

Directed polymers as a glass, pinning of elastic manifolds

Freezing transition of the directed polymer

Replica-related clone method to capture complex energy
landscapes

Pinning of elastic manifolds

Recapitulation

Specific models with exact solutions in the KPZ universality class:

- **Polynuclear growth** → map to problem of longest increasing subsequence of a permutation → purely combinatorial problem
- **Totally asymmetric exclusion process TASEP**
- **Directed polymer** in the limit of **delta-correlated disorder**: replica treatment for moments of the partition function Z^n in the long distance limit + scaling assumption for $\ln(Z)$ → $\theta = \frac{1}{3}$.

All predict **Tracy-Widom distribution** for the height.

Universality of the KPZ class beautifully confirmed in experiment.

Recapitulation

Directed polymer as a simple 'glass'

A partition function of a system with N degrees of freedom is a sum of $O(\exp(c N))$ terms! It usually takes exponential time to compute it exactly.

Monte Carlo simulations nevertheless allow to sample phase space, but only if the phase space is relatively easy to explore. It becomes very hard / slow when Z is extremely broadly distributed or when huge energy barriers have to be overcome.

Disordered systems with lots of **different low energy configurations** (= glasses) defy this usually. → **Hard problems!**

The **directed polymer** is an **interesting exception**: The partition function can be calculated recursively, in linear rather than exponential time in its length N !

TODAY: Reveal the glassy aspect of directed polymers & understand pinning + metastability

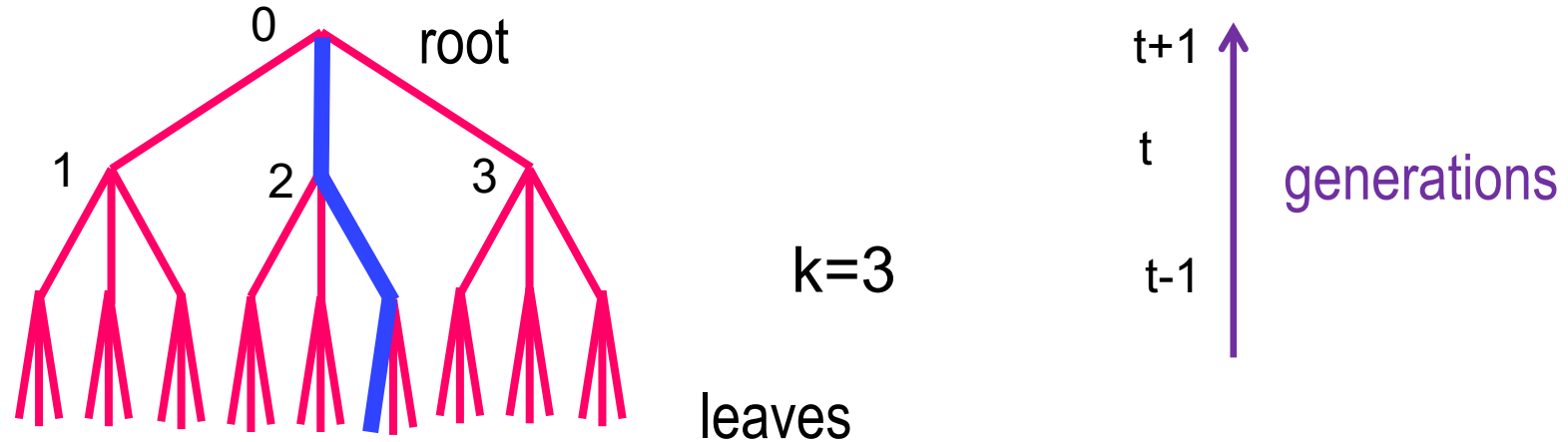
Directed polymer as a glass problem

So far: directed polymer in finite transverse dimension, mostly $d=1$

- A simpler, solvable case arises for a large number of transverse dimensions

Cayley tree:

Neglect that some second neighbors coincide in finite dimension



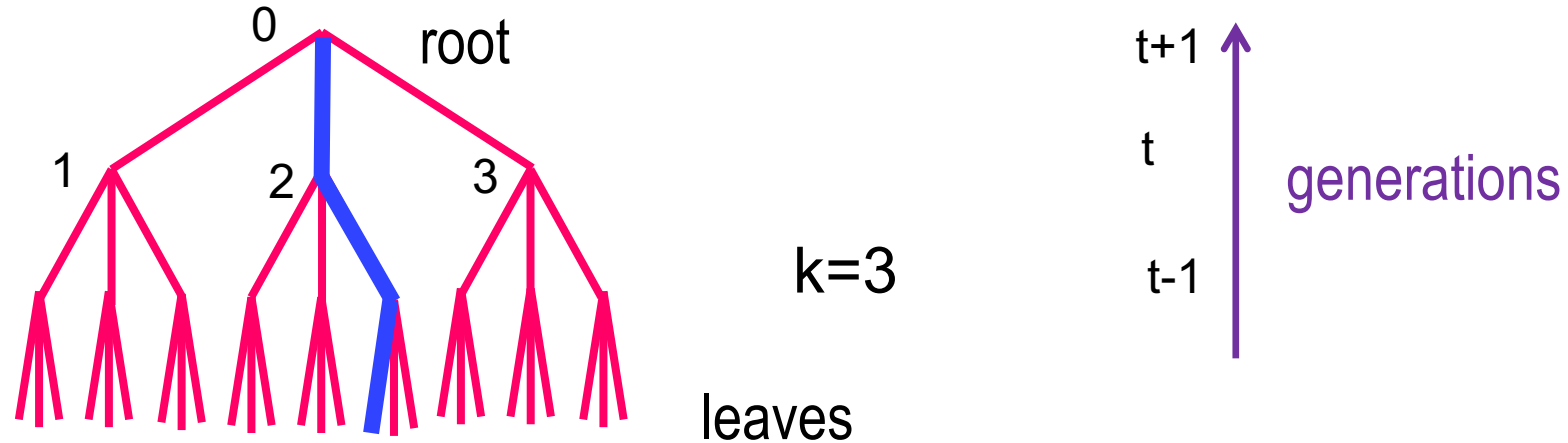
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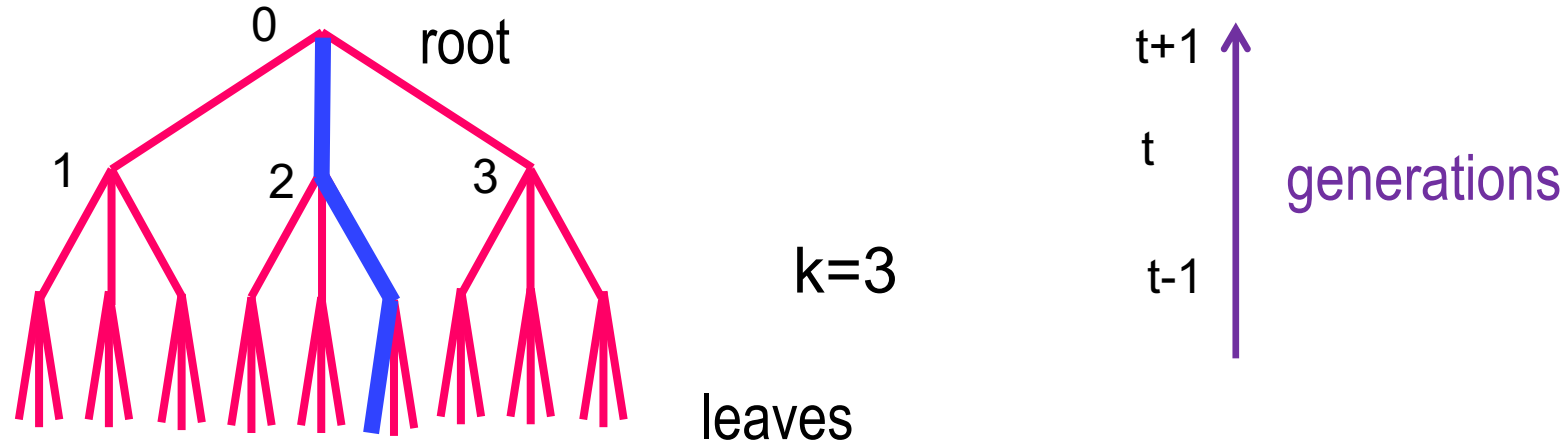
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- A polymer goes from any leaf (bottom to the top), its energy being the sum of site energies ε_i encountered along the path, $E = \sum_{i \in path} \varepsilon_i$
- Recursion equation for partition function of increasing generation:

$$Z_0(t+1) = e^{-\beta \epsilon_0} (Z_1(t) + \dots + Z_k(t)) \quad Z(0) \equiv 1$$

Directed polymer as a glass problem

Simple recursion for the average partition function (first moment of Z):

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$$\longrightarrow \overline{Z(t+1)} = \overline{k e^{-\beta\epsilon_0} Z(t)}$$

$$\longrightarrow \overline{Z(t)} = [\overline{k e^{-\beta\epsilon_0}}]^t \equiv \exp(-\beta f_{\text{ann}} t)$$

This computes the so-called “annealed” free energy f_{ann} (per unit length of the polymer)

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Annalaed versus quenched averages

Note: Computing the disorder average $\bar{Z}$ means to treat both, configurations of the polymer *and* disorder degrees of freedom, as dynamical variables! This thus treats the disorder as ‘annealed’ (that is, to be averaged over) in the relevant time window over which thermodynamic properties are probed:

$$-\beta F_{\text{ann}} := \log \bar{Z} \qquad f_{\text{ann}} = \lim_{N \rightarrow \infty} F_{\text{ann}}/N$$

To be contrasted with the **usual situation in disordered systems**: The **disordered potential/couplings are fixed** in time. In this case the relevant quantity is the free energy in a fixed (“quenched”) **disorder** realization. Since the free energy density in typical disorder configurations averages automatically over disorder in different portions of the sample (= **self-averaging**), the **free energy density tends to its disorder average**

$$-\beta F_{\text{qu}} := \overline{\log(Z)} = -\beta \bar{F} \qquad F/N \xrightarrow{N \rightarrow \infty} \bar{F}/N = F_{\text{qu}}/N$$

Annalaed versus quenched averages

- Jensen's inequality for convex functions implies that

$$\begin{aligned}\log Z_{\text{ann}} &\equiv \log \overline{Z} \geq \overline{\log Z} \equiv \log(Z_{\text{qu}}) \\ &\rightarrow F_{\text{ann}} \leq F_{\text{qu}}\end{aligned}$$

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- HOWEVER: It *can happen* (as we will show to be the case here) that at high temperatures, the annealed free energy *density* coincides with the quenched one:

$$\frac{F_{\text{qu}} - F_{\text{ann}}}{N} \xrightarrow{N \rightarrow \infty} 0 \quad \rightarrow \quad f_{\text{ann}} = f_{\text{qu}} \quad (T > T_c)$$

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Turns negative for

$$T < T_c = \frac{\Delta}{\sqrt{2 \log(k)}}$$

Entropy crisis and freezing transition

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Reason:

As usual, at low enough T , any moments of Z (here the first moment) start being dominated by rare disorder.

$S_{\text{ann}} < 0$ reflects that very rare disorder (exponentially rare in system size N , having probability $\sim \exp(s_{\text{ann}} N) \ll 1$) yields the dominant contribution to $\bar{Z} = \overline{\exp(-\beta f N)}$

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The freezing transition or (static) glass transition.

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The above strategy is an elegant, often used proof for the existence of a (glass) transition. -- To be determined: at which $T_g \geq T_c$ does the transition happen?

Entropy crisis and freezing transition

Generating deeper insight:

The clone method (R. Monasson (1995))

Picture of glasses: Disorder generates low-lying energy configurations (= “valleys”) separated from other configurations by barriers.

(Here “valleys” correspond just to the different polymer configurations. Two configurations always differ beyond some point in the tree and thus are typically very far from each other in phase space.)

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Idea: Instead of calculating the partition function for a single copy of the system, we imagine to weakly couple m copies of the system together, such that they all populate the same valley, but otherwise don't interact noticeably. → Separately weigh (valley) entropy and valley energy!

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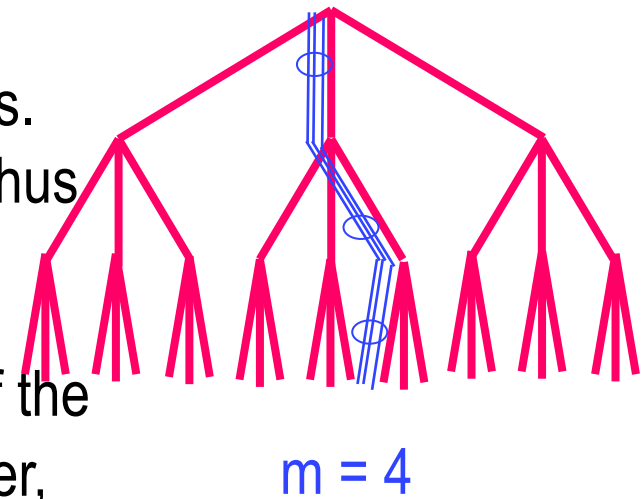
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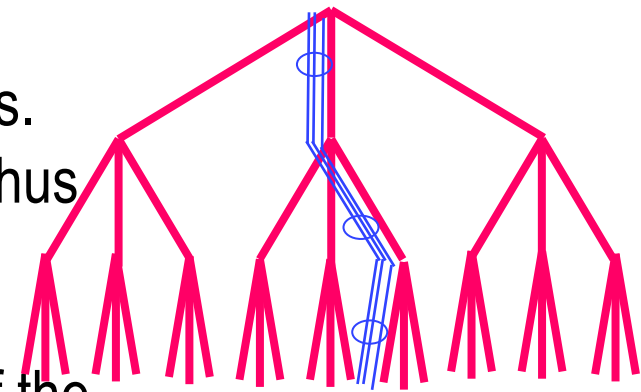
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But eventually:
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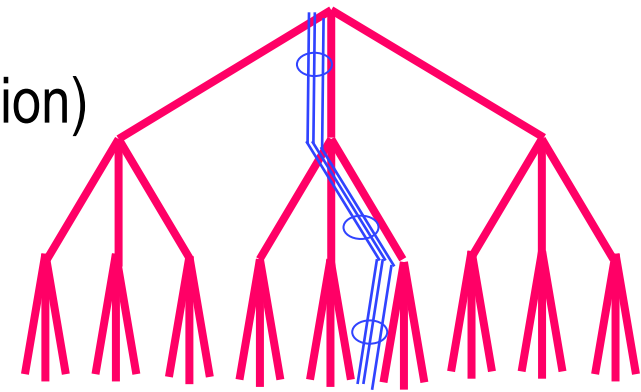
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In a glass: expect exponential number of “valleys” of a given (free) energy density f

