The Kardar-Parisi-Zhang equation and universality class

M. Kardar, G. Parisi and YC Zhang (PRL 1986)

From randomly growing non-equilibrium surfaces
to fluid dynamics and polymers in random media

Relations to random matrices, sequence alignment ...

Splendid recent progress in mathematical physics

References

- K. A. Takeuchi, *“An appetizer to modern developments in the KPZ universality class”*, Physica A 504, (2018), 77

Briefer account:

- T. Sasamoto, Prog. Theor. Exp. Phys. (2016), 022A01, The 1d KPZ equation: height distribution and universality

- M. Kardar in Statistical Physics, *“Directed paths in random media”*

More mathematical / technical:

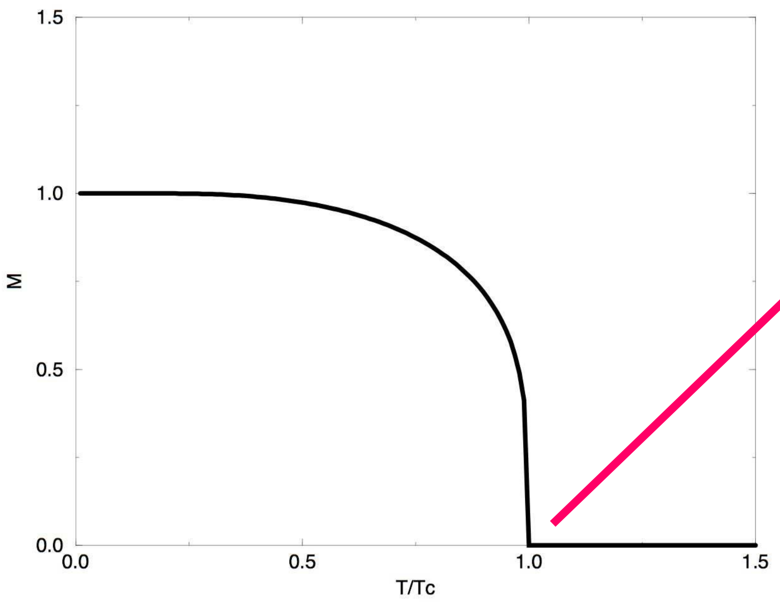
- T. Kriecherbauer and J. Krug, *“A pedestrian's view on interacting particle systems, KPZ universality and random matrices”*, J. Phys. A 43 (2010), 403001

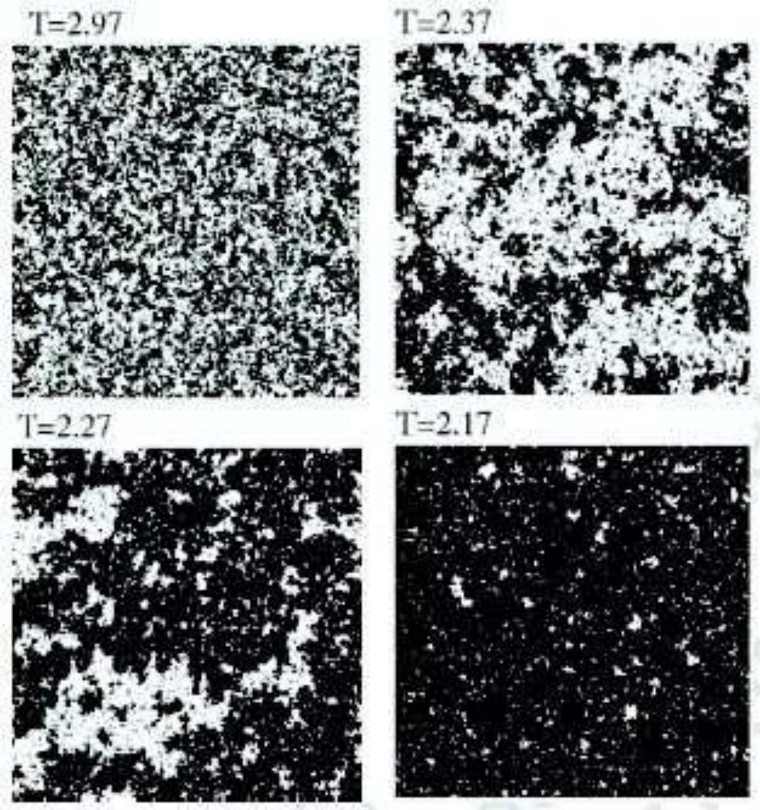
Universality, self-similarity and criticality beyond equilibrium phase transitions?

Second order phase transitions:

- Universality: at large scale: independence of microscopic details
- Self-similarity: critical exponents, scaling

2d Ising model

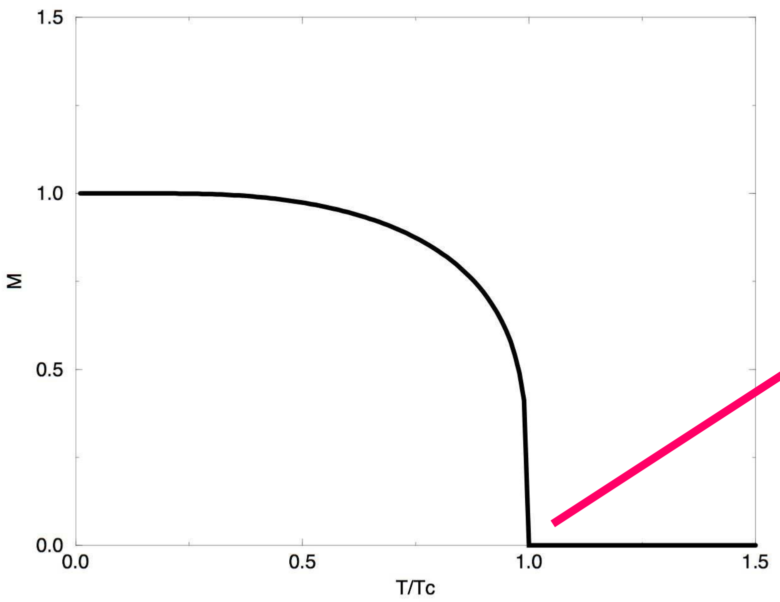




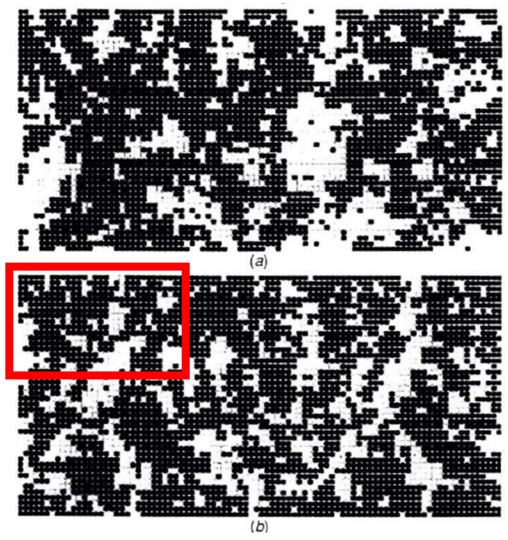
Fluctuations *on all scales*

Self-similarity:
(looks *statistically* the same after rescaling)

Self-similarity at criticality



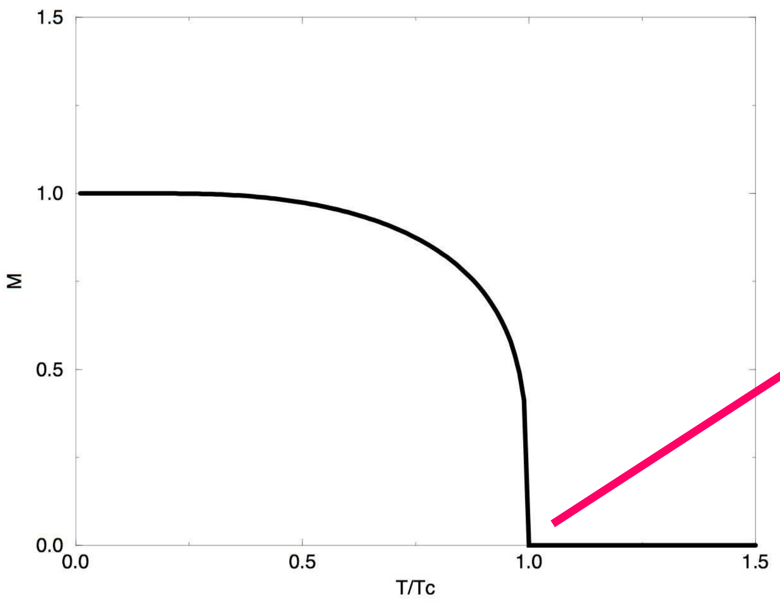
T = 0.99 Tc



Critical

Correlations at large scales

Self-similarity at criticality



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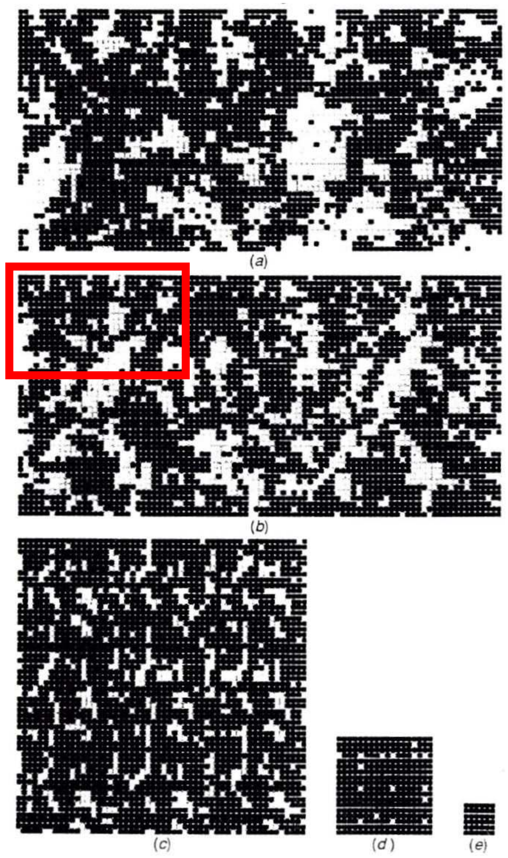

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Second order phase transitions:

- Universality: at large scale: independence of microscopic details
- Self-similarity: critical exponents, scaling

Does this exist

- out of equilibrium?
- in random, disordered systems?

- Does disorder matter? How so?
- Are there new aspects not present in clean systems?

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Second order phase transitions:

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Does this exist

- out of equilibrium?
- in random, disordered systems?

YES!

