each large group $\quad m_1 > m_2$



What to do in the limit $n \to 0$

$$1 < m_2 < m_1 < n \longrightarrow 0 < n < m_1 < m_2 < 1$$

revert the order! why?

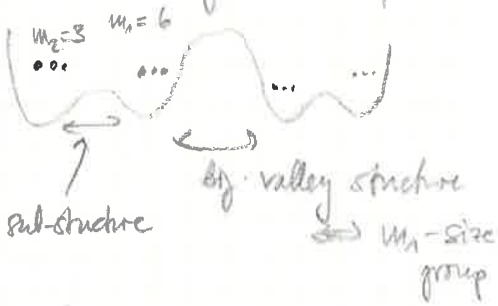
$$\overline{P(q)} = \lim_{n\to 0} \frac{1}{n-1}\left[(n-m_1)\,\delta(q-q_0) + (m_1-m_2)\,\delta(q-q_1) + (m_2-1)\,\delta(q-q_2)\right]$$

$$\rightarrow \quad m_1\,\delta(q-q_0) + (m_2-m_1)\,\delta(q-q_1) + (1-m_2)\,\delta(q-q_2)$$

$\overline{P(q)}$ must be positive! $\rightarrow$ $\underline{1 > m_2 > m_1 > 0}$

Small group inside large group: should have large overlap!

$\Rightarrow$ $\underline{q_2 > q_1 > q_0}$



Define "order parameter function":

$$q(x) \quad x \in [0,1]: \quad q(x) \begin{cases} = q_2 & m_2 < x < 1 \\ = q_1 & m_1 < x < m_2 \\ = q_0 & 0 < x < m_1 \end{cases}$$

(first line of q_{ab} read from right to left, or last line read left $\rightarrow$ right)

$$\hookrightarrow \quad \overline{P(q)} = \int_0^1 dx\, \delta(q - q(x)) = \left.\frac{1}{dq/dx}\right|_{q(x)=q} = \frac{dx}{dq} \;!$$

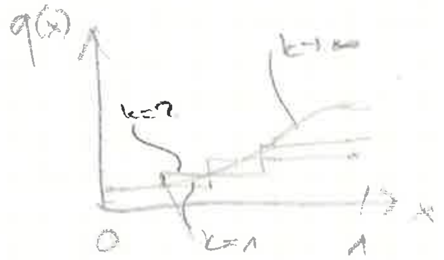
The function $q(x)$ captures the structure of low energy states ($\overline{P(q)}$)

If it is not constant $\Rightarrow$ replica symmetry is broken.

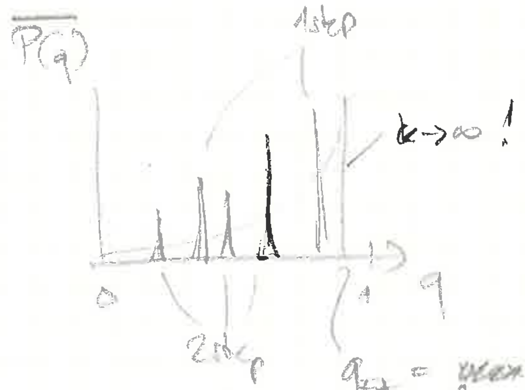
2 step solution: yet better, but need to go on!

↳ k-step with $k \to \infty$! $\quad S_{T=0}(k \to \infty) \to 0$!

↳ $q(x)$ becomes a smooth, increasing function:



$\Longleftarrow$



$q_{EA} = \lim_{k \to \infty} q_k$

Stability of saddle points?

gapless, but positive!

↳ very soft $\Longleftrightarrow$ $\dfrac{\partial^2 G^{TAP}(m_i)}{\partial m_i \partial m_j}$

is gapless, too!

$\Longleftrightarrow$ marginal stability

Remember on susceptibility: $\chi^{ZFC} \overset{?}{=} \beta(1-q_{\alpha\alpha}) \overset{?}{=} \beta(1-q_{EA})$ ✓

$\overset{!}{=} \beta\left(1-\max_{a \neq b} q_{ab}\right) = \beta(1-\bar{q})$!

thermodynamic (FC) susc.: $\dfrac{\partial m}{\partial h} = \lim_{n \to 0} \dfrac{\partial^2 f}{\partial h^2} = \dfrac{1}{n} \underbrace{\dfrac{\beta}{N} \sum_{i a, j b} \langle s_i^a s_j^b \rangle}_{\text{full solution} \Rightarrow \sim T \ ✓} \underset{i=j \text{ only}}{=} \beta\left(1 + \sum_{b (\neq a)} q_{ab}\right)$

Organisation of pure states?