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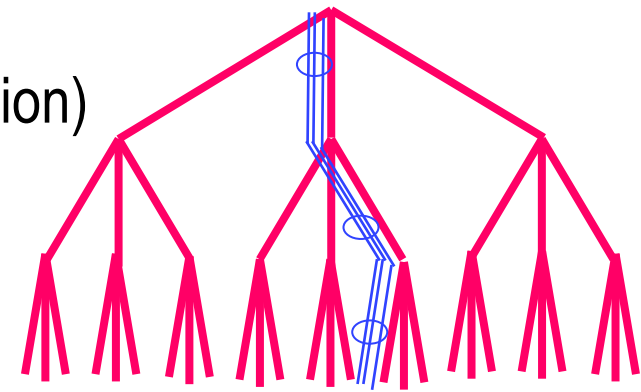
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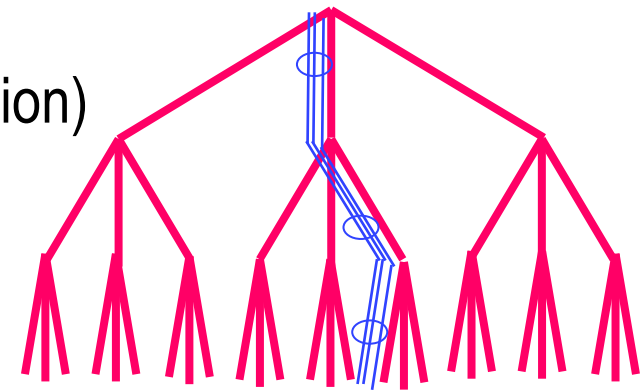
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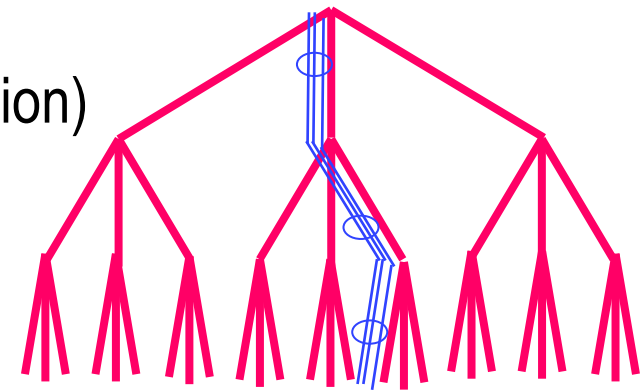
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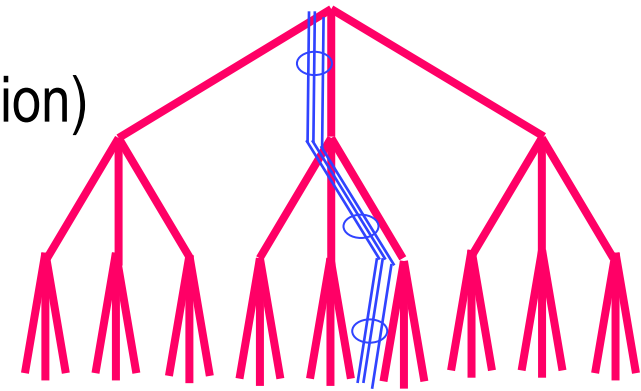
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“replicated” free energy”

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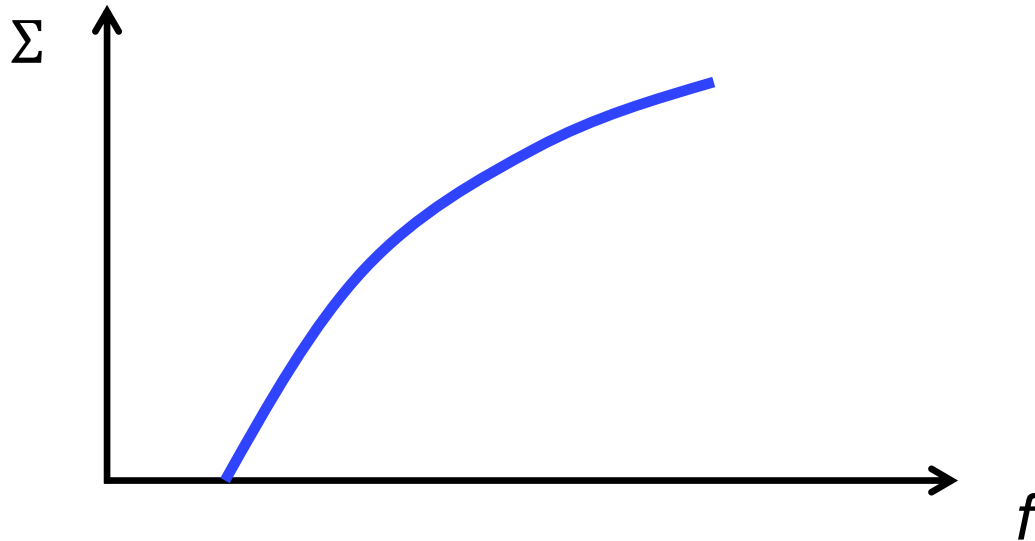
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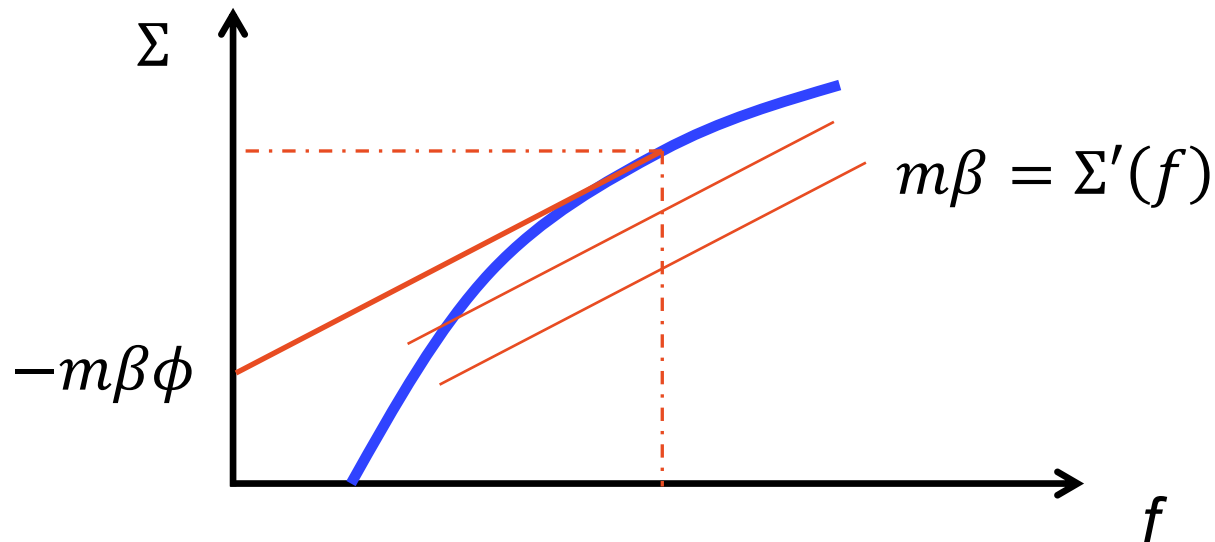


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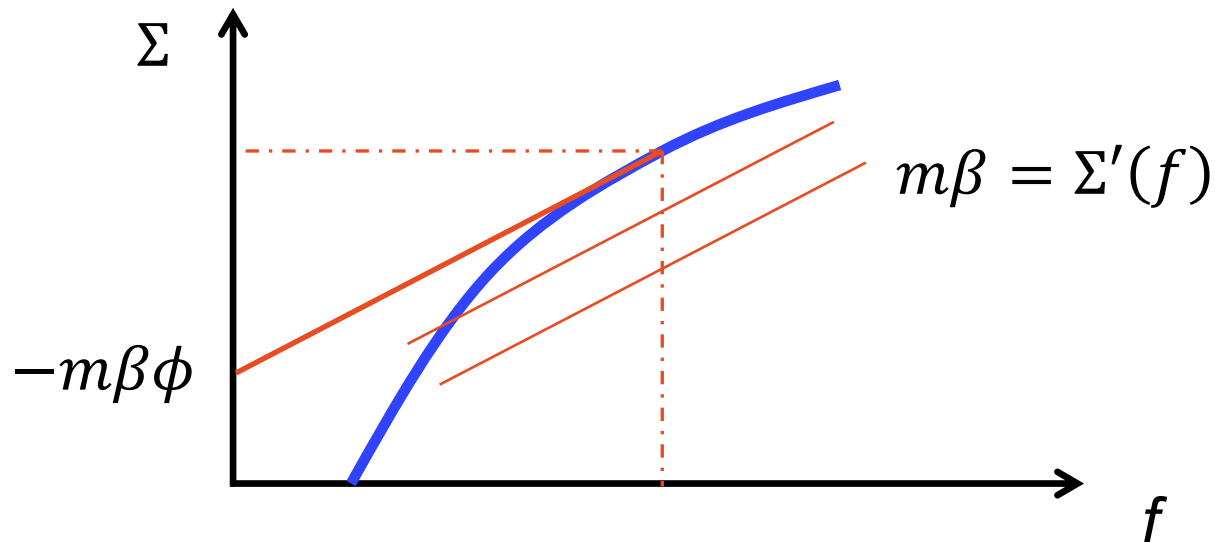


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cf. thermodynamic stability :

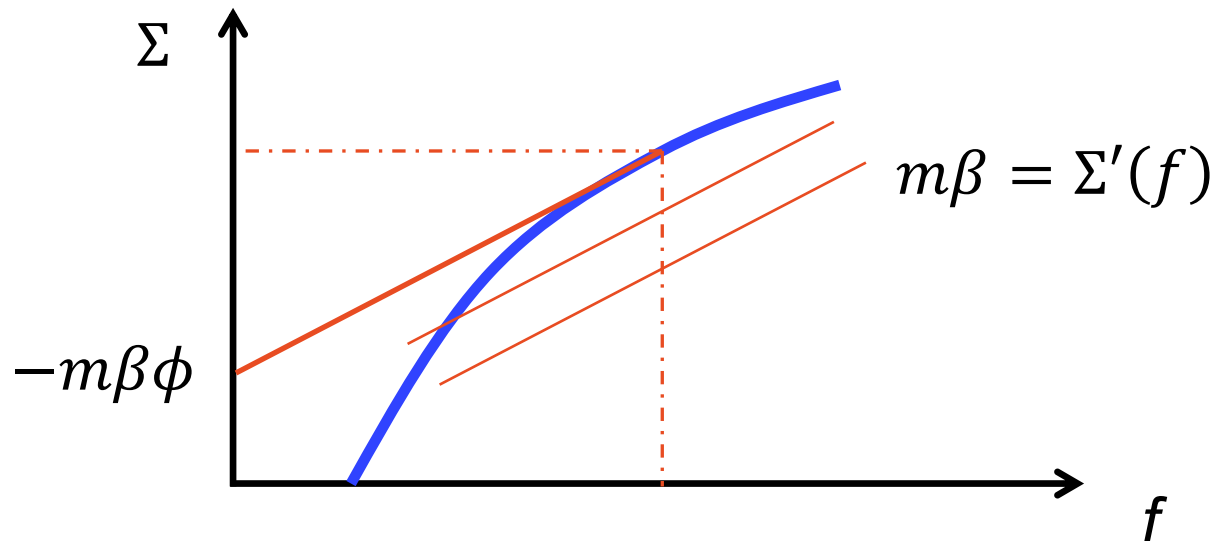
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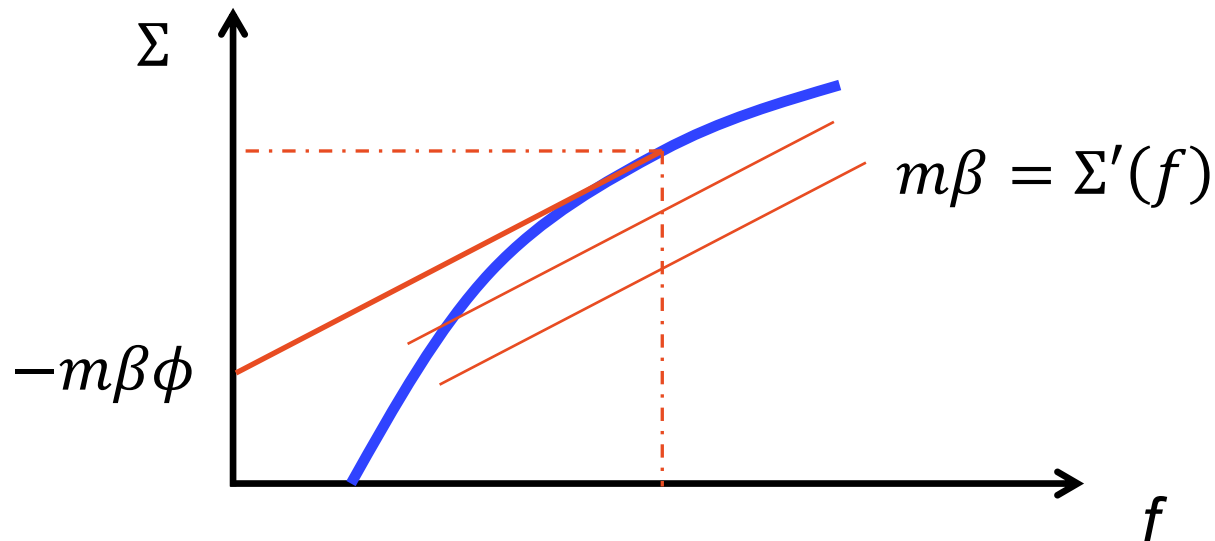
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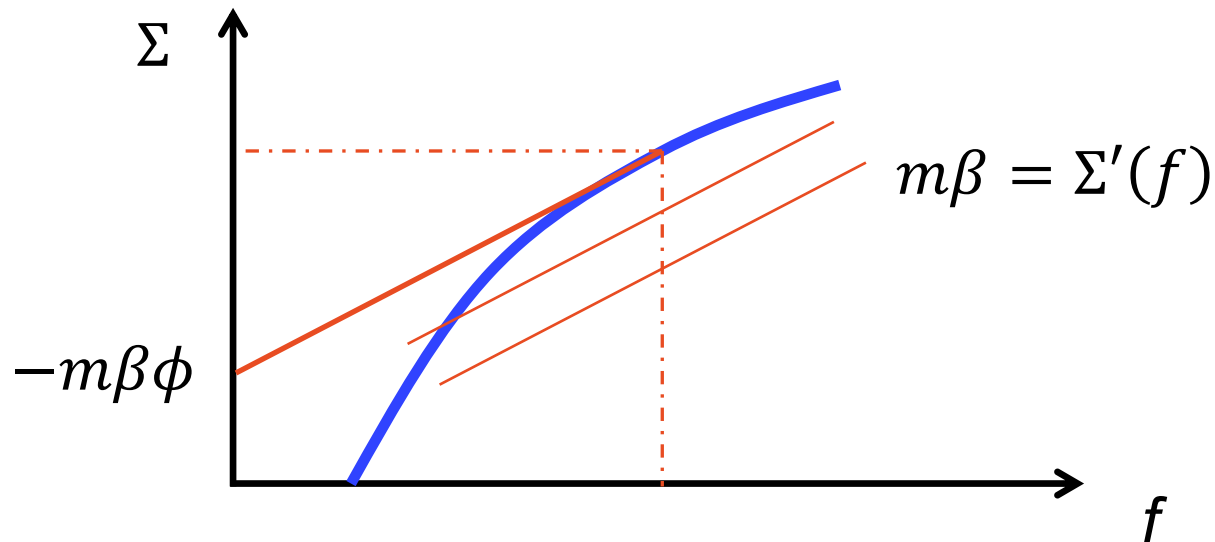
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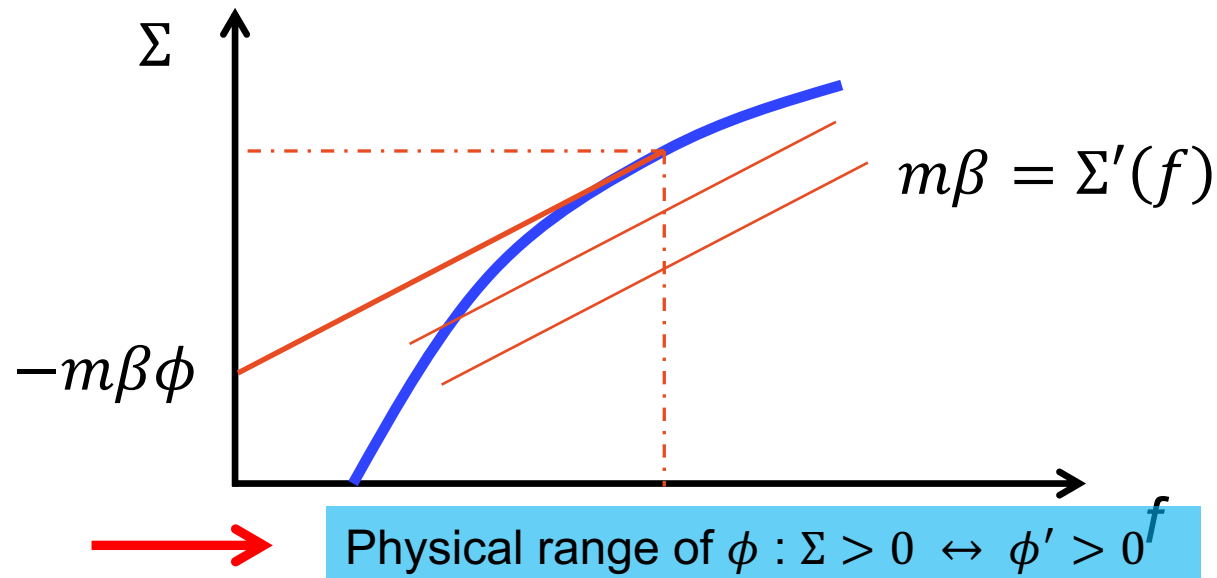
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Suppose we have calculated the replicated free energy $\phi(m)$ for a range of m (not necessarily integer)