- Does disorder matter? How so?
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Growing surfaces

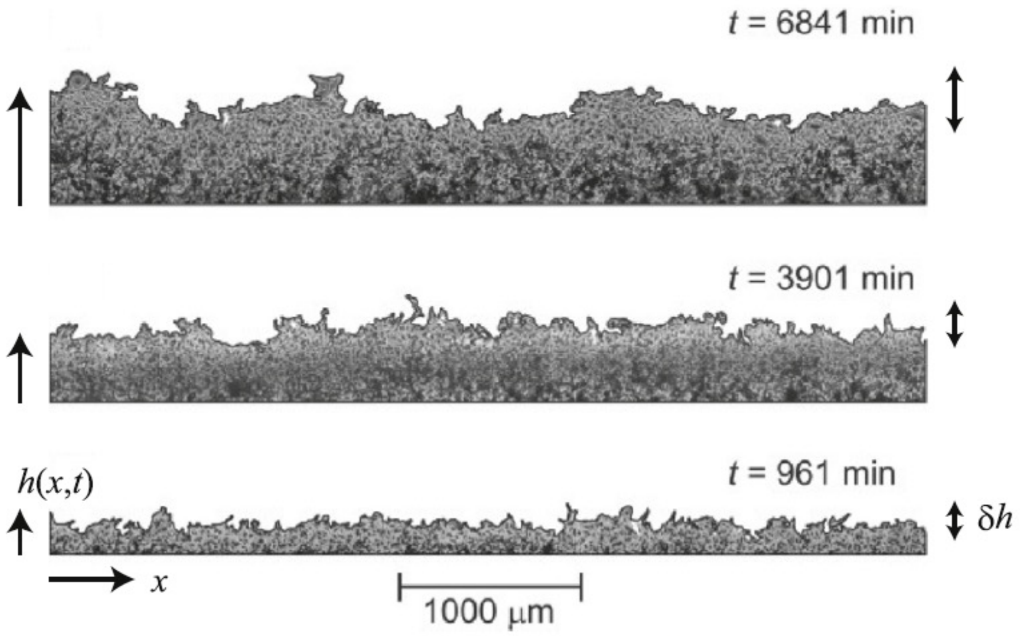
one of the simplest examples of this kind

(beyond simple diffusion / Brownian motion)

Rich *and* ubiquituos

Growing surfaces

See Takeuchi Review for details



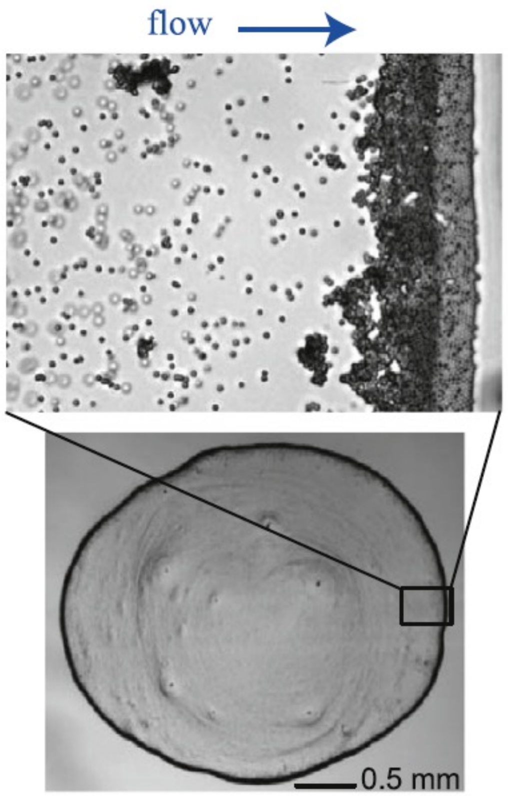
(a) Proliferating cancer cells.

1d surface of cancer cells growing on a Petri dish

Growing surfaces

See Takeuchi Review for details

Coffee ring effect:



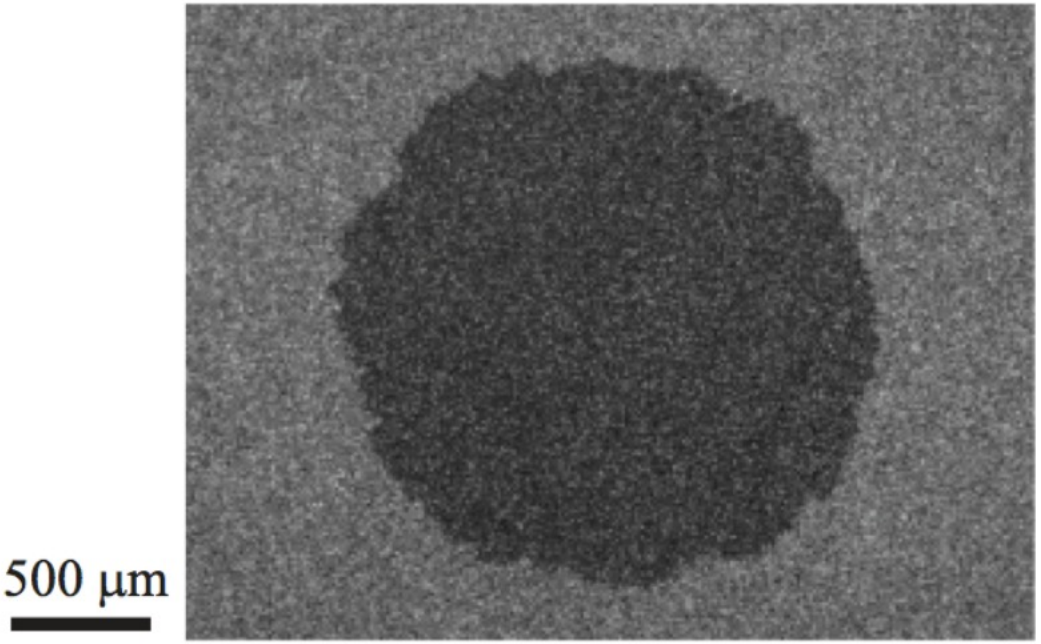
(b) Particle deposition in suspension droplet.

Colloid particles accumulating at the edge of a droplet
inward growth of colloid layer: roughening with time

Growing surfaces

See Takeuchi Review for details

Boundary between two (turbulent) liquid crystal phases



K. A. Takeuchi, M. Sano (J. Stat. Phys. 2012)

Growing surfaces

Physical examples:

- Growth of bacteria colonies
- Interfaces between two liquids; ink spreading on paper
- Slow combustion of paper
- Forest fire fronts
- Front of chemical reaction
- Motion of the front of a thin liquid film on surface
- Crystal growth

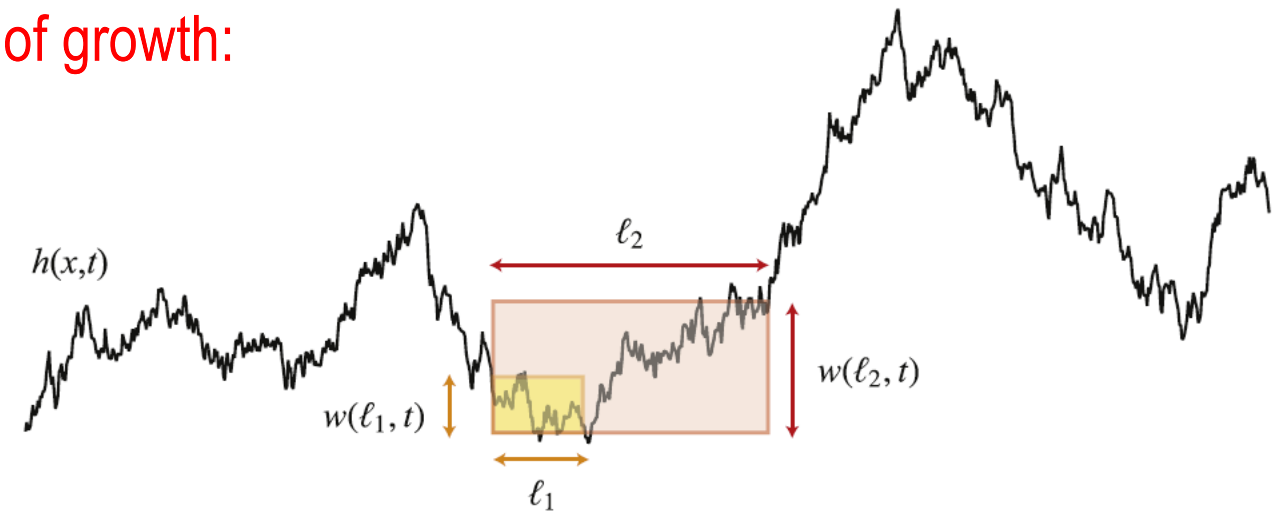
Ingredients: Growth under noise (heterogeneity) and short range interactions

Nontrivial relations to:

- Asymmetric simple exclusion process (hopping motion of hardcore particles)
- Longest growing subsequence in a random permutation, combinatorics

Universal scaling of surface roughness

Characterization of growth:



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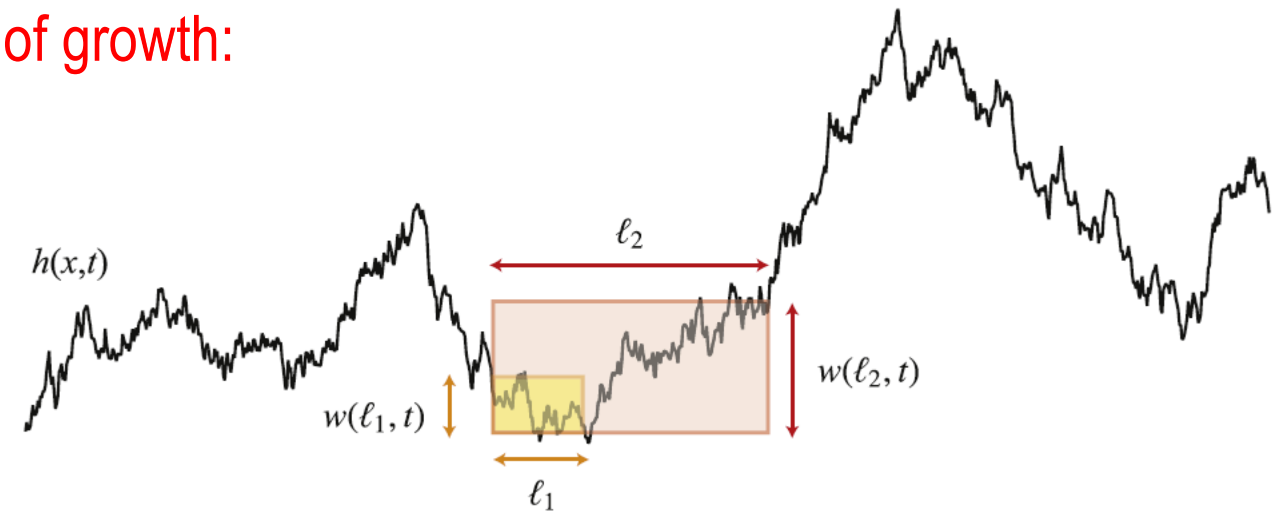

Scale invariance: $\longrightarrow$ Invariance of statistical properties under scale transformation

$w(\ell, t)$?