→ Ultrametric structure of pure states

Consider 3 copies: 1, 2, 3

overlaps: $q_{1,2} = \frac{1}{N} \sum_i \langle S_i^1 \rangle \langle S_i^2 \rangle$

$$P(q_{12}) = \sum_{\alpha,\beta} w_\alpha w_\beta \, \delta(q_{12} - q_{\alpha\beta}) = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \delta(q_{12} - q_{ab})$$

Similarly: $$P(q_{12}, q_{23}, q_{13}) = \sum_{\alpha\beta\gamma} w_\alpha w_\beta w_\gamma \, \delta(q_{12} - q_{\alpha\beta})\, \delta(q_{13} - q_{\alpha\gamma})\, \delta(q_{23} - q_{\beta\gamma})$$

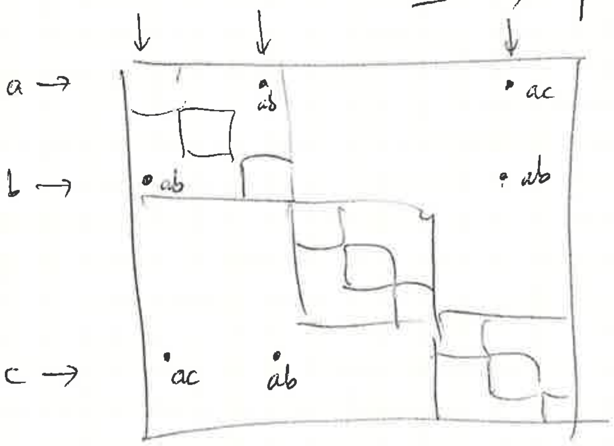
$$= \lim_{n\to 0} \frac{1}{n(n-1)(n-2)} \sum_{a\neq b\neq c} \delta(q_{12} - q_{ab})\, \delta(q_{13} - q_{ac})\, \delta(q_{23} - q_{bc})$$

sum over all saddle points

Now, Parisi's ansatz for the saddle point implies a very peculiar structure: $P(q, q', q'') \equiv 0$

<u>unless</u> i) $q = q' = q''$

<u>or</u> ii) $q = q' < q''$ or cyclic permutation hereof.



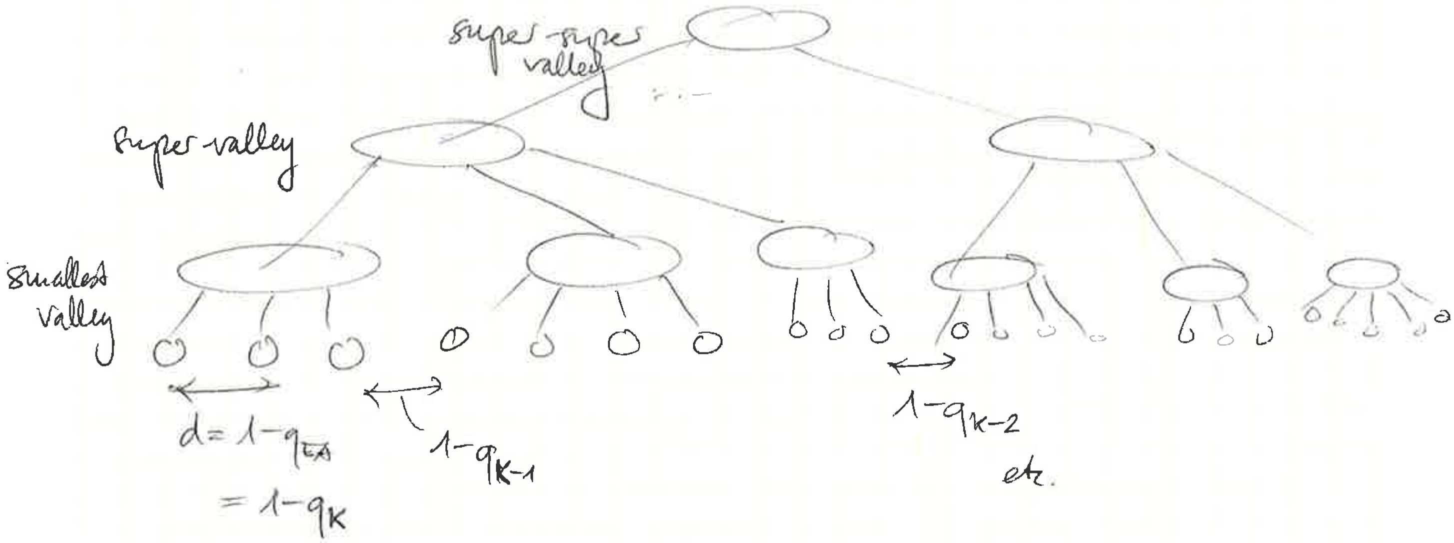
Phase space distance: $d_{ab} \equiv 1 - q_{ab}$:

Only possibility:

$$d_{ab} \leq d_{ac} = d_{bc}$$

closer together ⇒ c is equally far from both

⇒ "Ultrametric structure"



Distance $1-q_{ab}$ only depends on how high in the tree the first common ancestor lies!

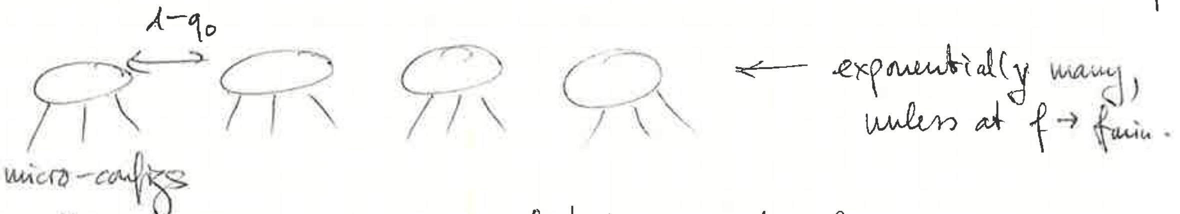
⇒ All pure states can be clustered hierarchically according to distance $1-q$. (strong numerical evidence for $N\to\infty$)

Note: this is very unusual!

SK model: infinitely many hierarchy levels, (for $N\to\infty$ only, of course)

⇔ p-spin like models with 1-step RSB: (coloring, 3SAT, ...)

just one level of hierarchy: pure states are all at the same distance $1-q_0$.



with overlap q_{EA}: depends on selected energy density

How does one solve saddle point equations in the limit $k \to \infty$ (continuous $q(x)$) ?

1) T close to T_c: q is small $\Rightarrow$ expand $G(q_{ab})$ $\Rightarrow$ Landau expansion
$\Rightarrow q_{EA} \sim |T_c - T|$

2) General: Perform Hubbard-Stratonovich in a differential manner:
(including $T \to 0$) $\quad \Delta m \to 0 \quad \Delta q \to 0$

$\Rightarrow$ Solve differential equation + self consistency condition; interesting self-similar structure in the $T \to 0$ limit.

Continuous RSB ($q(x)$) $\Rightarrow$ ensures marginal stability, including $\overline{\beta(1-m_i^2)^2} = 1$; $P(y) \sim |y|$ etc.

Permanent criticality for all $T < T_c$!

3) RSB transition exists <u>also</u> in a field! $H = -\frac{1}{2}\sum_{ij} J_{ij} s_i s_j - h \sum_i s_i$: $\boxed{\text{Almeida-Thouless line}}$

4) Interesting aspect of dynamics:

Decay of correlations from $C(t) \propto q_{EA} \to q < q_{EA} \to 0$

More precisely: wait t_w after quench, then look at $C(t; t_w)$

$\Rightarrow$ decay over times $t \gtrsim t_w$ (diverge when $t_w \to \infty$)

In equilibrium: $\frac{\partial \langle s_i \rangle}{\partial h_j} = \beta \langle s_i s_j \rangle_c \Rightarrow R(t; t_w) = \frac{\partial \langle s(t) \rangle}{\partial h(t_w)} = \frac{-1}{T_{eff}} \frac{\partial}{\partial t} \langle s(t) s(t_w) \rangle$

$\hookrightarrow$ generalizes to out-of-equilibrium dynamics $\beta \to \beta_{eff}(C)$

$\beta_{eff} = \beta \cdot x(q = C)$

$x < T \to T_{eff} = \frac{T}{x} > T$

Slow response feels a higher temperature!

$1/T_{eff} = \frac{\partial \Sigma_{valley}}{\partial f_{valley}} \Big|$ hierarchy scale where $q_{valley} = C$