($N \gg 1$) $\rightarrow -m\beta\phi(m) = \max_{f | \Sigma(f) \geq 0} [\Sigma(f) - m\beta f]$

Quenched free energy of the actual system:

$$e^{-\beta F} = \int_{\Sigma(f) \geq 0} df e^{N(\Sigma(f) - \beta f)}$$

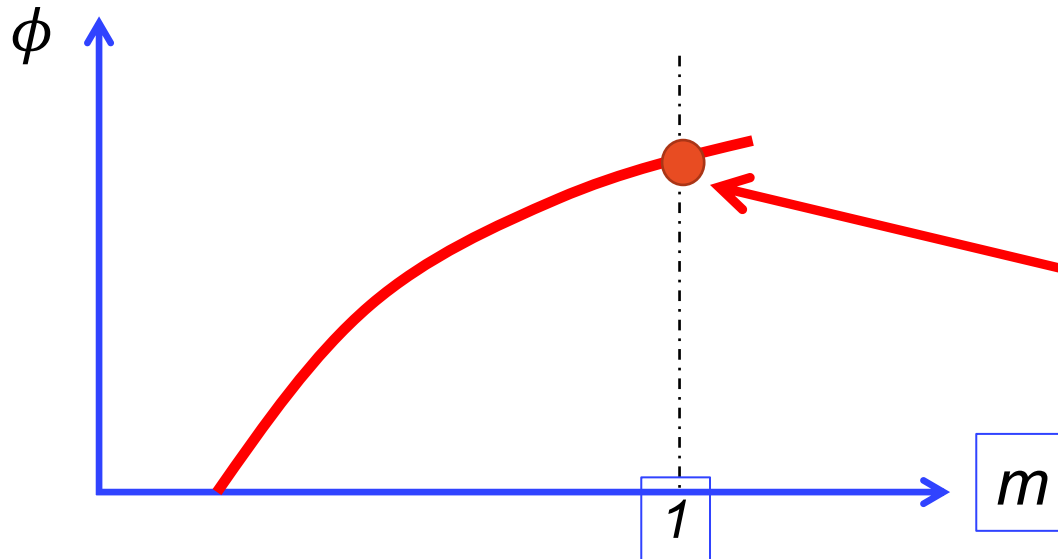
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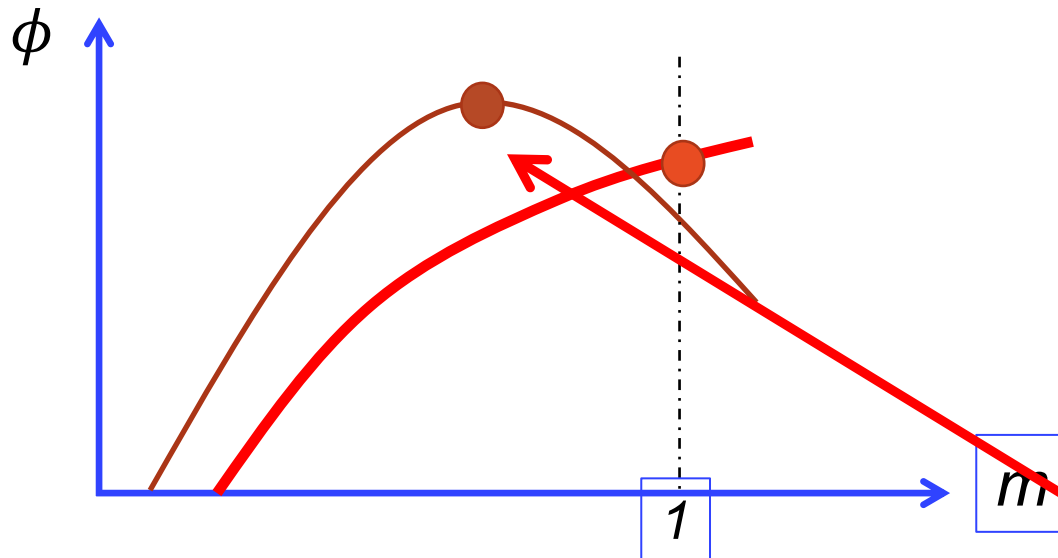
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else

$$= f|_{\Sigma(f)=0} = \phi|_{\phi'=0} = \max_m \phi(m).$$

Computing the complexity from cloning

Compact result:

$$F = F_{\text{qu}} = \max_{m \leq 1} \phi(m)!$$

The quenched free energy is the *maximum* of the free energy per clone, $\phi(m)$, in the range $m \leq 1$.

Computing the complexity from cloning

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Note:

Curiously, ϕ is to be *maximized* over the clone number m , in contrast to the minimum principle of F with respect to constraints:

The free energy of an unconstrained thermodynamic system is the minimum over the free energies of constrained systems.

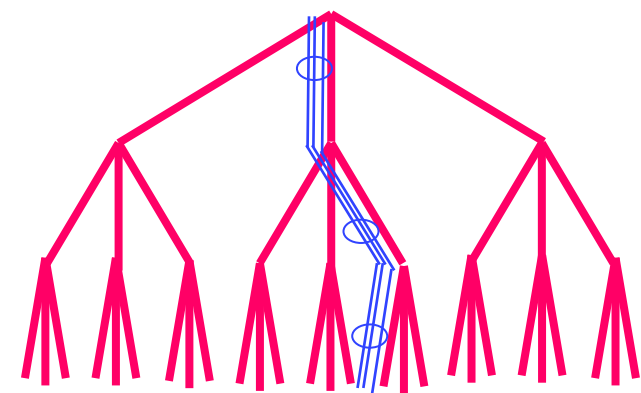
(Obviously, because constraints can only reduce the partition function and thus can only increase F).

But: Cloning is not a physical constraint, there is no contradiction with thermodynamic principles.

Entropy crisis and freezing transition

Application of the clone method to the directed polymer:

Simple, but instructive!



$m = 4$

$$Z_N^{(m)} = \int df e^{N\Sigma(f)} e^{-m\beta f N} \equiv e^{-\beta m \phi(m) N}$$

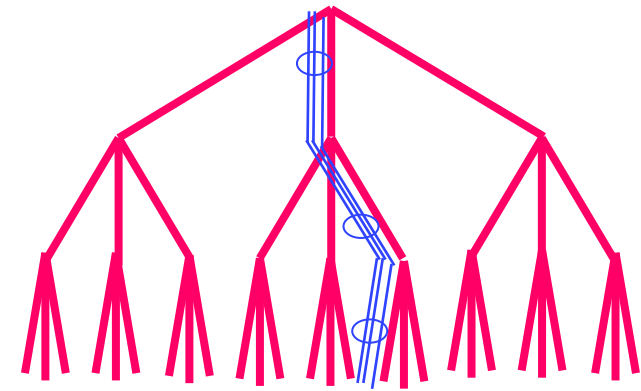
But eventually:
m non-integer, $m < 1$

Entropy crisis and freezing transition

Recursion for clones:

$$Z_{N+1}^{(m)} = e^{-\beta \epsilon_0 m} (Z_{N,1}^{(m)} + \dots + Z_{N,k}^{(m)})$$


Polymer length = tree generation



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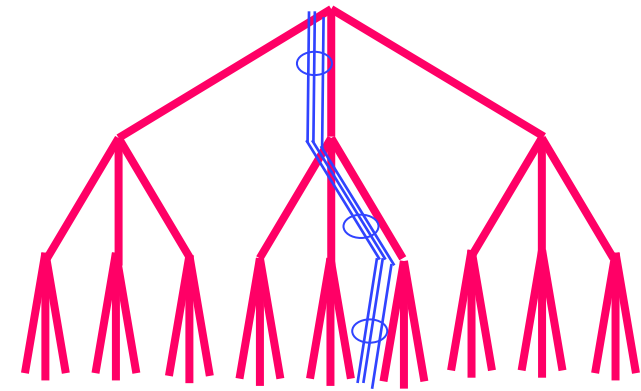
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$$\ln(Z_{N+1}^{(m)})/N = \ln(\overline{Z_{N+1}^{(m)}})/N$$



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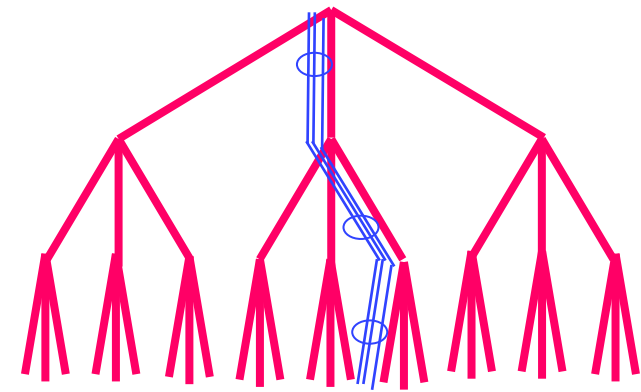
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$\rightarrow \overline{Z_{N+1}^{(m)}} = k \overline{e^{-\beta \epsilon_0 m}} \overline{Z_N^{(m)}}$

$\rightarrow \overline{Z_N^{(m)}} = (k \overline{e^{-\beta \epsilon_0 m}})^N$



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Entropy crisis and freezing transition

Replicated free energy

$$\phi(m) = -\frac{1}{\beta m} \log[\overline{ke^{-m\beta\epsilon_0}}] \stackrel{\text{in this simple case}}{=} f_{\text{ann}}(m\beta)$$

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$$\longleftrightarrow \mathcal{N}(f) = e^{N\Sigma(f)} = k^N e^{-N \frac{f^2}{2\Delta^2}} = k^N e^{-\frac{(Nf)^2}{2N\Delta^2}}$$