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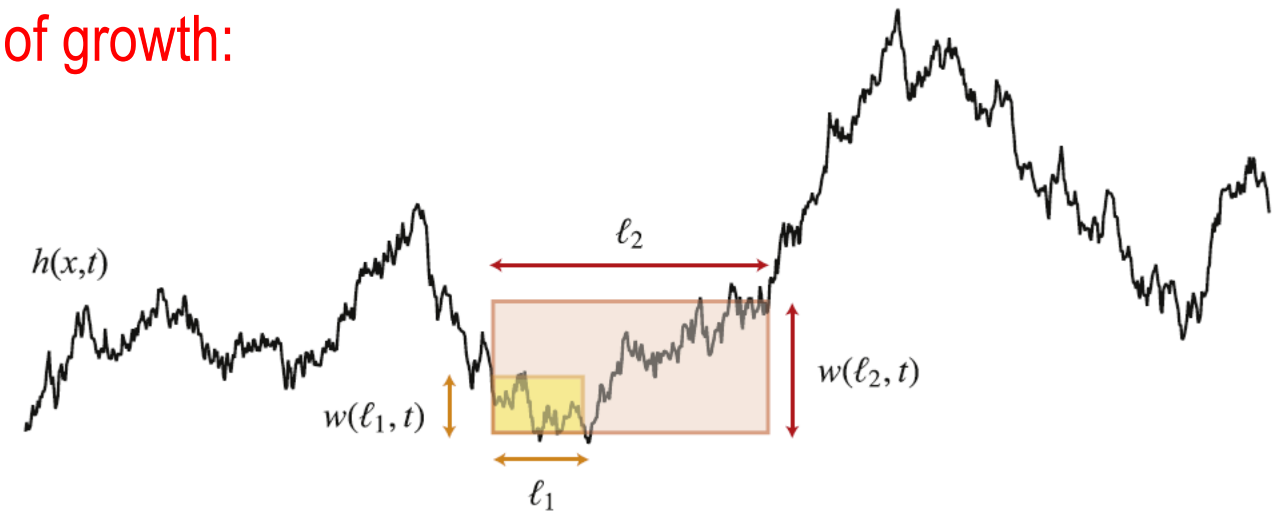

Scale invariance: → Invariance of statistical properties under scale transformation

$$w(\ell,t) \sim t^{\beta} F_w(\ell t^{-1/z}) \sim \begin{cases} \ell^{\alpha} & \text{for } \ell \ll \xi(t), \\ t^{\beta} & \text{for } \ell \gg \xi(t), \end{cases} \qquad \text{with } \xi(t) \sim t^{1/z} \ (z \equiv \alpha/\beta),$$

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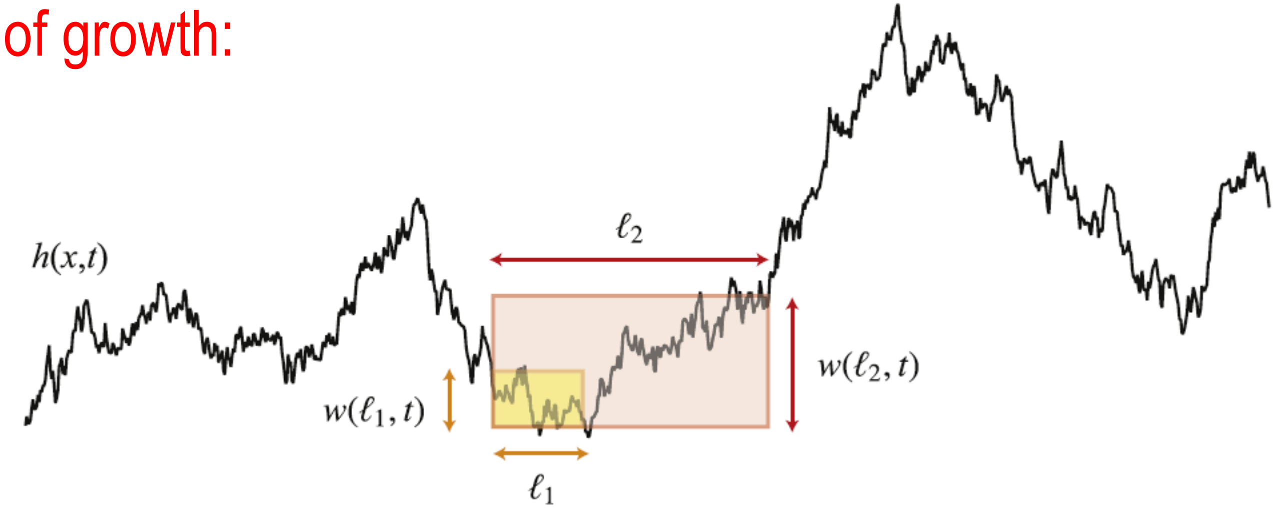

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$$x \mapsto bx, \quad t \mapsto b^z t, \quad \delta h \mapsto b^\alpha \delta h.$$

$$\delta h(x,t) \equiv h(x,t) - \langle h(x,t) \rangle$$

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α, β, z: Universal exponents

$\longrightarrow$ Universality classes of kinetic roughening

Linear growth: Edwards-Wilkinson

Equation of motion for growing surface:

- Time translation invariance $t \to t + \Delta t$
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- Height translation translation $h \to h + \Delta h$

Edwards-Wilkinson equation

Steady accumulation (with random noise) + diffusion

$$\frac{\partial}{\partial t} h(\boldsymbol{x}, t) = v_0 + \nu \boldsymbol{\nabla}^2 h + \eta(\boldsymbol{x}, t).$$

White noise: $\langle \eta(\boldsymbol{x}, t) \rangle = 0, \qquad \langle \eta(\boldsymbol{x}, t) \eta(\boldsymbol{x}', t') \rangle = D\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$

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Checked by explicit solution of linear equation in Fourier space (caution: usually too naive, see KPZ)

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From Langevin (stochastic diff Eq) to Fokker Planck (evolution of noise-averaged probability):

$$\frac{\partial X_i}{\partial t} = F_i[\{X_j\}] + \eta_i(t), \quad \longrightarrow \quad P(X,t) = \overline{\delta(X - X(t))}^{\eta} \;?$$

Fokker Planck from Langevin

$P(X,t) = \overline{\delta(X - X(t))}^{\eta}$?

$$\overline{\phi}(t) = \int \phi(X) P(X,t) dX \qquad \overline{\phi}(t+dt) = ?$$

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$$= \int \overline{\partial_{X_i}\phi(X)(F_i[X] + \eta_i)}^{\eta} P(X,t) dX$$
$$= \int \left[\partial_{X_i}\phi(X) F_i[X] + \frac{D}{2}\partial^2_{X_i}\phi(X)\right] P(X,t) dX$$
$$= \int \phi(X) \left[-\partial_{X_i}(F_i[X] P(X,t)) + \frac{D}{2}\partial^2_{X_i} P(X,t)\right] dX$$

$$\overline{g(X(t))\eta_i(t)}^{\eta} = \overline{g\left(X(t-\Delta t) + \int_{t-\Delta t}^{t} dt' \dot{X}(t')\right)\eta(t)}^{\eta}$$
$$= \overline{g(X(t-\Delta t))\eta(t)}^{\eta} + \sum_j \overline{\partial_{X_j} g(X(t-\Delta t)) \int_{t-\Delta t}^{t} dt' \eta_j(t')\eta_i(t)}^{\eta}$$
$$= 0 + \frac{D}{2}\partial_{X_i} g(X(t-\Delta t)) \approx \frac{D}{2}\partial_{X_i} g(X(t))$$

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Langevin to Fokker Planck

$$\frac{\partial}{\partial t} P[h(\boldsymbol{x}), t] = -\int \mathrm{d}^d \boldsymbol{x} \frac{\delta}{\delta h} (\nu \boldsymbol{\nabla}^2 h) P[h(\boldsymbol{x}), t] + \frac{D}{2} \int \mathrm{d}^d \boldsymbol{x} \frac{\delta^2}{\delta^2 h} P[h(\boldsymbol{x}), t]$$

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Stationary state: “free Gaussian field”
1d: like Brownian motion with x as time

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Non-linear growth: Kardar-Parisi-Zhang

Growth is typically non-linear!

Simple source of non-linearity: a sloped surface grows faster (in z) than a flat one:
because the surface per unit length (in x!) is larger (aggregeation is assumed prop. to surface!)

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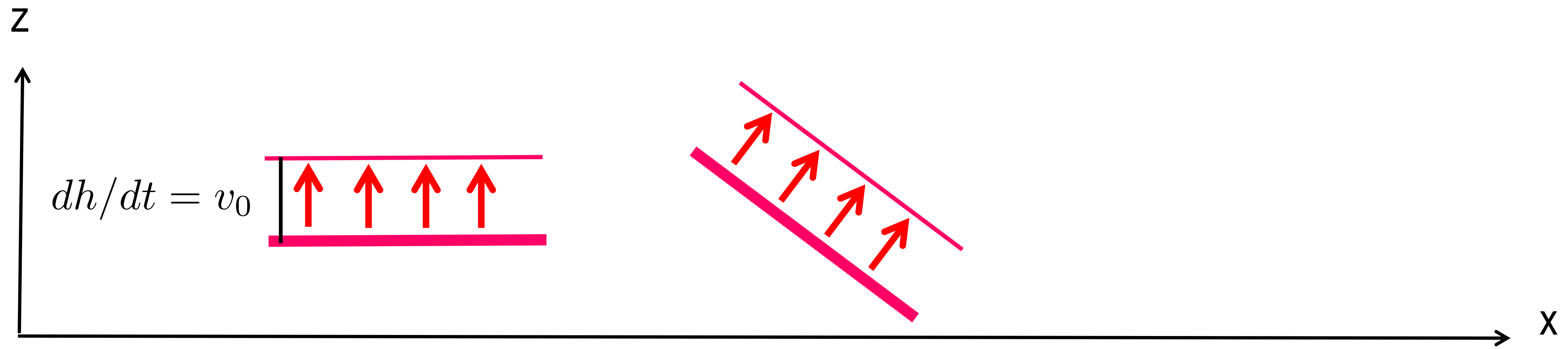

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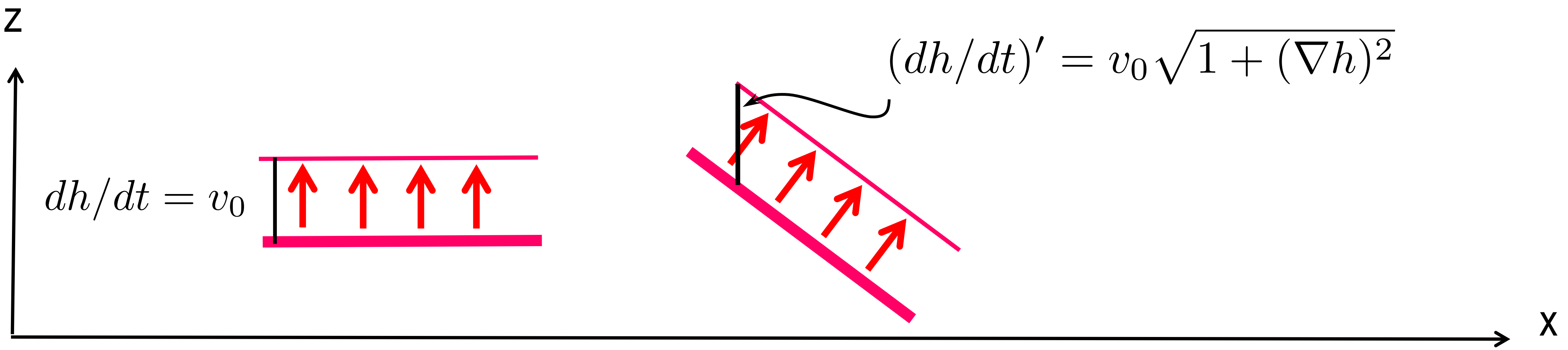

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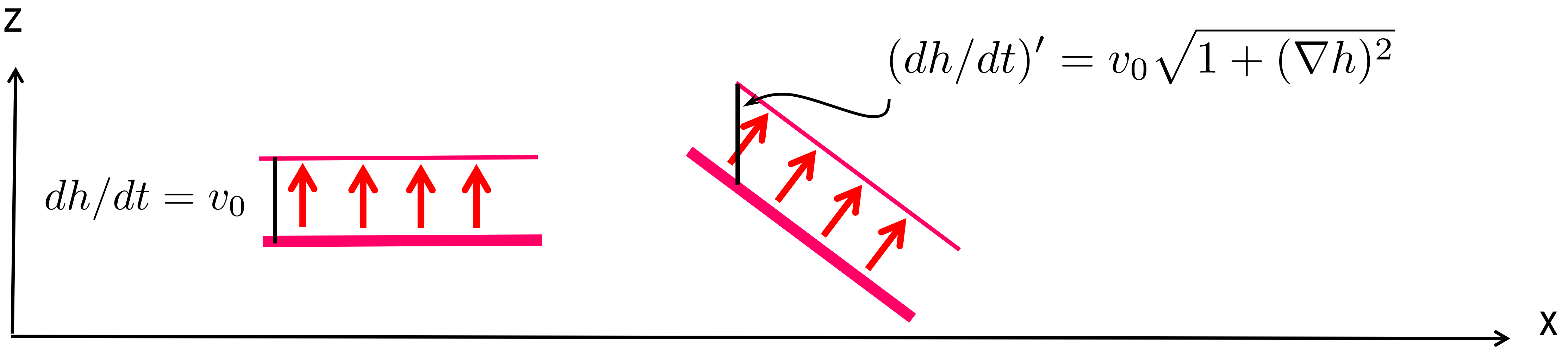

KPZ equation: $\frac{\partial}{\partial t}h(\boldsymbol{x},t) = \nu_0 + \nu\boldsymbol{\nabla}^2 h + \eta(\boldsymbol{x},t).$ $\longrightarrow$ $\boxed{\frac{\partial}{\partial t}h(\boldsymbol{x},t) = \nu_0 + \nu\boldsymbol{\nabla}^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla}h)^2 + \eta(\boldsymbol{x},t)}$

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- v_0 can be eliminated by Galilean transformation; but non-linearity breaks the invariance under $h \longrightarrow -h$. Genuine growth direction is singled out!
- The non-linearity is relevant in the RG sense (for d<= 2): $\lambda \rightarrow \lambda\, b^{\alpha+z-2}$

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$$\alpha = \frac{2-d}{2}, \quad \beta = \frac{2-d}{4}, \quad z = 2.$$

- Many other possible sources of non-linearity (e.g. interactions)
- The square of the gradient is the leading symmetry-allowed non-linearity
- v_0 can be eliminated by Galilean transformation; but non-linearity breaks the invariance under $h \longrightarrow -h$. Genuine growth direction is singled out!
- The non-linearity is relevant in the RG sense (for d<= 2): $\lambda \rightarrow \lambda\, b^{\alpha+z-2} = \lambda\, b^{1-\frac{d}{2}}$

KPZ equation: $\frac{\partial}{\partial t} h(\boldsymbol{x}, t) = v_0 + \nu \boldsymbol{\nabla}^2 h + \eta(\boldsymbol{x}, t). \longrightarrow$

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Non-linear growth: Kardar-Parisi-Zhang

Growth is typically non-linear!

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  $\longrightarrow$ EW fixed point ($\lambda = 0$) is unstable; changes universality class!

KPZ equation: $\frac{\partial}{\partial t} h(\boldsymbol{x}, t) = v_0 + \nu \nabla^2 h + \eta(\boldsymbol{x}, t).$ $\longrightarrow$ $\frac{\partial}{\partial t} h(\boldsymbol{x}, t) = v_0 + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\boldsymbol{x}, t)$

Kardar-Parisi-Zhang universality class

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Universal exponents α, β, z ?

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- Indeed: ν and D flow under scale transformation (renormalization)

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Universal exponents α, β, z ?

- 4 terms, but only 3 exponents: naive scaling analysis does not work!
- Indeed: ν and D flow under scale transformation (renormalization)
- But: λ does not flow : a symmetry protects it!

This is easiest to see from a mapping to fluid dynamics.

Relation to particle flows: stirred Burger's equation

KPZ equation

$$\partial_t h = \nu \vec{\nabla}^2 h + \frac{\lambda}{2}(\vec{\nabla} h)^2 + \eta$$

Define: Velocity field

$$\mathbf{v} = -\lambda \nabla h$$

$$\longrightarrow \quad \partial_t \mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{\nabla}{2}(\mathbf{v}^2) - \lambda \nabla \eta$$

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$$\frac{1}{2}\nabla(\mathbf{v}^2) = (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v}) \to (\mathbf{v}\cdot\nabla)\mathbf{v}$$

Burger's equation
(= Navier Stokes)
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Shear viscosity (dissipation)

Convective derivative

Random stirring force

Burgers vs Navier-Stokes

Burger's equation

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Identical to incompressible $[\rho = cst.]$ Navier-Stokes equations, without pressure gradient)

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→ While here *v* is a gradient: $\mathbf{v} = -\lambda\nabla h$ → No curl → no eddies!

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"Burger's turbulence": Does not describe real turbulence, but Burger's equation has applications to large scales of galaxies, and interesting short scale singularities (shock wave generation)

Galilean invariance of fluid dynamics

Burger's equation $$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \nabla^2 \mathbf{v} - \lambda \nabla \eta$$

For any physical fluid dynamics:

Invariance under Galilean transformation: For any relative velocity v_0

$$\mathbf{v}_{\mathrm{new}}(\mathbf{x}, t) = \mathbf{v}_0 + \mathbf{v}(\mathbf{x} - \mathbf{v}_0 t, t)$$

also satisfies Burger's equation!

Note: With a shifted noise realization, but having the same statistics:

$$\langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t')\rangle_{\mathrm{new}} = \langle \eta(\boldsymbol{x} - \boldsymbol{v}_0 t, t)\eta(\boldsymbol{x}' - \boldsymbol{v}_0 t', t')\rangle = D\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t').$$

$\longrightarrow$ "*statistical* Galilean invariance"

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Related symmetry for height model?

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Related symmetry for height model:

$$(\mathbf{v}_0 = -\lambda\mathbf{s})$$

$$h_{\mathrm{new}}(\mathbf{x},t) = h(\mathbf{x}+\lambda\mathbf{s}t, t) + \mathbf{s}\cdot\mathbf{x} + \frac{\lambda}{2}\mathbf{s}^2 t$$

also satisfies the KPZ growth equation

$$\partial_t h = \nu\vec{\nabla}^2 h + \frac{\lambda}{2}(\vec{\nabla}h)^2 + \eta$$

$\longrightarrow$ "Statistical tilt symmetry"

Galilean invariance of fluid dynamics

Implication of symmetries

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \nabla^2 \mathbf{v} - \lambda \nabla \eta$$

Physical Galilean invariance always has prefactor 1.
This is obviously preserved under renormalization (integrating out small scales)

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Scale invariance of KPZ equation

$$b^{\alpha - z} = b^{2\alpha - 2} \rightarrow \alpha + z = 2$$

KPZ exponents

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$$\delta h \sim t^\beta \equiv t^{\alpha/z}$$

$$z = 2 - \alpha;$$

Statistical tilt symmetry

$$\beta = \frac{\alpha}{z} = \frac{\alpha}{2-\alpha}$$

(by definition)

d=1 : exactly known

$$\alpha = 1/2, \quad \beta = 1/3, \quad z = 3/2$$

d=2 : $\alpha = 0.3869;\ \beta = 0.2398,\ z = 1.6131$

d>2 : weak $\lambda \longrightarrow$ RG-irrelevant $\longrightarrow$ flow to Edwards-Wilkinson

strong λ: transition to a strong coupling fixed point!

KPZ exact exponents in d = 1

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(Huse-Henley-Fisher PRL ’85)

How do we know?

KPZ in d=1 – exact exponents

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Check: ($\longrightarrow$ Exercise): The Fokker-Planck equation for KPZ now reads

$$\frac{\partial}{\partial t} P[h(\boldsymbol{x}), t] = -\int \mathrm{d}^d\boldsymbol{x} \frac{\delta}{\delta h}\left[\nu \boldsymbol{\nabla}^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla} h)^2\right] P[h(\boldsymbol{x}), t] + \frac{D}{2}\int \mathrm{d}^d\boldsymbol{x} \frac{\delta^2}{\delta^2 h} P[h(\boldsymbol{x}), t]$$

In d = 1 (and only there)

$$P_{\mathrm{stat}}[h(\boldsymbol{x})] \propto \exp\left[-\int \mathrm{d}^d\boldsymbol{x} \frac{\nu}{D}(\boldsymbol{\nabla} h)^2\right]$$

is *still* the stationary distribution for $\lambda \neq 0$: $\quad \partial_t P[\{h(x)\}, t] = 0$

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d=1 : exactly known $\alpha = 1/2, \quad \beta = 1/3, \quad z = 3/2$ (Huse-Henley-Fisher PRL '85)

Check: ($\longrightarrow$ Exercise):

$\int dx\,(dh/dx)^2$ is indeed scale invariant with $2\alpha - 1 = 0$
$\rightarrow \alpha = 1/2$

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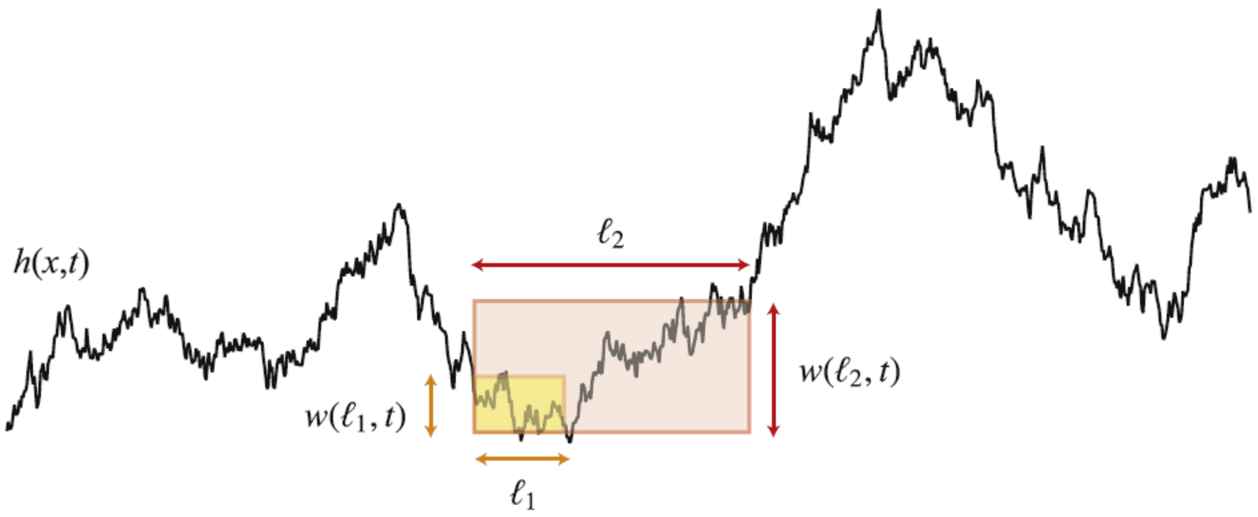

- Start from flat interface at t = 0
- At later times t: small scales

$$w \sim \ell^{1/2}$$

$$(\ell \ll \xi(t) \sim t^{1/z} = t^{2/3})$$

look like random walks!