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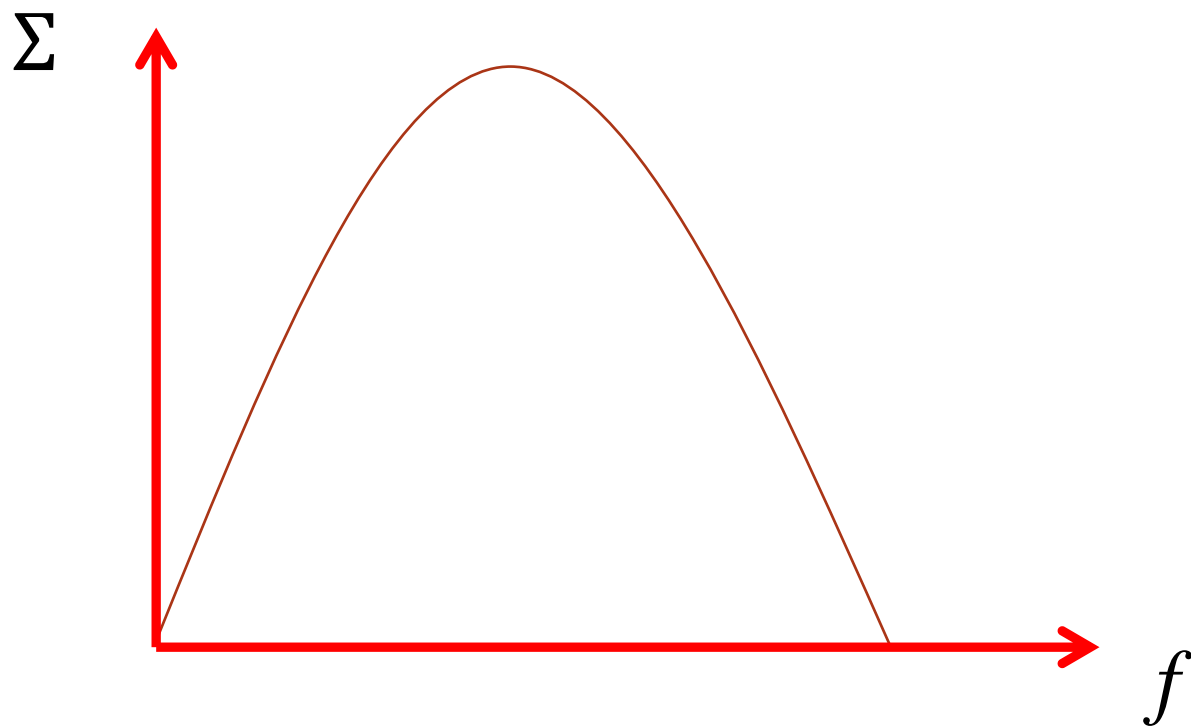
Natural interpretation:

Total No. of paths

Probability that path has
energy density = f

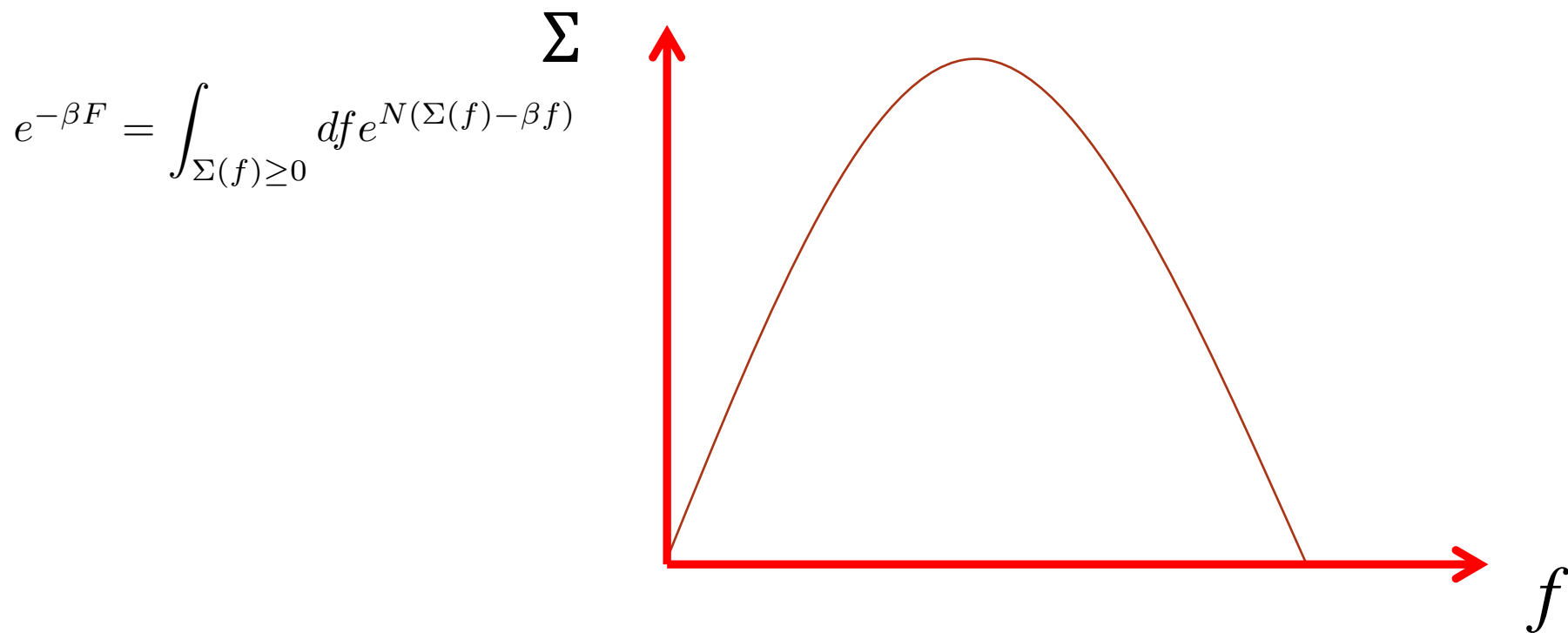
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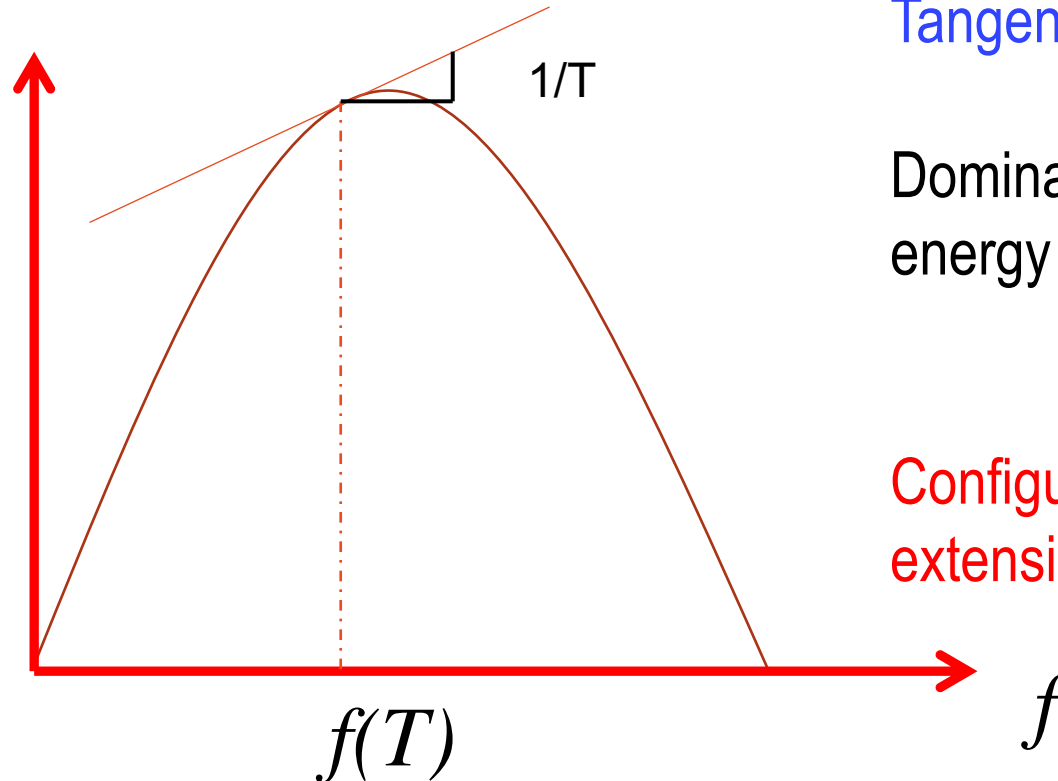
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Tangent construction:

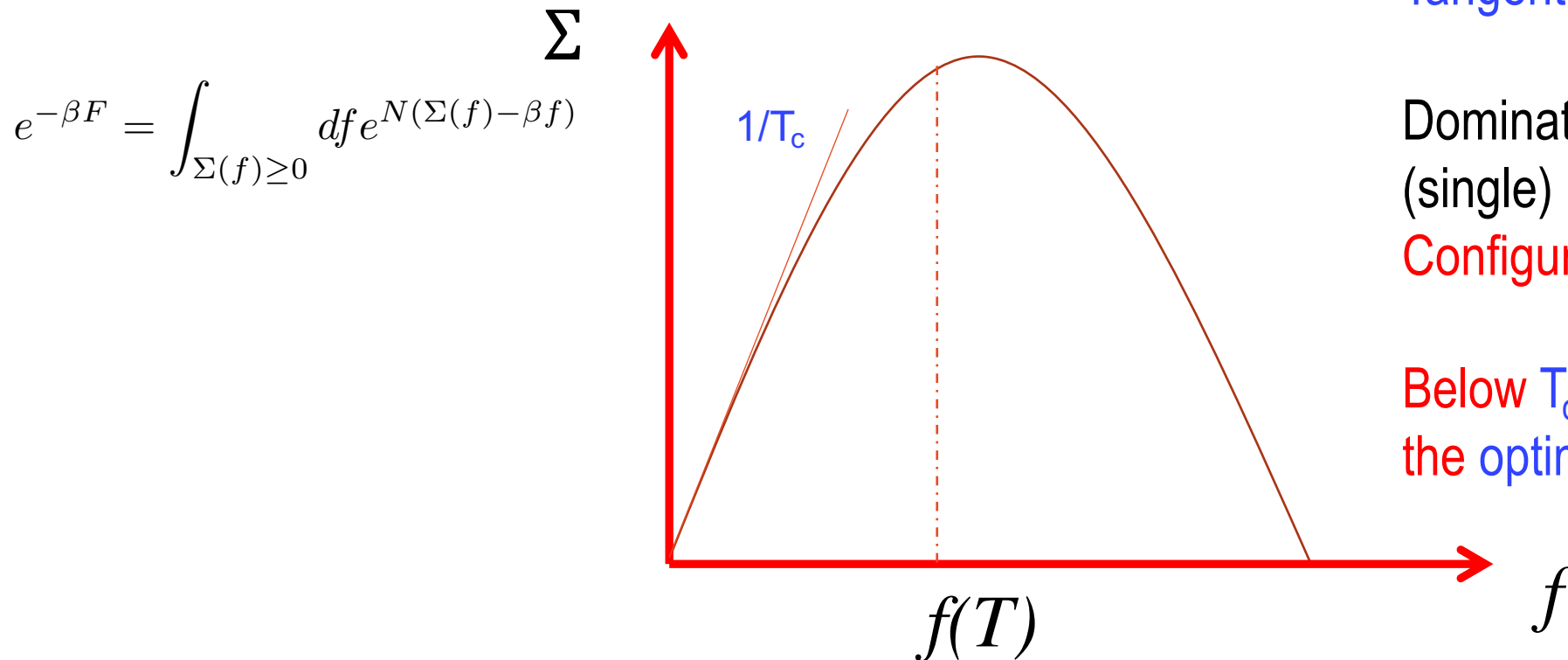
Dominating paths at $T > T_c$ have energy density $f(T)$:

$$\Sigma'(f(T)) = 1/T$$

Configurational entropy $N\Sigma$ is extensive!

Entropy crisis and freezing transition

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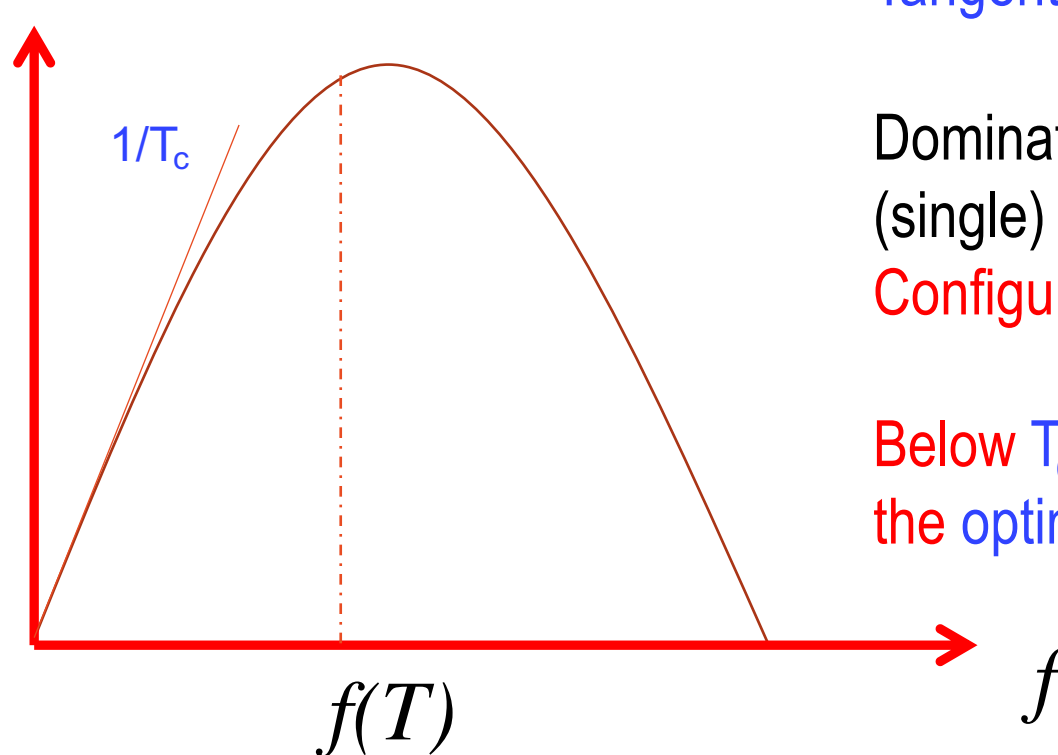
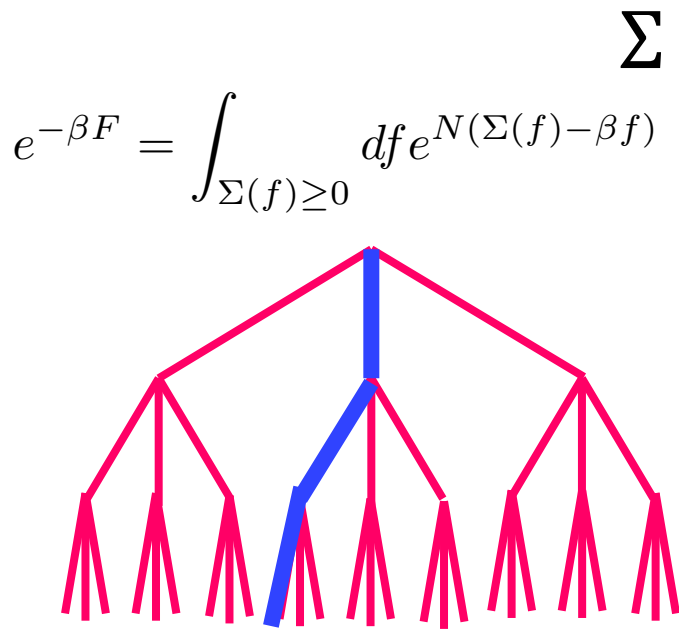
Tangent construction:

Dominating path at $T = T_c$ is the
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Configurational entropy vanishes!