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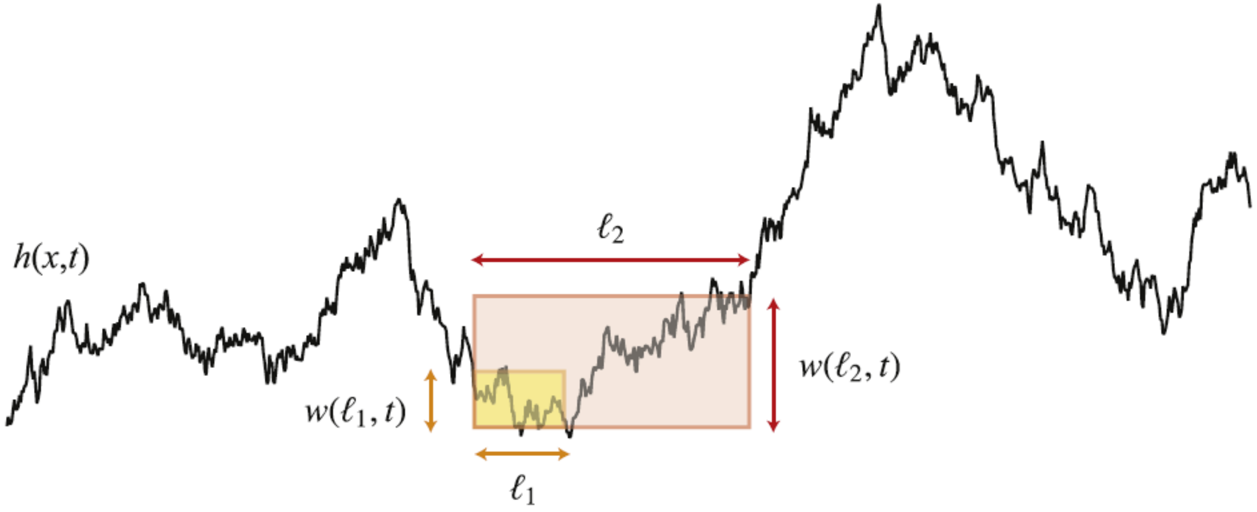

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- At large scales $(\ell \gg \xi(t))$

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Alternative argument for exponent $\beta = 1/3$:

Microscopic lattice models (e.g. exclusion processes) lead to a non-linear fluid dynamics, but do *not* contain fluctuating forces (D) / diffusion (ν) a priori.
(They are often added phenomenologically in the spirit of fluctuating hydrodynamics, obeying a fluctuation-dissipation relation)

Still they reflect the stationary distribution $\langle (h(x) - h(x'))^2 \rangle = \frac{D}{2\nu} |x - x'| = A|x - x'|$
where only the ratio $A = \frac{D}{2\nu}$ enters as a parameter of the model.

Typical fluctuation of h(x,t)? Independent of x, depends only on t, λ, A $\quad [\lambda] = \frac{[x]^2}{[h][t]} \quad [A] = \frac{[h]^2}{[x]}$

Only combination: $\longrightarrow$ $h(t) \sim (\lambda t A^2)^{1/3}$ $\longrightarrow$ $\beta = 1/3$

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Only dimensionally correct combination:

$$h(t) \sim (\lambda t A^2)^{1/3} \longrightarrow \boxed{\beta = 1/3}$$

Mapping to directed polymers: the Cole-Hopf transformation

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Mathematical issue with the KPZ equation in the continuum:

White noise η is non-differentiable

$\longrightarrow$ $\nabla \mathrm{h}$ becomes singular

$\longrightarrow$ $(\nabla \mathrm{h})^2$ is a priori ill-defined

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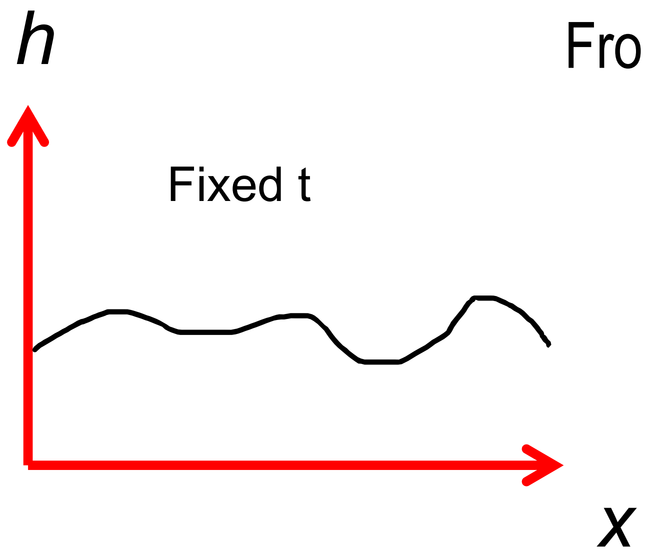
Possible remedies:

- Work with finite-in-time correlated, colored noise
- Use discretized lattice models
- Or: Mapping that removes the non-linearity: Cole-Hopf transformation

Mapping to directed polymers: The Cole-Hopf transformation

From growing surface to a directed polymer



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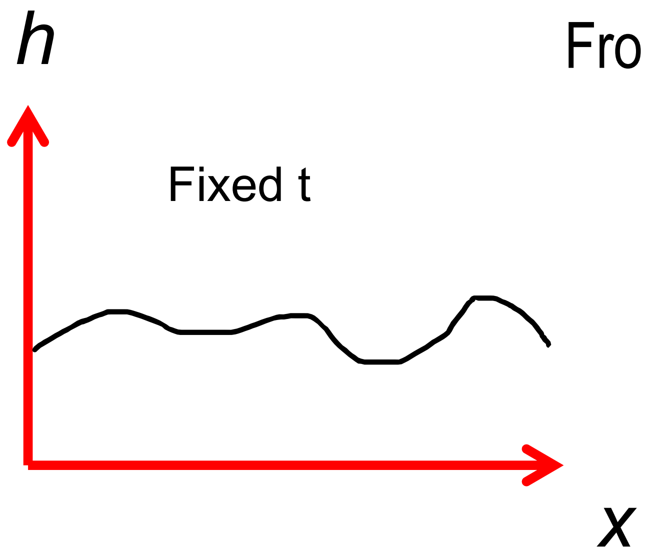

From growing surface to a directed polymer

$$h(x,t) \leftrightarrow F(x,\ell)$$

$$t \leftrightarrow \ell$$

Interpret height at time t as free energy of a *directed* polymer starting at the origin and ending at lateral displacement x after a longitudinal distance $\ell \equiv t$

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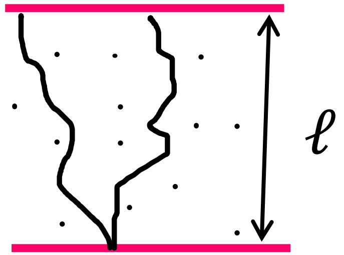

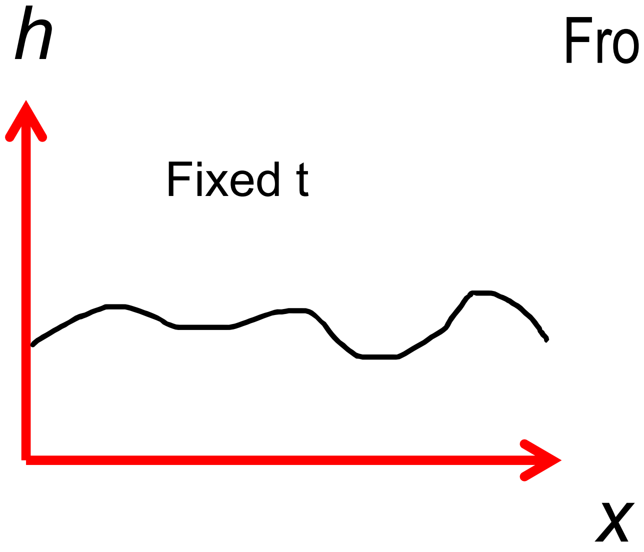

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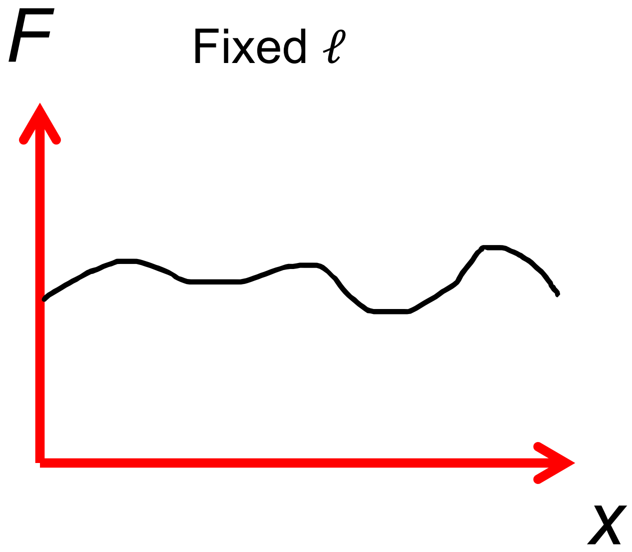

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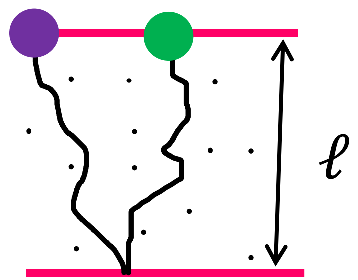
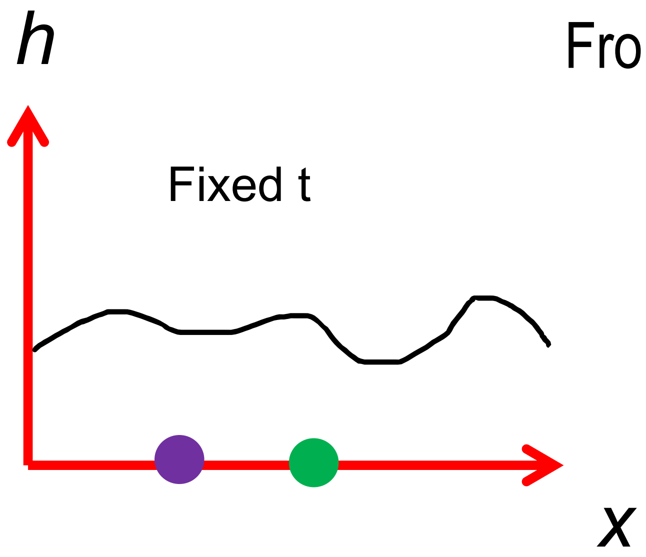
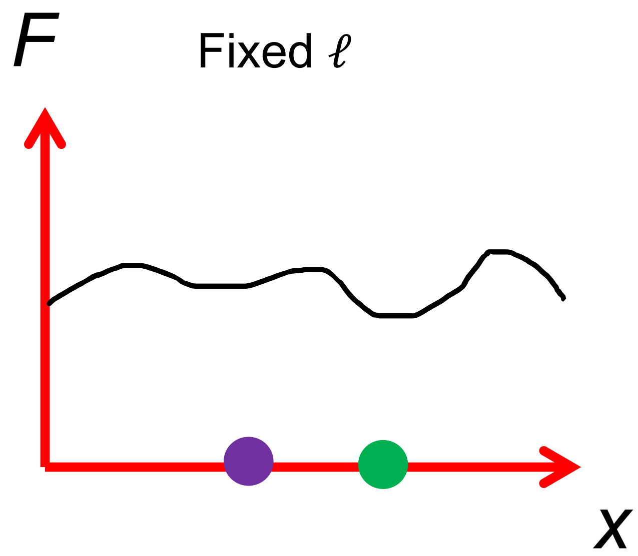


From growing surface to a directed polymer



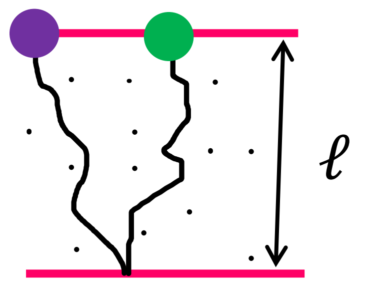
$$h(x,t) \leftrightarrow F(x,\ell)$$

$$t \leftrightarrow \ell$$



Interpret height at time t as free energy of a *directed* polymer starting at the origin and ending at lateral displacement x after a longitudinal distance $\ell \equiv t$

Mapping to directed polymers: The Cole-Hopf transformation



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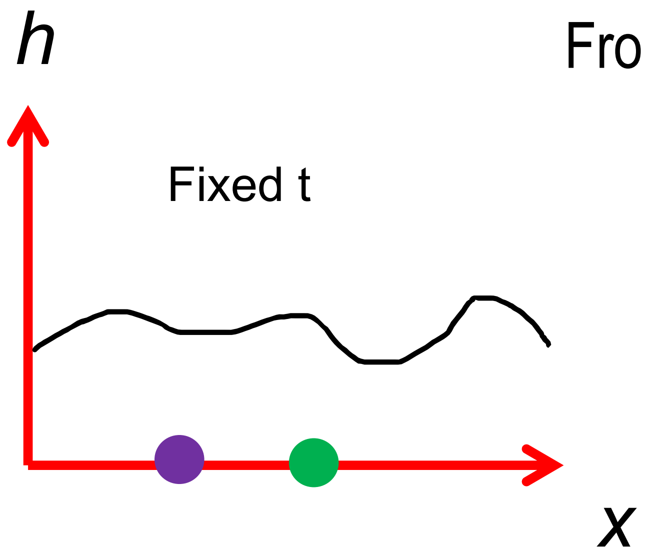

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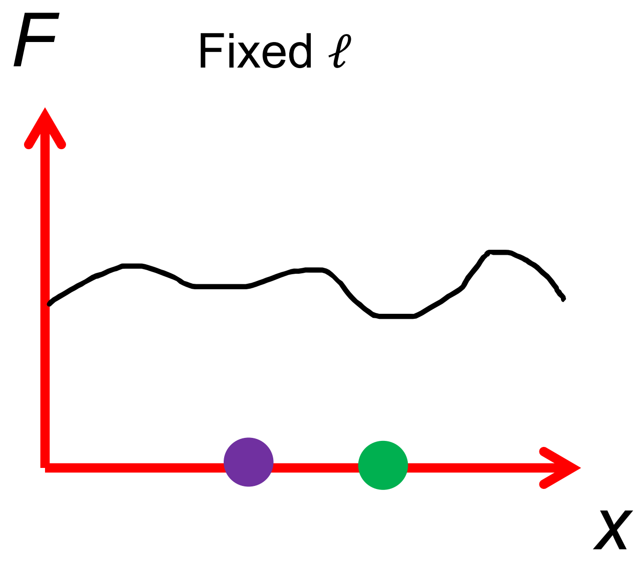

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$\rightarrow$ Partition function: $Z(\boldsymbol{x},t) \equiv \exp\left[\frac{\lambda}{2\nu} h(\boldsymbol{x},t)\right]$

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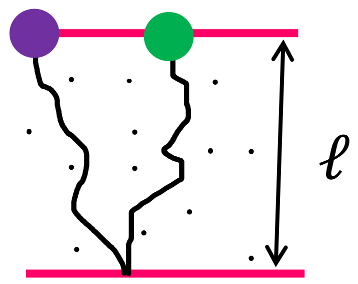
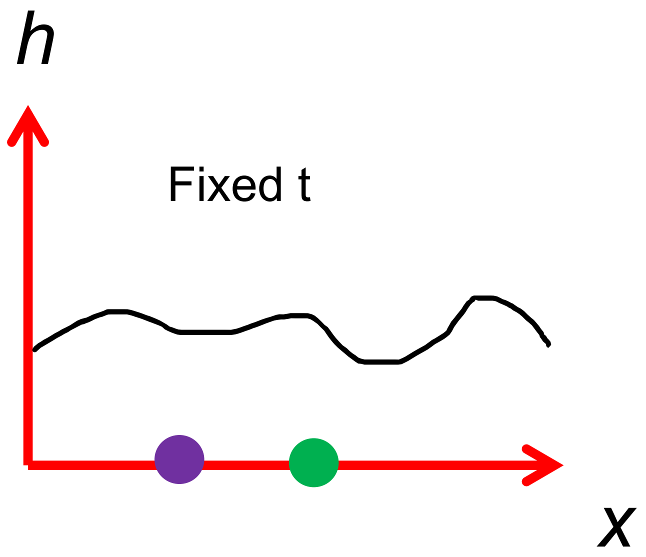
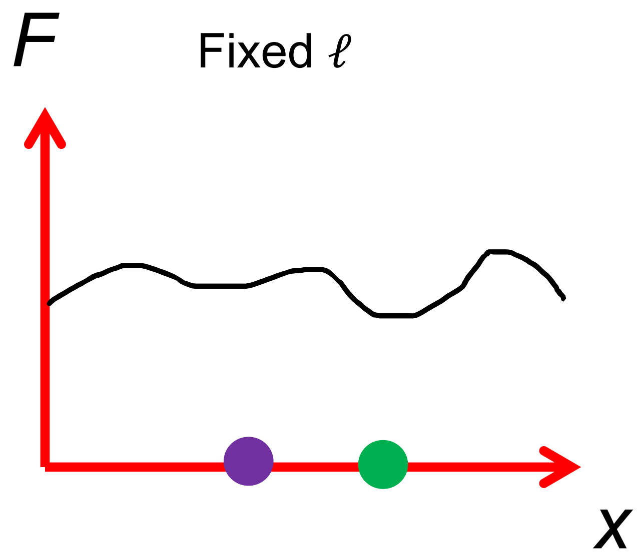




$$\partial_t h = \nu \vec{\nabla}^2 h + \frac{\lambda}{2}(\vec{\nabla} h)^2 + \eta$$

$$\partial_t Z(x,t) = \nu \partial_x^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t)\eta(x,t)$$



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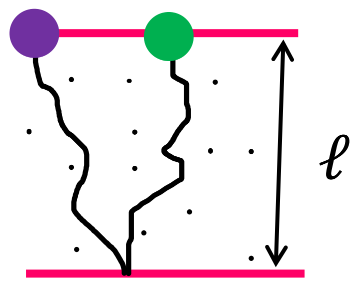

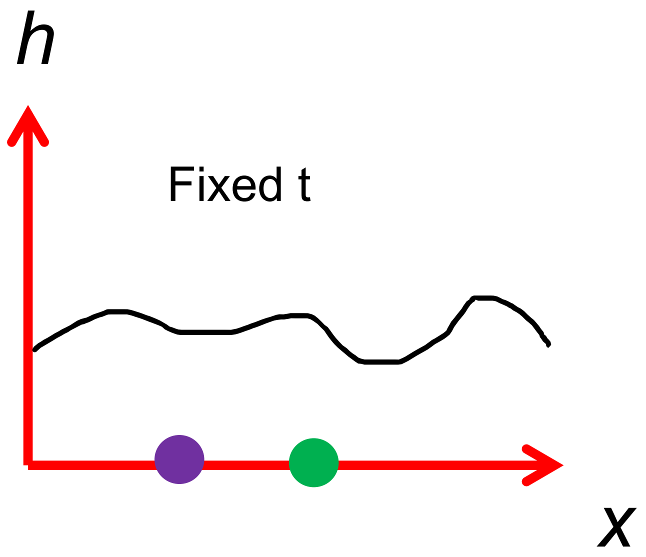

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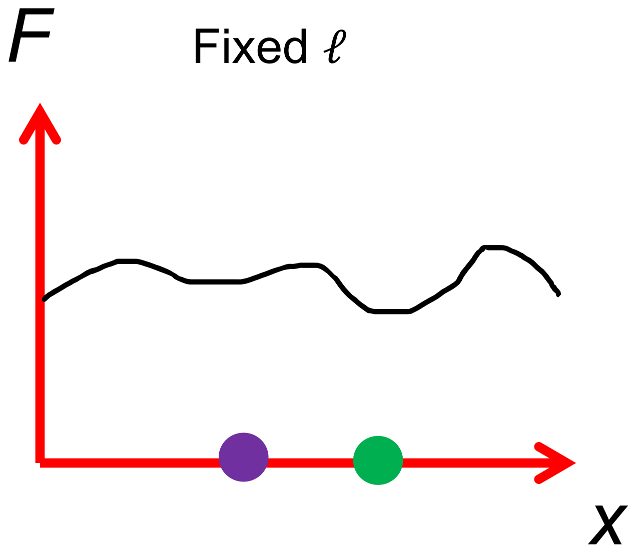
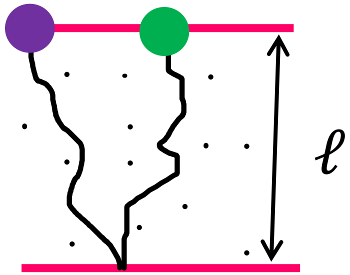
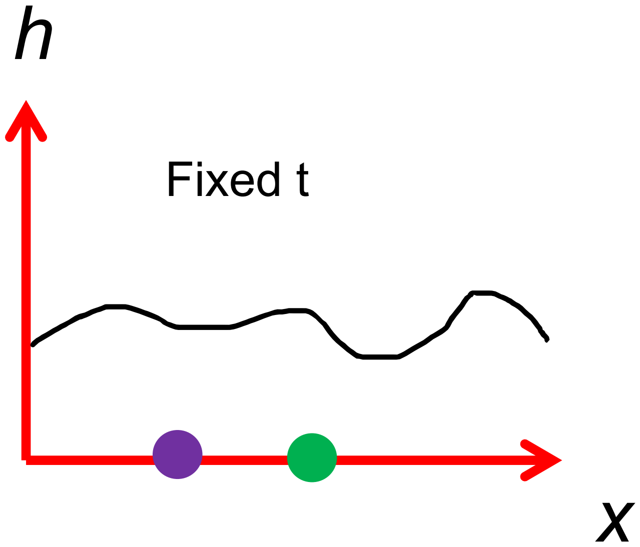


$\rightarrow$ Stochastic, **linear** diffusion or heat equation, but with **multiplicative** noise!

Mapping to directed polymers: The Cole-Hopf transformation





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$$\downarrow$$

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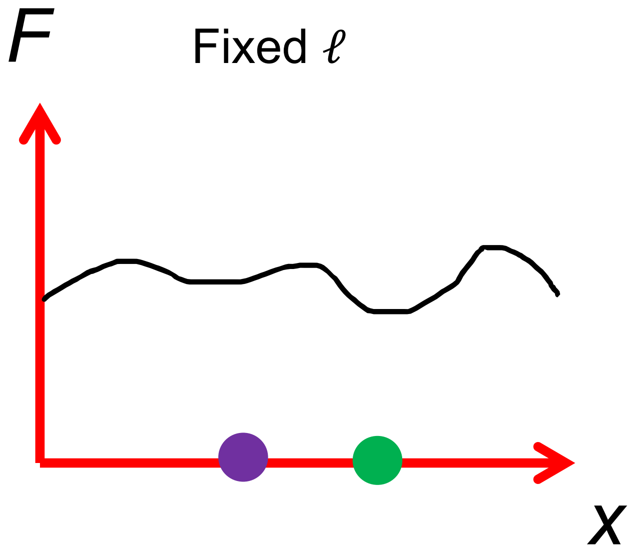
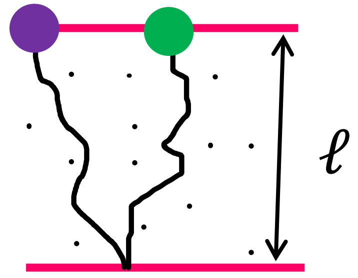


$\rightarrow$ Stochastic, **linear** diffusion or heat equation, but with **multiplicative** noise!

Rem: Well-defined with Itô's prescription for $Z(x,t)\eta(x,t) = Z(x,t) \lim_{dt \to 0^+} \frac{1}{dt} \int_0^{dt} \eta(x, t+\delta) d\delta$

"Evaluate noise at slightly later time" $\rightarrow$ avoid correlation btw Z and η !