Below T_c the polymer is frozen in
the optimal path.

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Exercise

1. Redo the DPRM analysis for non-Gaussian disorder assuming a small third or fourth cumulant of the distribution of local energies, neglecting higher cumulants. How come the results are different? Why does the CLT not apply?

2. Compare the DPRM on the Cayley tree to the random energy model (REM):

The REM is a system assumed to have 2^N configurations. Each of those has a random energy, drawn independently from a Gaussian distribution with zero mean and variance $N\Delta^2$.

- Write down the partition function of this system, analyze it by a saddle point method. Identify the freezing transition T_c . Compare to the DPRM!
- What happens at the freezing transition? What is the likelihood to find the system in the lowest energy configuration for $T > T_c$ and for $T < T_c$?
- Imagine to do Monte Carlo for this REM: In each step you propose a new configuration. Accept the move with probability $\exp(-\max(E_{\text{new}} - E_{\text{old}}, 0)/T)$. Imagine to start in low energy configuration with $E/N = e < 0$. How long will it take roughly to find yourself in a new state with $e' < 0$? Argue that this takes exponentially long (in N) time, at any temperature! Thus even above the freezing temperature the dynamics of such a system is very slow or 'glassy', even though thermodynamically it is still in the high temperature phase!

Rigorous solution by Derrida and Spohn

J. Stat. Phys. '88

- The above heuristic methods can be shown to give the rigorously exact free energy density.
- Instead of a discretely branching tree, it is nicer to consider a continuous branching process. Then the problem maps onto the KPP equation - a simple nonlinear equation known for wave propagation phenomena.
- Exact results for the wave propagation problem allow one to obtain exact results for the DPRM on the tree

Rigorous solution by Derrida and Spohn

J. Stat. Phys. '88

- At generation t from the leaves one defines the partition function $Z(t)$ and the quantity

$$G_t(x) = \overline{\exp(-e^{-\beta x} Z(t))} \stackrel{x \rightarrow \infty}{\approx} 1 - \overline{e^{-\beta x} Z(t)}$$

- One can show that the discrete tree recursion implies that

$$G_{t+1}(x) = \int dV \rho(V) [G_t(x + V)]^k$$

This can be shown to tend to a travelling wave solution of the form $w(x-ct)$.
(as suggested by the solution on the continuously branching tree)

Determining the speed $c > 0$ is equivalent to computing (minus) the free energy density f (since $\exp(-\beta x) Z(t) \sim \exp(-\beta(x + ft))$).

This turns out to be exactly the f obtained with our less rigorous methods above.

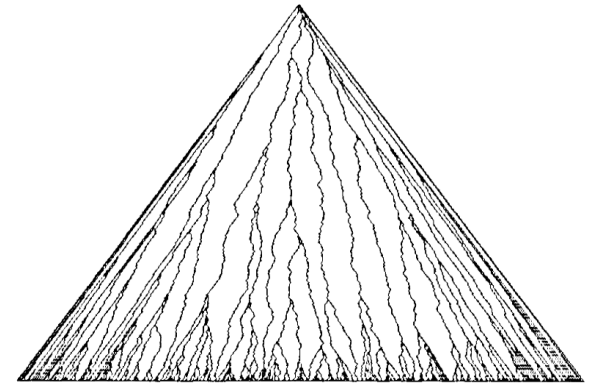
Freezing of f at low T : $\leftrightarrow$ wave must travel with minimal speed $c = -f(T_c)$.

How does the glassy aspect of
an elastic line or interface
affect their motion?

Pinning of elastic manifolds

“Glassy” features of the directed polymer:

- Deep energy valleys: domination by one single valley at low T ,
→ thermal fluctuations are frozen out, disorder dominates;
Competing low energy configurations are far apart



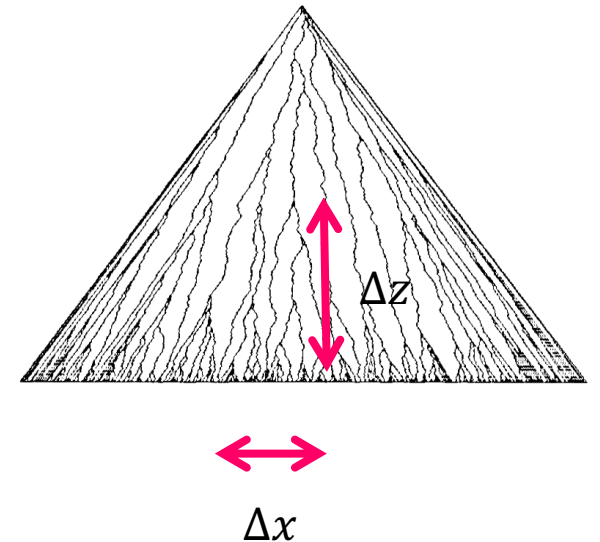
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To change from one configuration to another one has to cross large barriers: to move a portion of the polymer sideways by Δx requires typically detaching a length $\Delta z = \Delta x^{1/\zeta}$.



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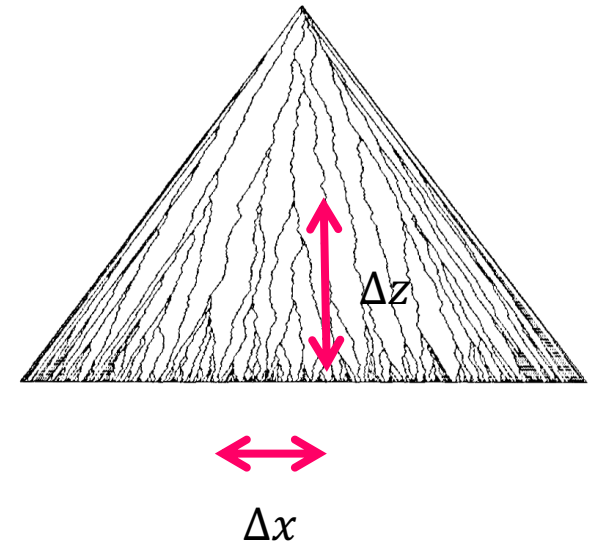
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Intermediate configurations have excess

energy scaling as $\Delta E \sim \Delta z^\theta \sim \Delta x^{(2\zeta+d-2)/\zeta}$

(in $d=1$: $\sim \Delta x^{(\frac{1}{3})/(\frac{2}{3})} \sim \Delta x^{\frac{1}{2}}$)



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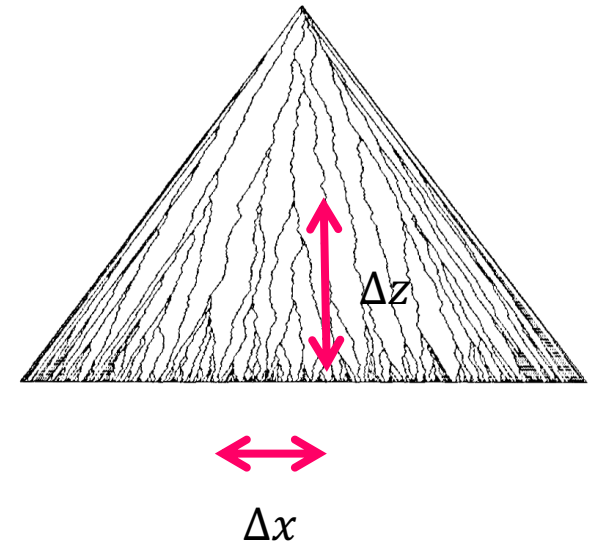
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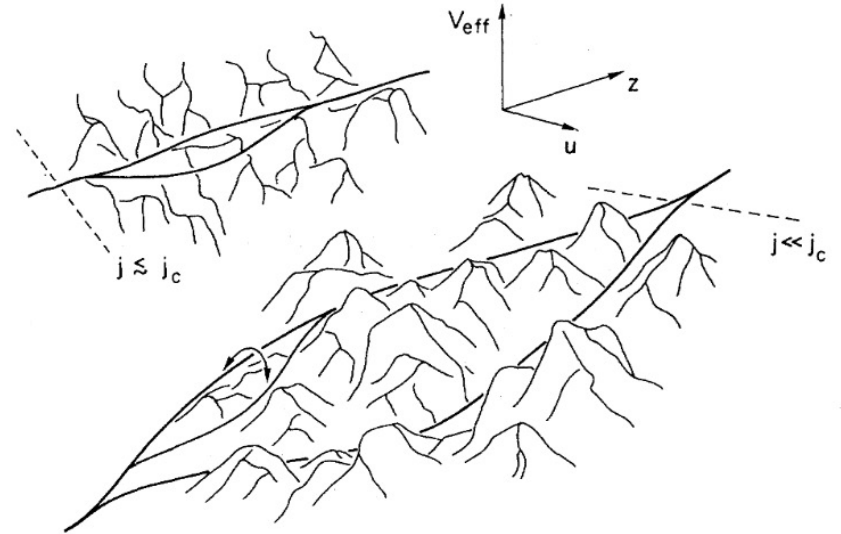
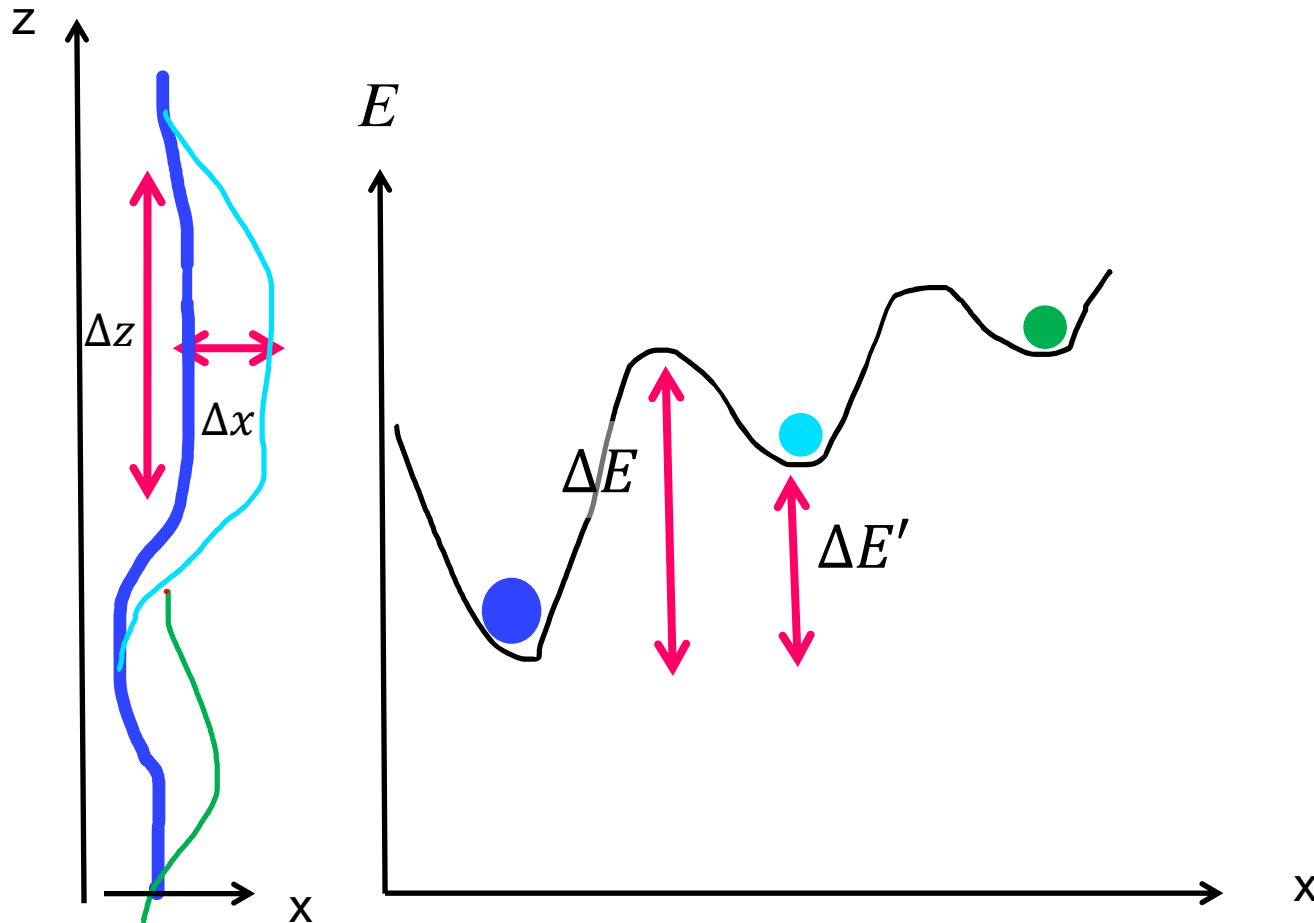
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→ Growing energy barriers. How do these affect motion under a force?