Solution of the stochastic heat equation



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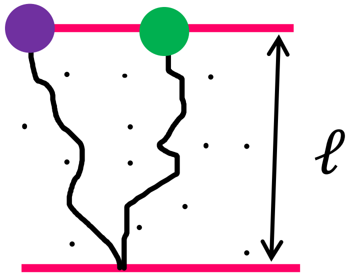
Solving the linear stochastic heat equation?

Solution of the stochastic heat equation



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Like in QM:

$$i\partial_t \psi(x,t) = -\frac{1}{2m}\partial_x^2 \psi(x,t) + V(x)\psi(x,t)$$

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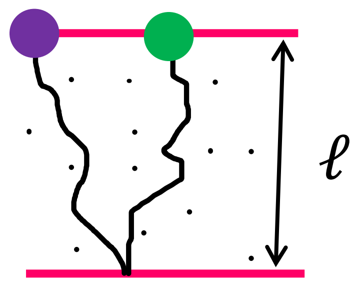

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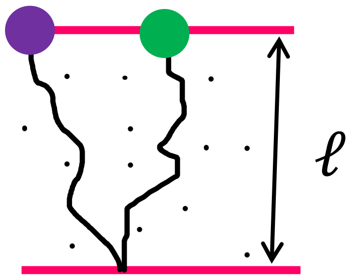

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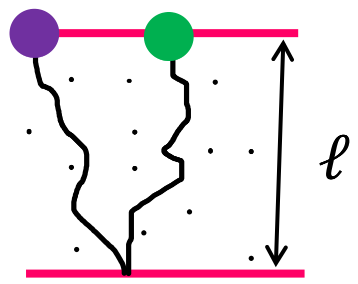
Feynman-Kac

$$\psi(x,t) = \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[i\frac{m}{2}\int_0^t \dot{u}^2 dt - i\int_0^t V(u(t))\right] = \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[i\int_0^t \mathcal{L}(u(t))\right]$$

Solution of the stochastic heat equation



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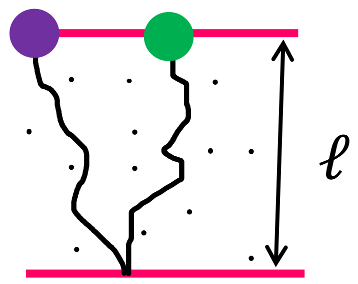

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$$T \equiv 2\nu$$

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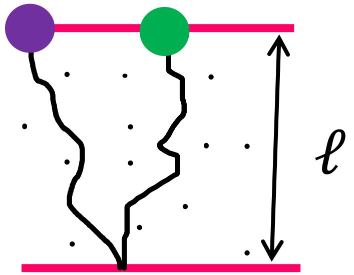

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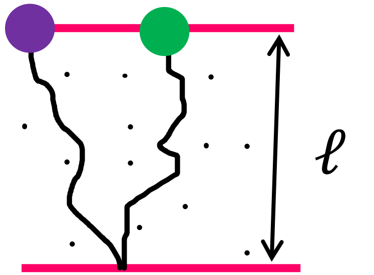
Solution of the stochastic heat equation



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Z: partition function of an elastic string (=dir. polymer) in the random potential $V(x,t) = -\lambda\eta(x,t)$!

Dictionary DPRM - KPZ

$$Z(x,t) = \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[-\frac{1}{T}\left(\frac{1}{2}\int_0^t \dot{u}^2 dt - \int_0^t \lambda\eta(u(t),t)\right)\right]$$

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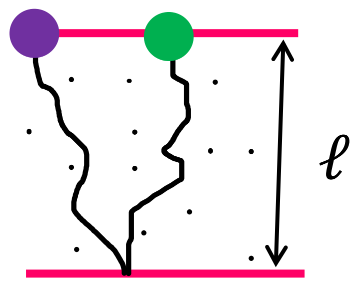

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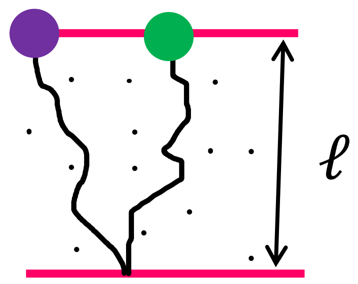

KPZ growth	**Directed polymer**
Time t	Longitudinal direction ℓ = t
Height h $\delta h \sim t^{\beta=1/3}$	Free energy F $\delta F \sim \ell^{\theta=1/3}$
Spatial position x $x \sim t^{1/z=2/3}$	Typical lateral displacement of the polymer $x \sim \ell^{\zeta=2/3}$

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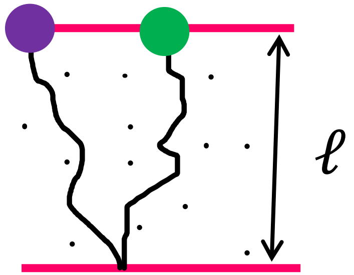

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Roughness exponent of the polymer: $\zeta = 2/3 > \zeta_{\mathrm{RW}} = 1/2$

Disorder dominates over elastic entropy!

Dictionary DPRM - KPZ

$$Z(x,t) = \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[-\frac{1}{T}\left(\frac{1}{2}\int_0^t \dot{u}^2 dt - \int_0^t \lambda\eta(u(t),t)\right)\right]$$

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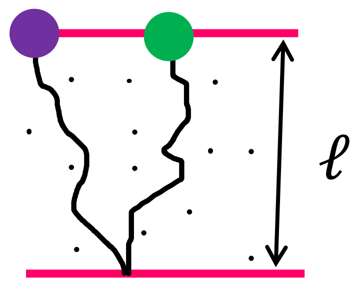

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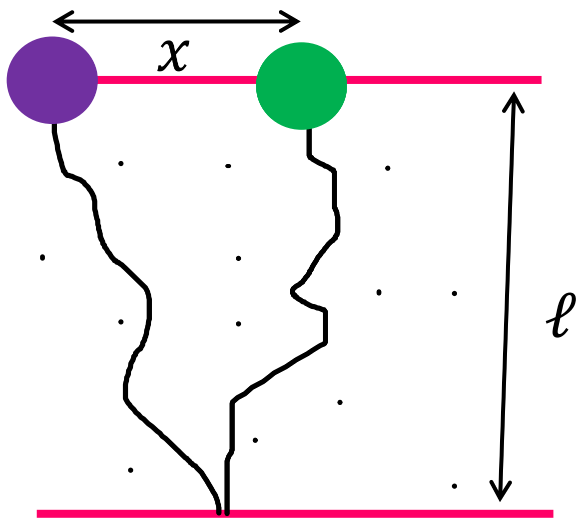
Statistical tilt symmetry: $\alpha + z = 2 \to \beta \equiv \frac{\alpha}{z} = \frac{2-z}{z} \to \boxed{\theta = 2\zeta - 1}$

Dictionary DPRM - KPZ

Simple interpretation:



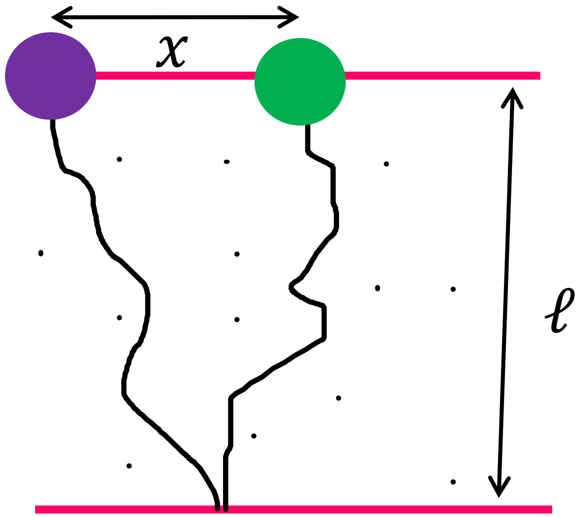
$$E_{\rm el} \sim \ell \cdot (x/\ell)^2 \sim x^2/\ell \sim \ell^{2\zeta-1} \leftrightarrow \delta F \sim \ell^{\theta}$$

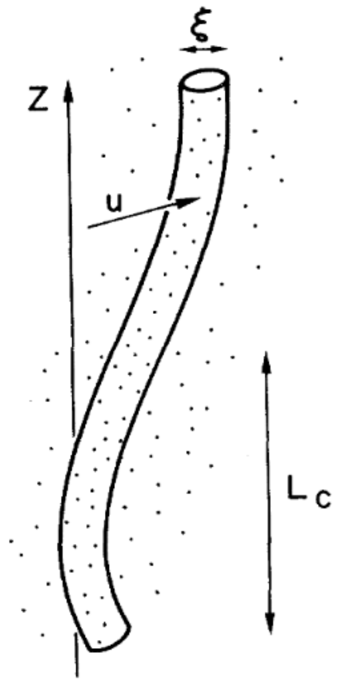
Disorder deforms the polymer until the elastic energy starts competing with gained potential energy

Statistical tilt symmetry: $\alpha + z = 2 \to \beta \equiv \dfrac{\alpha}{z} = \dfrac{2-z}{z} \to \theta = 2\zeta - 1$