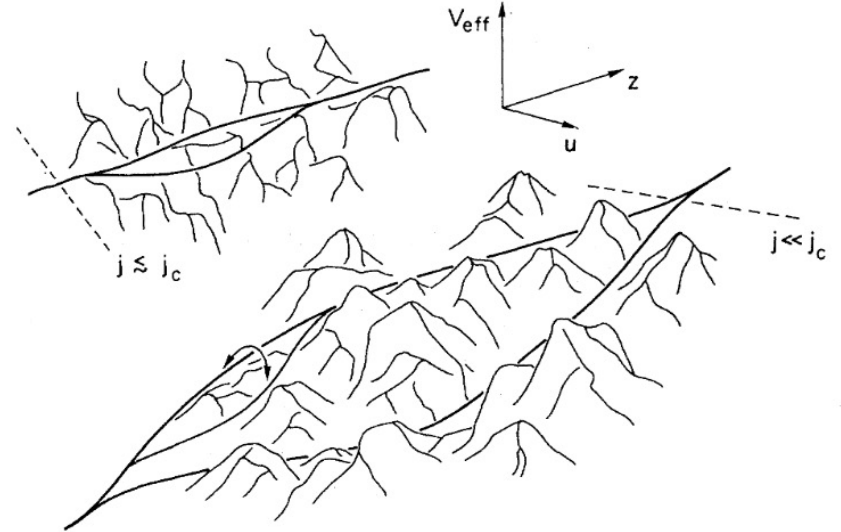
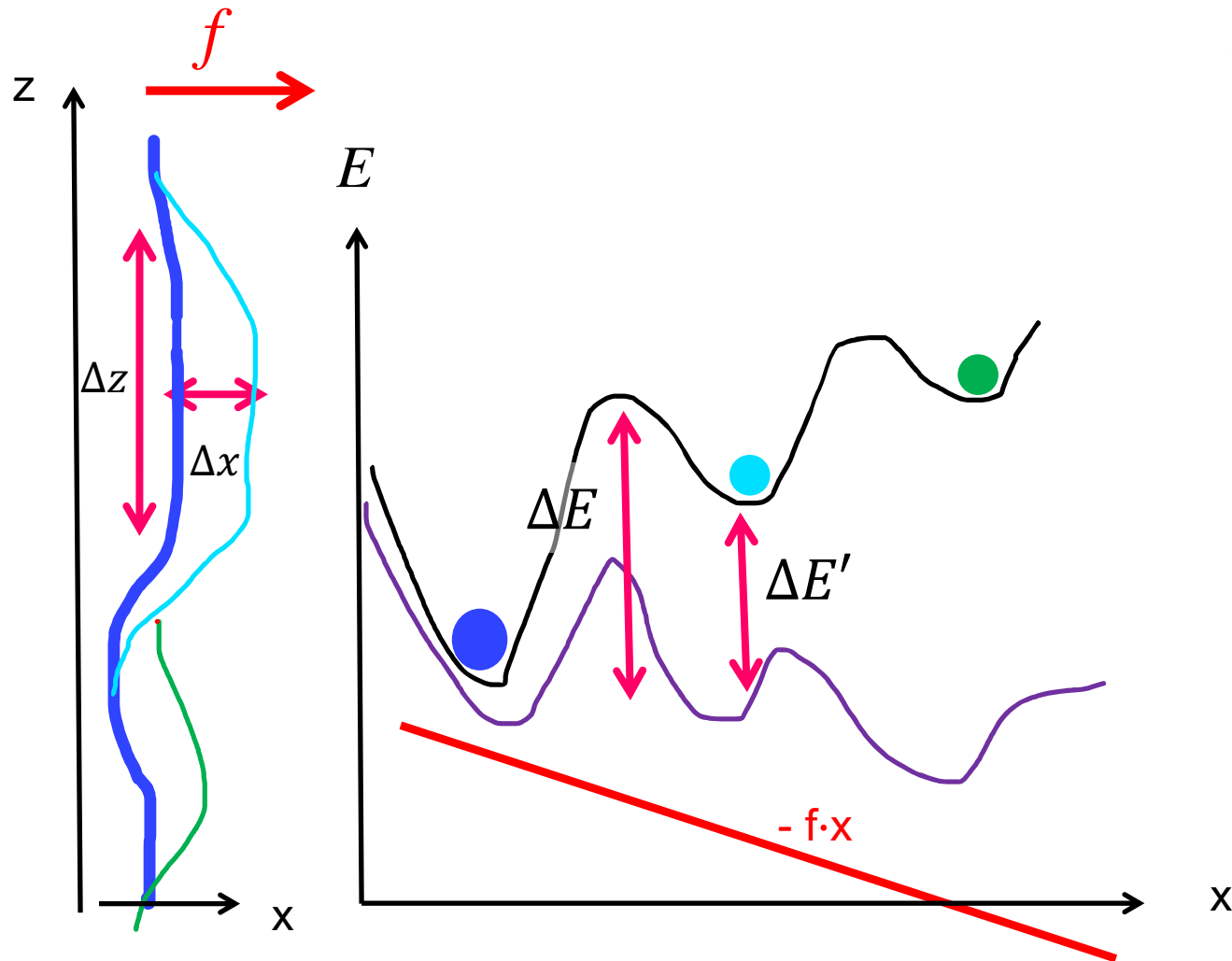
Thermal creep

Applying a force f , what are the minimal barriers to cross?



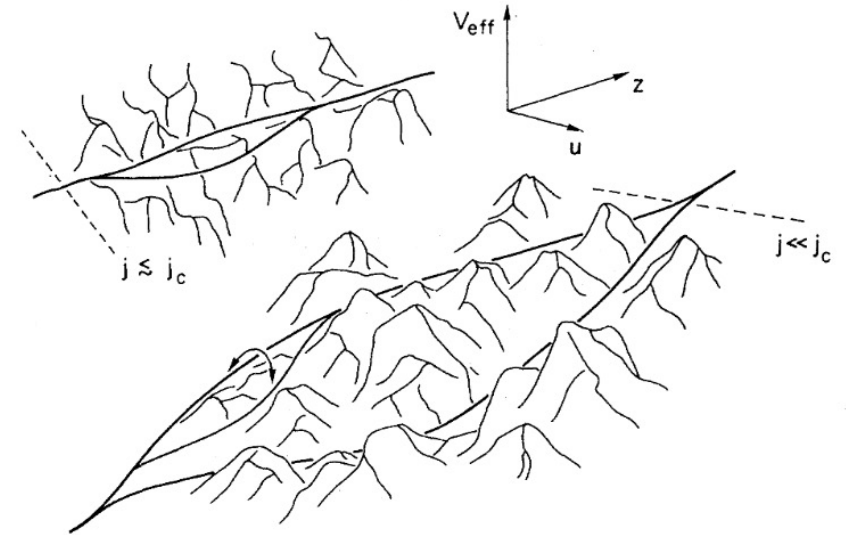
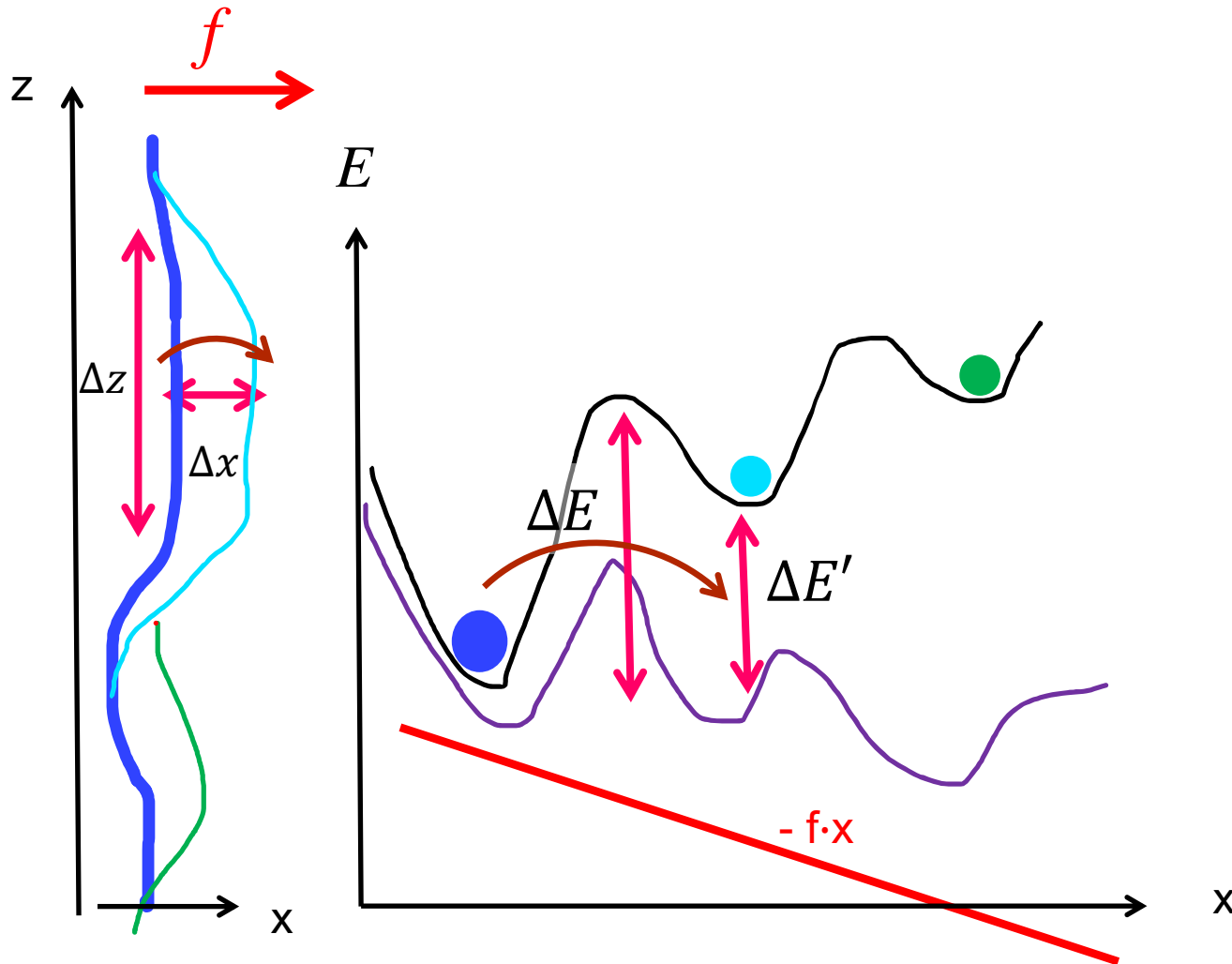
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Barrier: $\Delta E \sim \Delta E' \sim \Delta z^{\theta=(d-2+2\zeta)}$

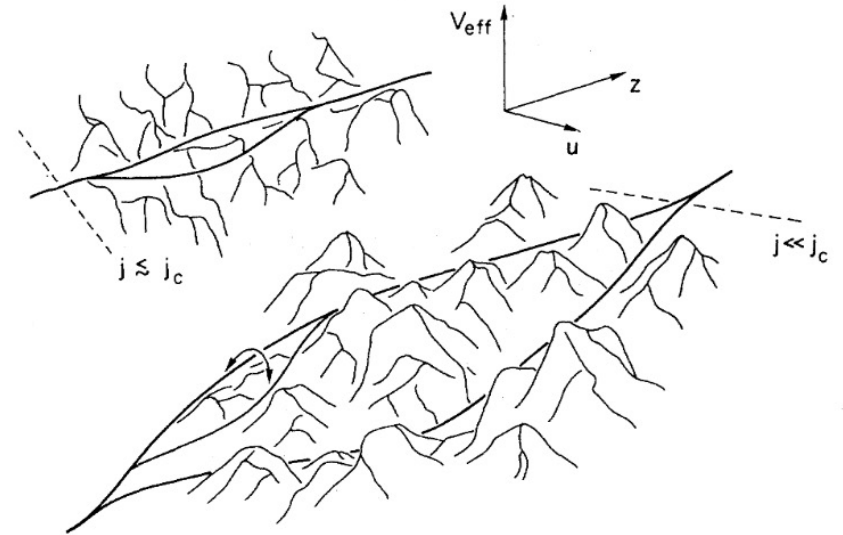
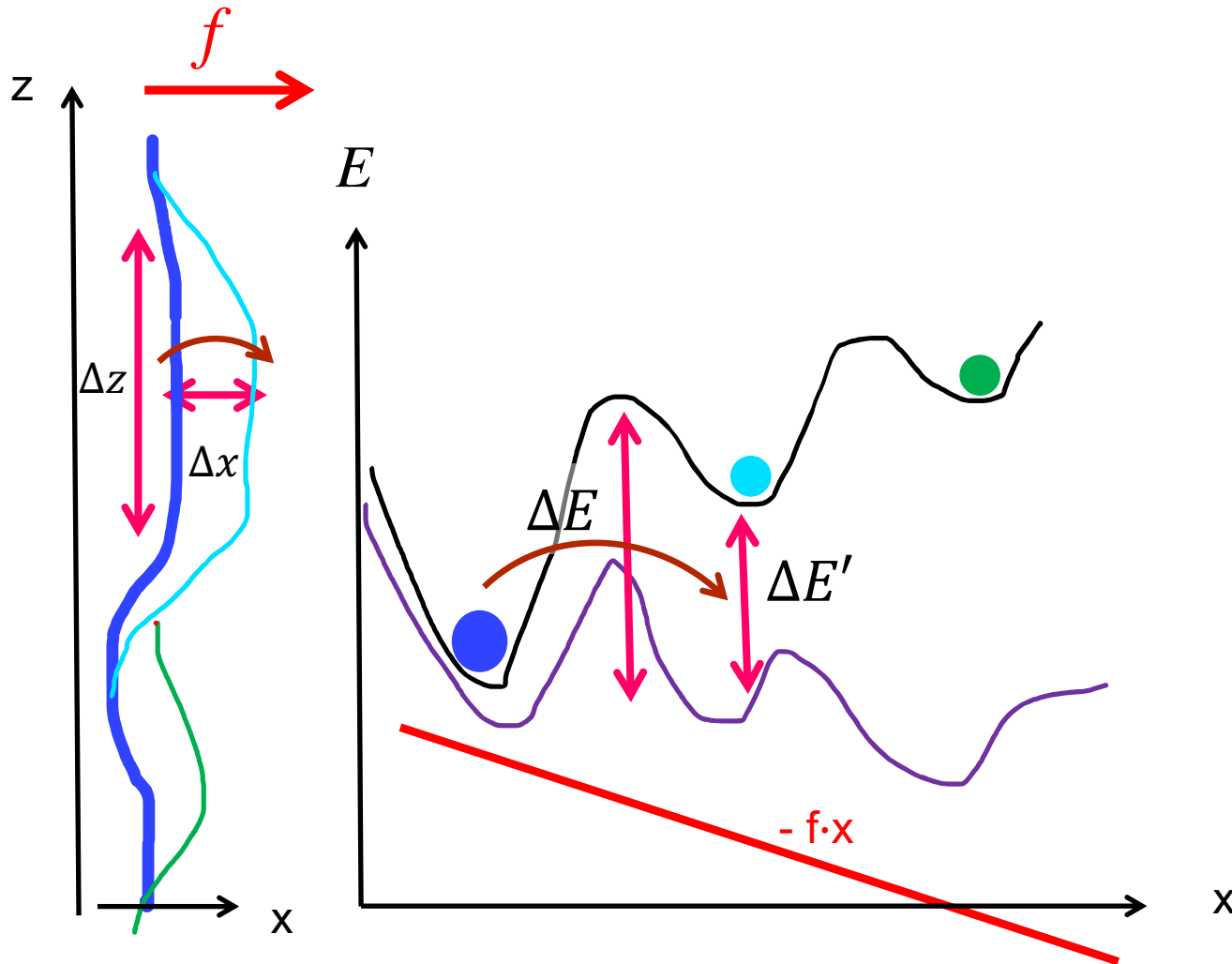
Energy gain: $E_g \sim \Delta z^d f \Delta x \sim \Delta z^d f \Delta z^{\zeta}$

$$E_g > \Delta E \rightarrow \Delta z \geq f^{-1/(2-\zeta)}$$

$$\rightarrow \Delta E \geq f^{-\theta/(2-\zeta)}$$

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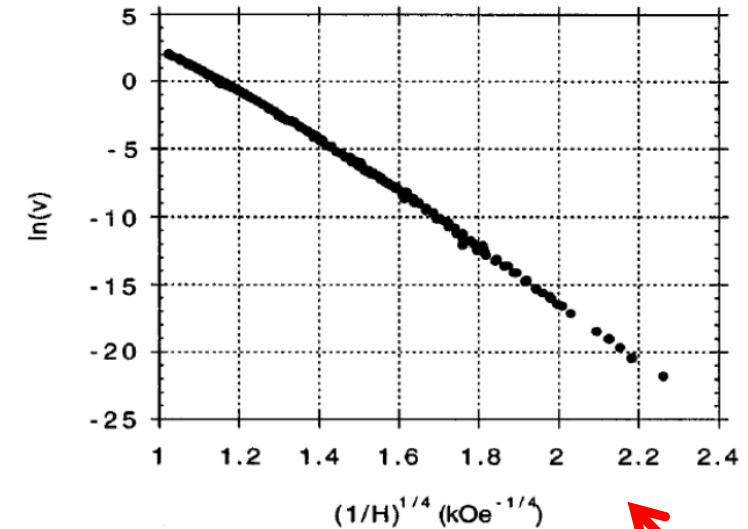
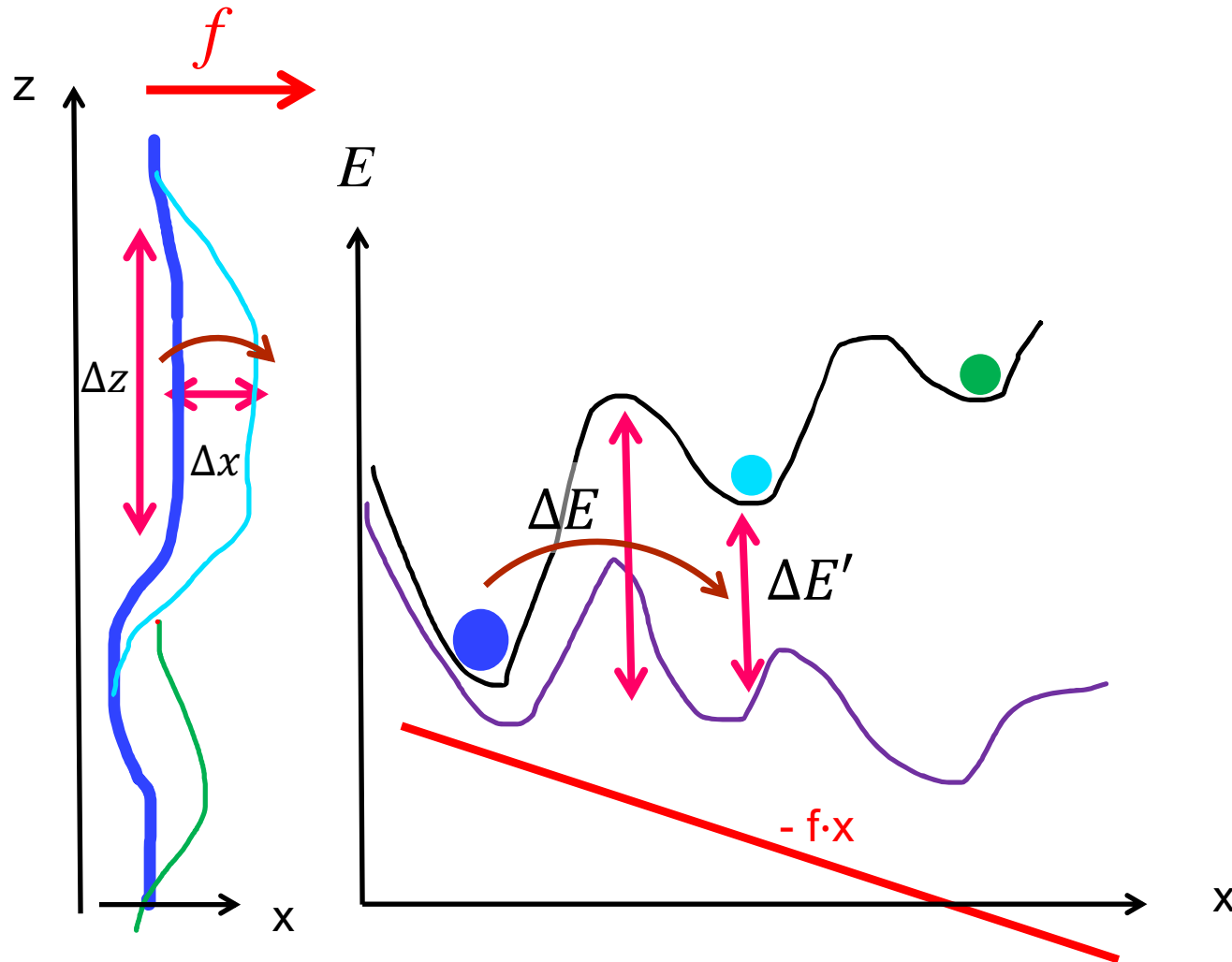
$$E_g > \Delta E \rightarrow \Delta z \geq f^{-1/(2-\zeta)}$$

$$\rightarrow \Delta E \geq f^{-\theta/(2-\zeta)}$$

$$\rightarrow \text{creep motion: } v \sim \exp\left(-\frac{c f^{-\theta/(2-\zeta)}}{T}\right)$$

Thermal creep

Applying a force f , what are the minimal barriers to cross?



$d = 1+1$: domain wall in magnetic film
pushed by magnetic field H

$$f^{-\theta/(2-\zeta)} = f^{-1/3/(2-2/3)} = f^{-1/4}$$

→ creep motion: $v \sim \exp\left(-\frac{c f^{-\theta/(2-\zeta)}}{T}\right)$

What about motion at $T = 0$?

Naively: our reasoning shows that for any force f there are barriers. So, at $T=0$, is there never any motion for any force?

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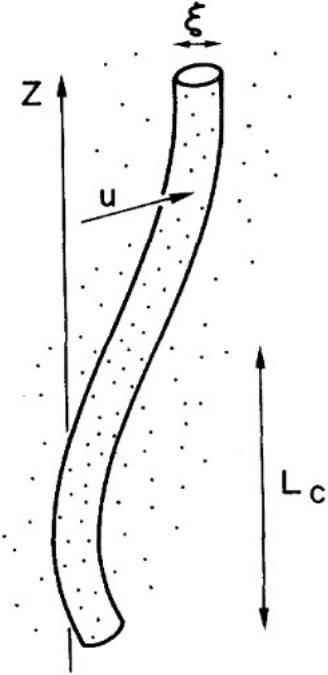
Naively: our reasoning shows that for any force f there are barriers. So, at $T=0$, is there never any motion for any force?

Not necessarily: The larger the force the smaller the scale $L(f)$.

Thus the question is:

Are there metastable states/ barriers at all scales,
or is there a smallest scale which determines a maximal force
that suffices to unpin the interface?

Minimal displacement for metastability to set in?

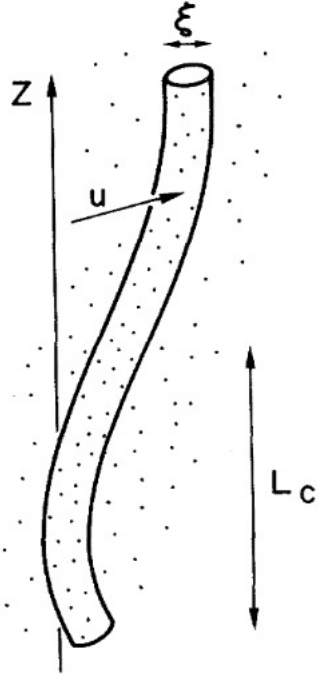


Caution: The random manifold regime only holds beyond some minimal length scale where the correlations in the potential are not felt anymore!

What is this length scale, and what happens before?

Revisit disorder more carefully!

Minimal displacement for metastability to set in?



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What is this length scale, and what happens before?

Recall: Elasticity and disorder compete

$$H_{\text{el}} = \int d^d z \frac{c}{2} (\nabla \mathbf{u})^2 \quad H_{\text{pin}} = \int d^d z V(\mathbf{u}(\mathbf{z}), \mathbf{z})$$

Disorder correlations:

$$\overline{V(\mathbf{u}, \mathbf{z}) V(\mathbf{u}', \mathbf{z}')} = K(|\mathbf{u} - \mathbf{u}'|) \delta^d(\mathbf{z} - \mathbf{z}').$$



Force correlations: $\mathbf{f} = -\nabla V$

$$\overline{\mathbf{f}^\alpha(\mathbf{u}, \mathbf{z}) \mathbf{f}^\beta(\mathbf{u}', \mathbf{z}')} = -\partial^\alpha \partial^\beta K(|\mathbf{u} - \mathbf{u}'|) \delta^d(\mathbf{z} - \mathbf{z}') \equiv \Delta^{\alpha\beta}(|\mathbf{u} - \mathbf{u}'|) \delta^d(\mathbf{z} - \mathbf{z}').$$

Short scale roughness: Larkin model

Equilibrium configuration satisfies the force balance:

$$-c\nabla_{\mathbf{z}}^2\mathbf{u}(\mathbf{z}) = \mathbf{f}_{\text{pin}}(\mathbf{u}(\mathbf{z}), \mathbf{z}) = -\nabla_{\mathbf{u}}V_{\text{pin}}(\mathbf{u}(\mathbf{z}), \mathbf{z}).$$

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Neglect u -dependence on the right ($u \ll \xi$): Solve for response to the pinning forces:

$$\mathbf{u}^\alpha(\mathbf{z}) = \int d^d z' G^{\alpha\beta}(\mathbf{z} - \mathbf{z}') f_{\text{pin}}^\beta(0, \mathbf{z}').$$

with the elastic Green's function

$$G^{\alpha\beta}(\mathbf{z} - \mathbf{z}') = \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot(\mathbf{z}-\mathbf{z}')}}{c q^2} \delta_{\alpha\beta}$$

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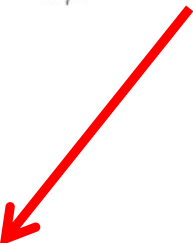
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Short range force correlator



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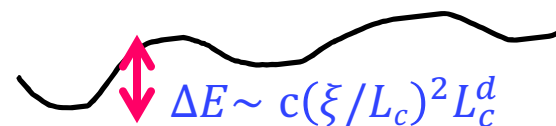
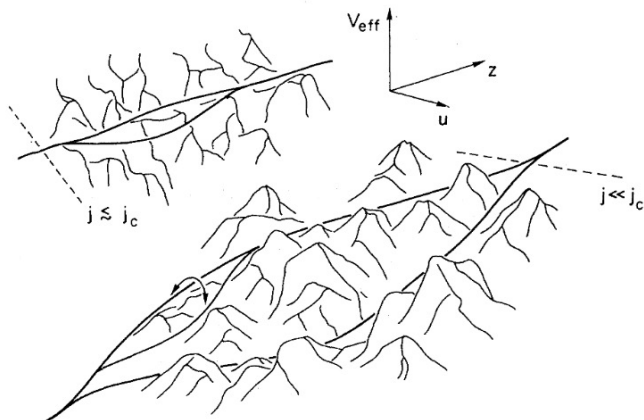
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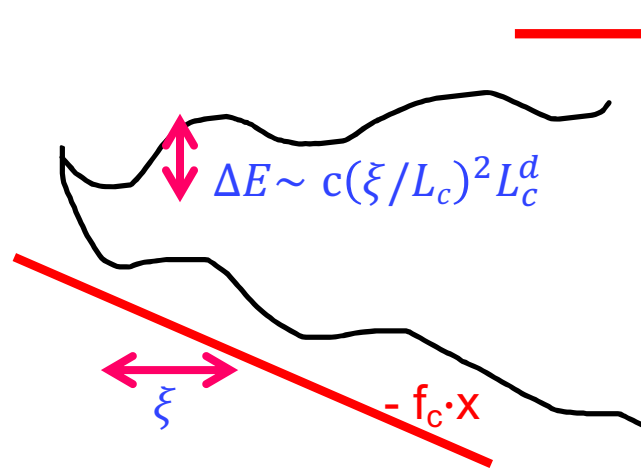
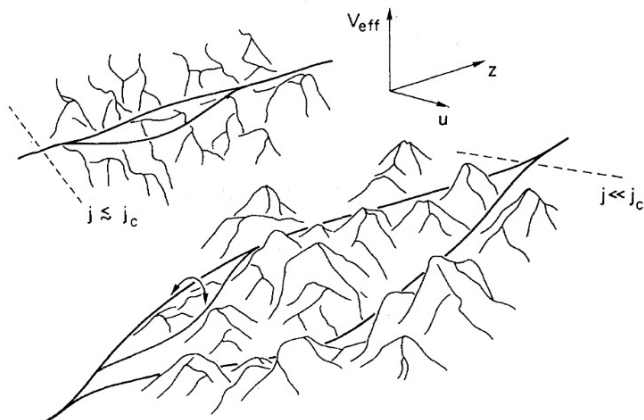
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$\rightarrow$ To move the manifold at $T = 0$ (downward sliding) ?