Directed polymers in physics

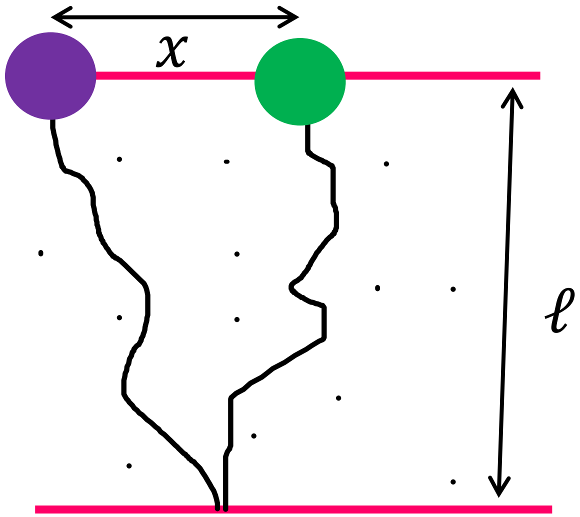
Vortices in superconductors

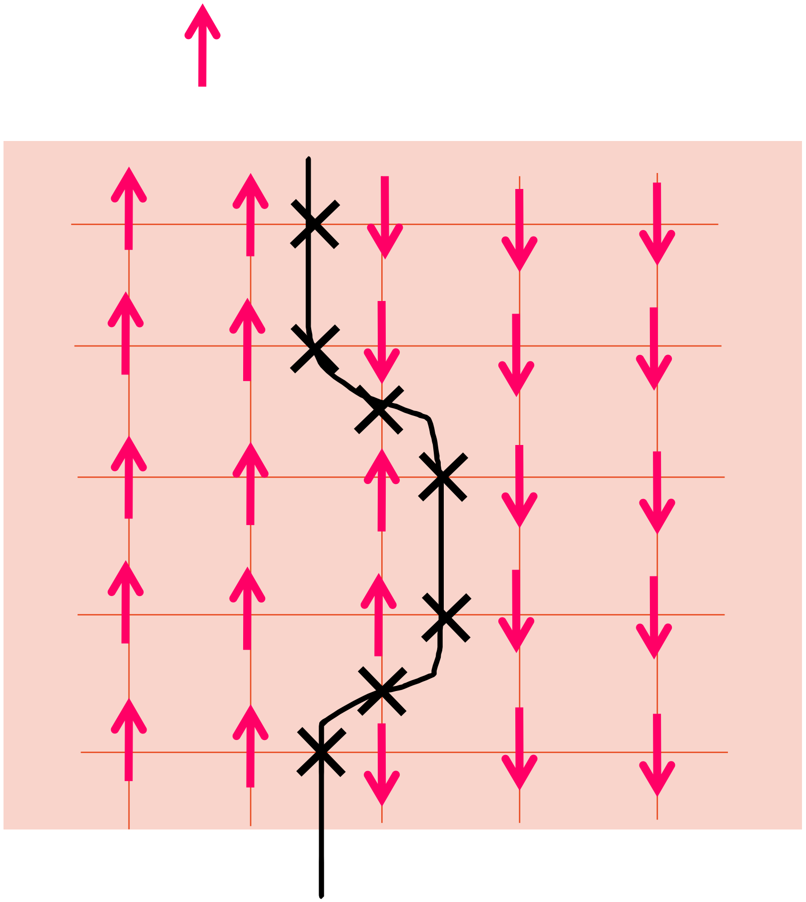




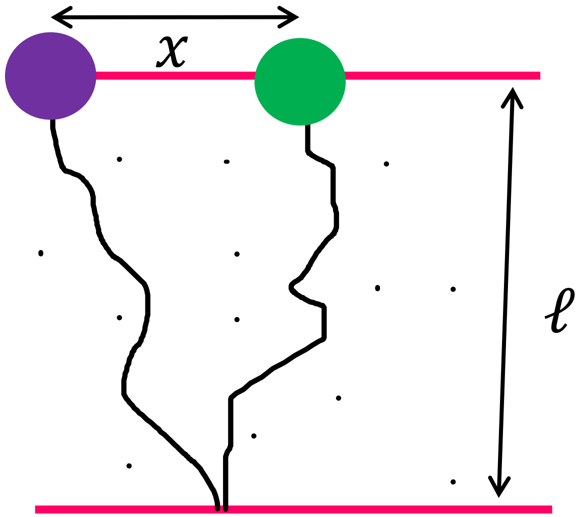
Directed polymers in physics

Domain walls in random bond Ising ferromagnets





Directed polymers in physics



- Spin-spin correlators in random magnets
- Decay of strongly localized quantum wavefunctions
- etc

Exact solutions and universality of KPZ

Several exact solutions of solvable models:

- Replica approach to directed polymers
- Polynuclear growth model
- Asymmetric exclusion processes

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They allow to compute specific quantities of interest, that are universal for all models in the KPZ class:

Like Onsager's 2d Ising solution , but now for KPZ!

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- Height distribution $P\left(\frac{h(x,t)-tv_0}{t^{1/3}}\right)$
- Two point correlation function $\langle \delta h(x,t)\delta h(x',t')\rangle$ (explicit but complicated)

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- Height distribution $P\left(\frac{h(x,t)-tv_0}{t^{1/3}}\right)$: Tracy Widom distribution!
  Like the maximal eigenvalue of a Gaussian random matrix! -- WHY??
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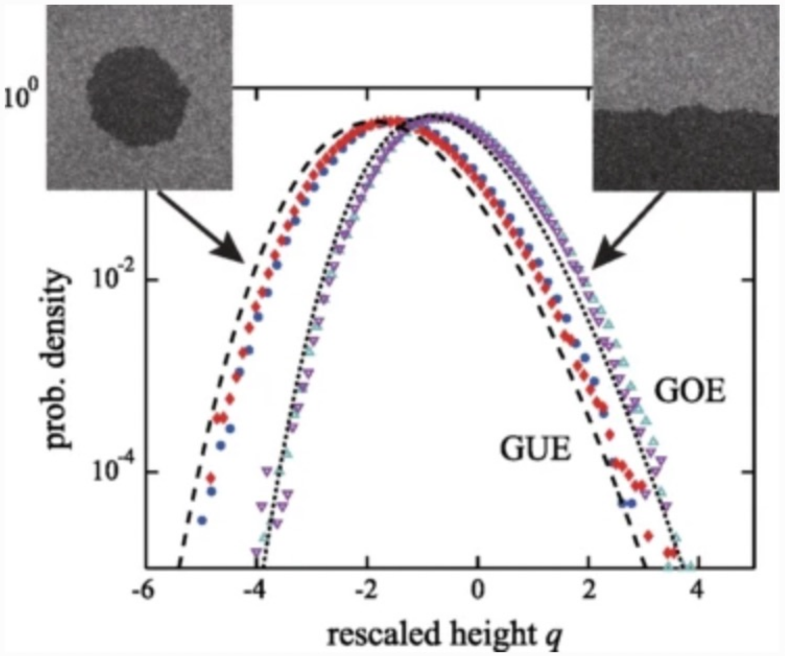
Tracy-Widom distribution of height

K. A. Takeuchi, M. Sano
(J. Stat. Phys. 2012)

Evidence for Geometry-Dependent Universal Fluctuations of the Kardar-Parisi-Zhang Interfaces in Liquid-Crystal Turbulence

Fig. 8



Histogram of the rescaled local height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3}$ for the circular (*solid symbols*) and flat (*open symbols*) interfaces. The *blue circles* and *red diamonds* display the histograms for the circular interfaces at t=10 s and 30 s, respectively, while the *turquoise up-triangles* and *purple down-triangles* are for the flat interfaces at t=20 s and 60 s, respectively. The *dashed* and *dotted curves* show the GUE and GOE TW distributions, respectively, defined by the random variables χ_{GUE} and χ_{GOE}. (Color figure online)

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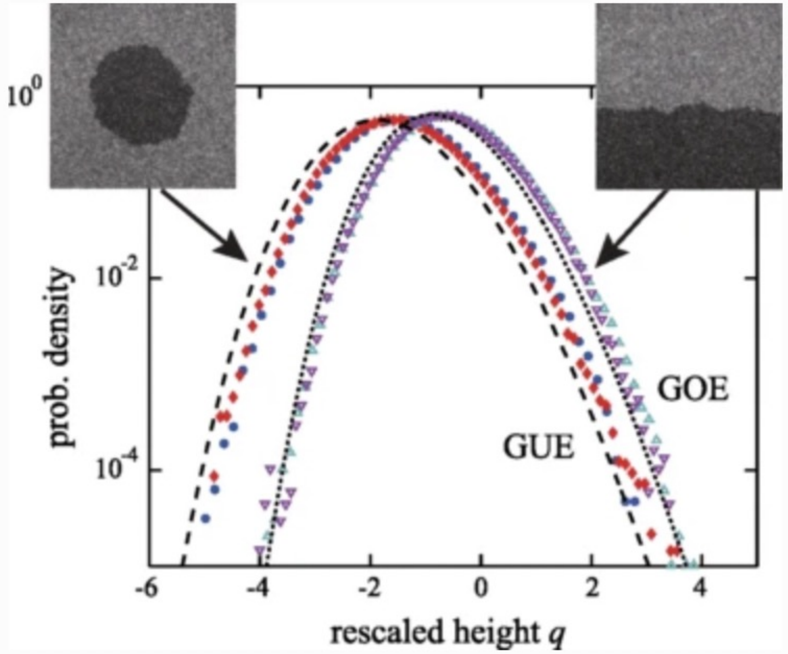

Histogram of the rescaled local height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3}$ for the circular (*solid symbols*) and flat (*open symbols*) interfaces. The *blue circles* and *red diamonds* display the histograms for the circular interfaces at t=10 s and 30 s, respectively, while the *turquoise up-triangles* and *purple down-triangles* are for the flat interfaces at t=20 s and 60 s, respectively. The *dashed* and *dotted curves* show the GUE and GOE TW distributions, respectively, defined by the random variables χ_{GUE} and χ_{GOE}. (Color figure online)

Mirror symmetry of flat surface
↔
Time reversal symmetry of random matrix GOE

Exercises

KPZ : symmetry and exponents

- Check the invariance of the KPZ Equation under tilt symmetry
- Derive the Fokker Planck equation from the Langevin equation. First for a discrete set of variables h_i , then for a function h(x). Apply this to the KPZ equation.
- Show that the free Gaussian field is the stationary solution of the FokkerPlanck equation for $\lambda = 0$. Show that in $\mathrm{d} = 1$ it is also stationary for any nonlinearity $\lambda > 0$.

Exercises

Flory exponents for pinned elastic manifolds

Consider the generalization of the directed polymer problem (elastic line (d=1)) with D transverse dimensions (d = dimension of the space in KPZ growth).
Now let us look at a more general elastic manifold $\mathrm{M} = \mathbb{R}^{\mathrm{d}}$ (surface d=2, crystalline solid (d=3)), where each point z has a displacement field u(z) ∈ $\mathbb{R}^{\mathrm{D}}$). (The standard D=1 KPZ Eq. maps to the d=1, D=1 directed polymer).

This system is subject to elastic energy $$H_{\mathrm{el}} = \int \mathrm{d}^d z \, \frac{c}{2} (\nabla \mathbf{u})^2$$

and a random disorder pinning energy $$H_{\mathrm{pin}} = \int \mathrm{d}^d z \, V(\mathbf{u}(\mathbf{z}), \mathbf{z})$$

We assume Gaussian correlated disorder of variance $$\overline{V(\mathbf{u},\mathbf{z}) V(\mathbf{u}',\mathbf{z}')} = K(|\mathbf{u}-\mathbf{u}'|)\delta^d(\mathbf{z}-\mathbf{z}')$$ and zero mean.

Exercises

- Show that in the absence of disorder the manifold has "thermal roughness"

$$\left\langle [\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{z}')]^2 \right\rangle_{\mathrm{th}} \sim |\mathbf{z} - \mathbf{z}'|^{2\zeta_{\mathrm{th}}} \qquad \zeta_{\mathrm{th}} = \frac{2-d}{2} \qquad (d \leq 2)$$

- Replicate the system m times and average the partition function over the disorder to obtain the replicated Hamiltonian for m copies:

$$H_m = \int \mathrm{d}^d z \left[\frac{c}{2} \sum_{a=1}^{m} (\nabla \mathbf{u}_a)^2 - \frac{1}{2T} \sum_{a,b=1}^{m} K(|\mathbf{u}_a(\mathbf{z}) - \mathbf{u}_b(\mathbf{z})|) \right]$$

- Assume the correlator decays as a power law $K(u) \sim u^{-\beta}$
  Argue that a short range correlator would naively correspond to $\beta = D$.

Exercises

- Assume that at large scale a self-similar regime exists where on the longitudinal distance L transverse fluctuations scale as L^{ζ}. Determine how one should rescale the temperature, $T \sim L^{\theta}$, to keep the term H_{el}/T invariant!

- Demand now that the replicated disorder term also remain scale invariant to obtain an expression for $\zeta(d, D)$.

  This kind of estimate is similar to Flory's on polymer physics.
  It assumes that the correlator *K* does not flow with the scale *L*.

- Discuss your result for *D* = *1*. For which d is your expression meaningful, and in which dimension is disorder perturbatively relevant? (compare to KPZ!)

Exercises

- The above estimate is expected to be exact for small enough β (fat power law tail). An analysis of the renormalization of *K* (next time) suggests that this only holds up to the critical value $\beta_c = D/2$ (not until $\beta = D$!) For faster decaying correlators with $\beta > \beta_c$ one expects the roughness as for $\beta > \beta_c$.

- Compare the obtained estimate $\zeta(d = 1, D = 1{,}2)$ with the exact value

$$\zeta(d = 1, D = 1) = 2/3$$

and the numerical value

$$\zeta(d = 1, D = 2) = 0.62$$

as well as for 2d interfaces (*D* = 1):

$$\zeta(d = 2, D = 1) \approx 0.3126$$

This line of reasoning is due to T. Halpin-Healey (1990)