Apply force f_c that overcomes the barrier to the next

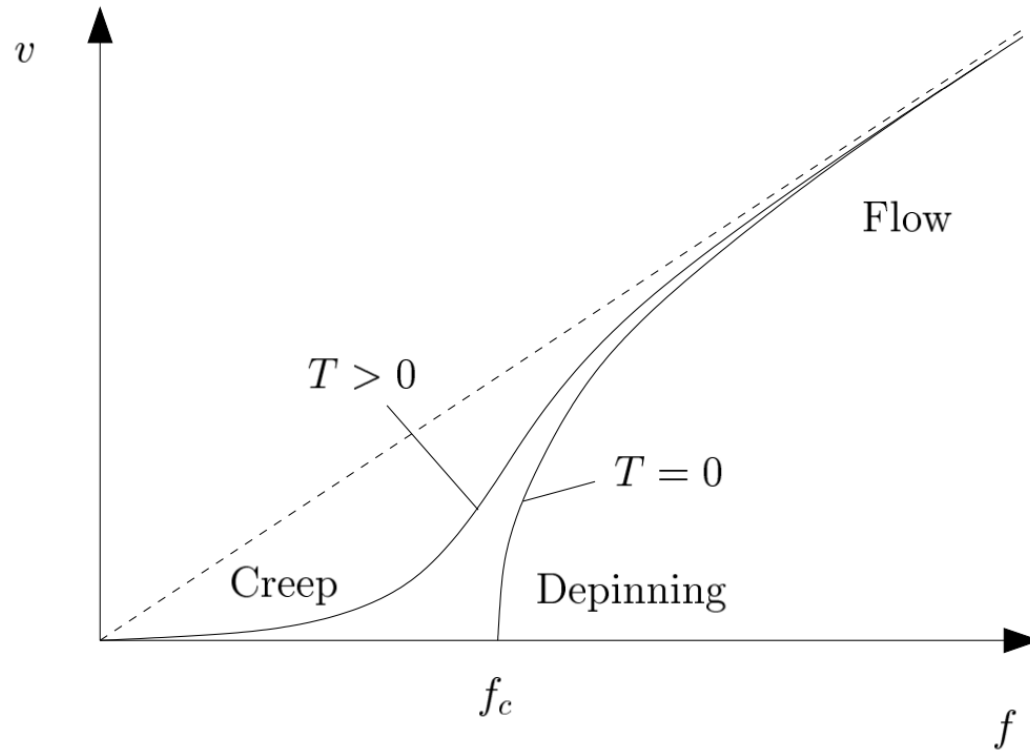
minimum: $\Delta E \sim f_c \xi L_c^d$

$\rightarrow f_c = c \xi / L_c^2 \sim \xi^{1-4/(4-d)}$!

Depinning and flow

What happens (at $T = 0$) as one approaches and crosses f_c ?

Depinning and flow

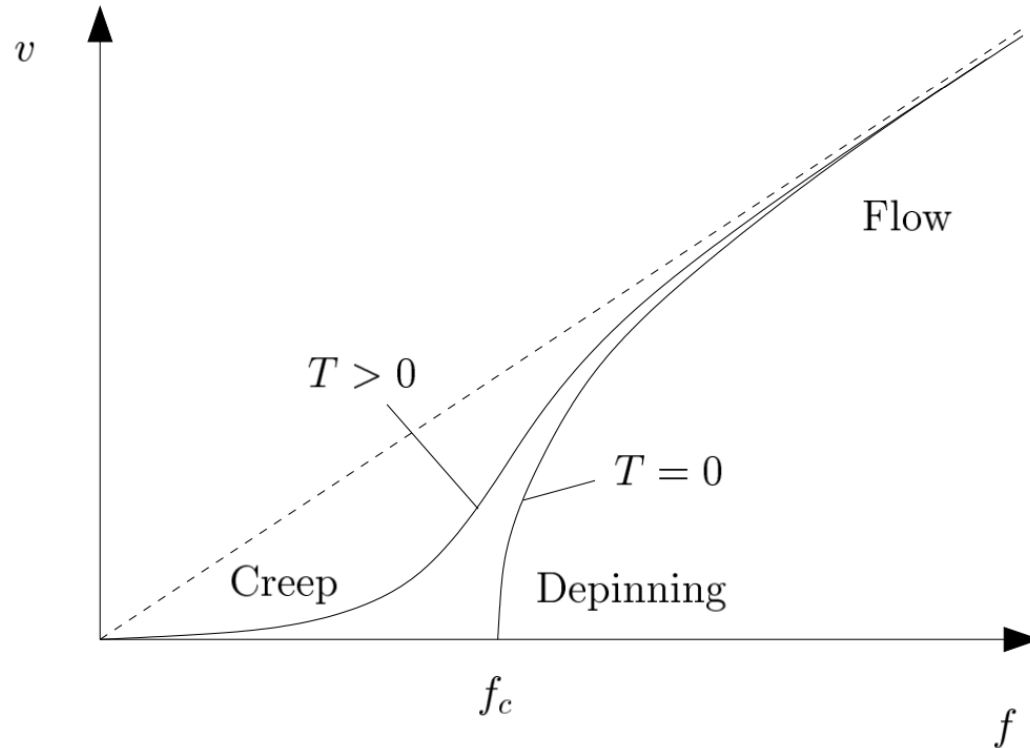


Depinning: a dynamical critical phenomenon!

Second critical exponent β

$$v = (f - f_c)^\beta$$

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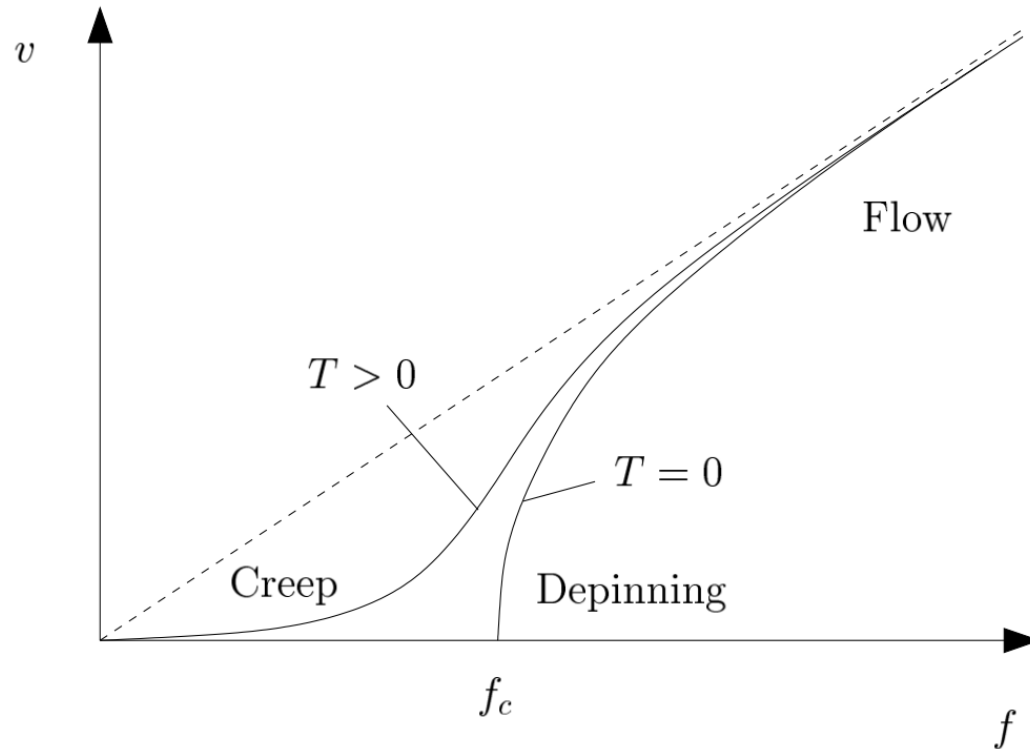
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Diverging correlation length and time

$$\xi_z \sim (f - f_c)^{-\nu}$$

$$\tau \sim \xi_z^z$$

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Scaling relations:

$$v \sim \xi_z^\zeta / \tau \sim \xi_z^{\zeta - z}$$

$$f - f_c \sim \xi_z^\zeta / \xi_z^2 \longrightarrow \nu = \frac{\beta}{z - \zeta} = \frac{1}{2 - \zeta}$$

2 independent exponents are left!

Large scale roughness

Large scale roughness beyond 1+1 dimensions? Capturing metastability in field theory?

- Replicate action (elasticity+ disorder)

- Disorder average
$$H_m = \int d^d z \left[\frac{c}{2} \sum_{a=1}^m (\nabla \mathbf{u}_a)^2 - \frac{1}{2T} \sum_{a,b=1}^m K(|\mathbf{u}_a(\mathbf{z}) - \mathbf{u}_b(\mathbf{z})|) \right]$$

- Require first and second term in H_m/T to scale similarly (Ex 1 - Flory)

$$K(u) \sim |u|^{-\alpha} \quad L^{d-2+2\zeta} \leftrightarrow L^d L^{-\alpha\zeta} / (T \sim L^{(d-2+2\zeta)})$$

$$\rightarrow \zeta = \frac{4-d}{4+\alpha} \quad \alpha = ?$$

Random bonds vs random fields

Two universality classes of disorder

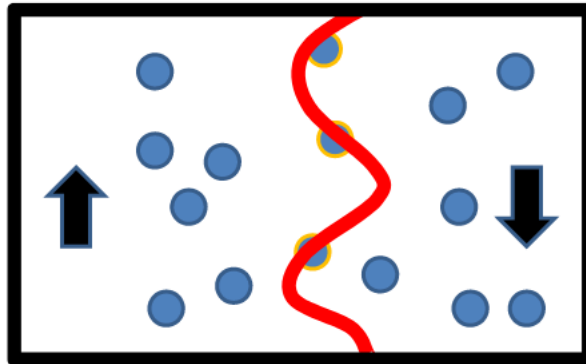
So far considered:

“Random bond (RB)” disorder : The elastic object feels a potential where it passes through

→ $K(u)$ is short ranged (typical scale ξ) → $K(u) \sim 1/|u|$

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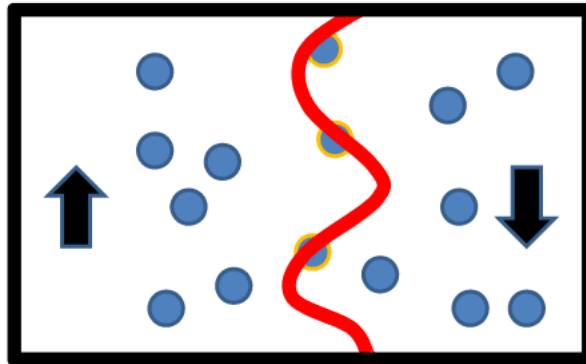
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Also relevant:

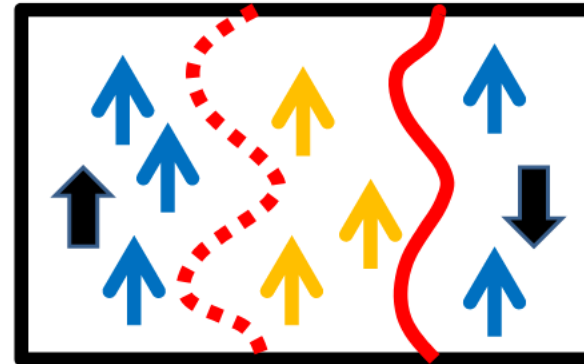
“Random force (RF)” : The elastic interface separates two phases (e.g. up / down magnet). The random disorder couples oppositely to the two phases → $K(u) \sim |u|$

$$\alpha = -1$$

RB



RF



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Can one do better? Yes!

Renormalization group

$$\zeta_{short} = \frac{4-d}{2} \quad \longleftrightarrow \quad \text{Disorder is relevant in } d \leq d_{uc} = 4!$$

Renormalization group treatment for $d = 4 - \varepsilon$?

Yes: Renormalize replicated action!

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→ *Cannot* expand $K(u)$ in a power law series and truncate, but must renormalize the full function! → **functional RG**

Renormalization group

FRG flow equation for rescaled force correlator

$$\Delta(u) = -K''(u)$$
$$\tilde{\Delta}_l(u) = \frac{A_d \Lambda^d}{(c\Lambda^2)^2} e^{(4-d-2\zeta)l} \Delta_l(u e^{\zeta l}) \quad \Lambda = \text{short-scale cutoff}$$

$$\partial_l \tilde{\Delta}_l(u) = (\epsilon - 2\zeta) \tilde{\Delta}_l(u) + \zeta u \tilde{\Delta}'_l(u) + \tilde{\Delta}''_l(u) (\tilde{\Delta}_l(0) - \tilde{\Delta}_l(u)) - \tilde{\Delta}'_l(u)^2$$

Interesting aspect:

After renormalization to a finite length scale of order L_c the renormalized correlator becomes non-analytic at $u = 0$: $\Delta_l(u) = \Delta_l(0) - a|u|$ Larkin scale!

$$l_c = \frac{1}{\epsilon} \ln \left(1 + \frac{\epsilon}{3|\tilde{\Delta}''_0(0)|} \right) \longleftrightarrow L_c = \Lambda^{-1} e^{l_c} \approx \frac{1}{\Lambda} \left(\frac{\epsilon}{3\tilde{\Delta}''_0(0)} \right)^{1/\epsilon} = \left(\frac{\epsilon}{3A_d} \frac{c^2}{\Delta''_0(0)} \right)^{1/\epsilon}$$

The non-analytic term $-a|u|$ reflects emergence of degenerate local minima at the scale L_c !