Random matrix theory

History: Complex spectra of excited nuclei: Many energy levels for many nuclei ⇒ any organizing principle?

Interesting: Despite all differences in level density, composition etc, surprising universality about spacings between levels!

Wigner 1951: Complex nuclear many body Hamiltonian: When written in some basis of states $|m\rangle \to \langle n|H|m\rangle \equiv H_{mn}$ will be essentially random. The statistics of H_{nm} should not really depend on the choice of basis (for a chaotic system!)
⇒ Study random matrices ⇒ an extremely usefull rich field!

Beyond nuclear physics:
- random single particle Hamiltonians:
  Quantum billiards (→ random scattering between waves)
  ⇒ very complex, chaotic eigen- functions.

here the basis does matter!
- Anderson localisation; MBL; many body delocalisation
- electrons in quantum dots
- sound waves in disordered materials
- Coupling matrices of random magnets (J_{ij} of spin glasses!)
- random sequences of numbers: apparently
  - prime numbers
  - zeros of the Riemann ζ function

  appear to be spaced as eigenvalues of random matrices!

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Invariant ensembles of matrices

Idea: study the statistical properties of an ensemble of random matrices.

Wigner's (+Dyson) perspective: Any basis can be chosen to represent the Hamiltonian of a complex, chaotic system, and the choice should not matter ⇒ consider ensembles with probabilities $P(M)$ that are invariant under basis changes! $M \to TMT^{-1}$ T: basis change

Important class of random matrix ensembles: M: $N\times N$ matrix

$$P(M) \propto \exp\left[-\mathrm{Tr}\left[V(M)\right]\right]$$

$V(M)$ = arbitrary function of M (analytic at $M=0$)

e.g. $V(M) = M^2$ or M^p or $aM + bM^2 + cM^4 \ldots$

Trace ensures invariance under conjugation (basis change)

General $V(M)$ ⇒ matrix elements are correlated among each other.

BUT: for $V(M) = aM^2 \Rightarrow \mathrm{Tr}(aM^2) = a\sum_{ij=1}^{N} |M_{ij}|^2$

and Hermitian $M = M^\dagger$

$$\Rightarrow P(M) \propto \exp\left(-a\sum_{ij}|M_{ij}|^2\right) = \prod_{ij} \exp\left(-a\,|M_{ij}|^2\right)$$

Matrix elements are independent, Gaussian variables!

⇒ Gaussian random matrix ensembles of Wigner + Dyson!

⚠ Ensembles can be Gaussian, with independent M_{ij}, but not invariant

example: $P(M) = \exp\left(-\sum_{ij}\frac{|M_{ij}|^2}{A_{ij}}\right)$ with A_{ij} dependent on i,j

Rosenzweig-Porter: $A_{ij} = \begin{cases} \Lambda a & n = m \\ a & n \neq m \end{cases}$

Physics: invariant ensembles ⇔ extended, chaotic wavefunctions, non-localized

non-invariant: chosen basis is important, reflects locality (e.g.) ⇒ used to describe multifr. + localisation alit.

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Joint probability distribution of eigenvalues

Consider $N \times N$ Hermitian matrices $H = H^\dagger$.

Interest: eigenvalues E_n and eigenvectors $\mathcal{U} = \{\psi_1, \dots, \psi_N\}$ of H : $H\mathcal{U} = \mathcal{U}E$ $\quad E = \mathrm{diag}(\{E_n\})$

$\Leftrightarrow H = \mathcal{U}E\mathcal{U}^\dagger \quad \mathcal{U}$: unitary

H has $N + (N-1) + \dots + 1 = \frac{N(N+1)}{2}$ independent entries (for real H)

$\Rightarrow$ instead describe H by E_n (N values)

and $\frac{N(N-1)}{2}$ independent degrees of freedom of ψ_n's!

Gaussian ensemble : $P(H) \propto \exp\left(-a \sum_n E_n^2\right)$ (generally: $\propto \exp(-\sum_n V(E_n))$)

BUT: Jacobian for variable change

$$\{H_{mn}\} \longrightarrow \{E_n, \psi_m\} \text{ or } \{E_n, \mathcal{U}\}$$

$$dH_{nm} = [J] \begin{pmatrix} dE \\ df = \mathcal{U}^\dagger d\mathcal{U} \end{pmatrix} = [J]\, dx$$

$df = \mathcal{U}^\dagger d\mathcal{U} = -d\mathcal{U}^\dagger \mathcal{U} = -df^\dagger$: AntiHermitian matrix that captures possible deviations from $\mathcal{U}$.

$|\det J| = ?$

$$\mathrm{Tr}\, dH^2 = \sum_{ij} dH_{ij}^2 = \sum_{k,\ell,n} dx_k^\dagger \underbrace{J^\dagger_{kn} J_{n\ell}}_{J^\dagger J} dx_\ell = \sum_{k\ell} dx_k^\dagger D_{k\ell}\, dx_\ell$$

$$\Rightarrow |\det J| = \sqrt{\det D}$$

Now: $dH = d\mathcal{U}E\mathcal{U}^\dagger + \mathcal{U}\,dE\,\mathcal{U}^\dagger + \mathcal{U}E\,d\mathcal{U}^\dagger$; no $d\mathcal{U}\,dE$ cross-terms!

$$\Rightarrow \mathrm{Tr}(dH)^2 = \mathrm{Tr}\left(2E\,df\,E\,df + dE^2 + 2\,d\mathcal{U}E^2 d\mathcal{U}^\dagger\right)$$

$$= \mathrm{Tr}\left(2E\,df\,E\,df + dE^2 - 2\,df^2E^2\right)$$

$$= \sum_{n \neq m} 2E_n\, df_{nm} E_m\, df_{mn} - 2\,df_{nm}\,df_{mn} E_n^2 + \sum_n dE_n^2$$

$$= 2\sum_{n>m} |df_{nm}|^2 \underbrace{\left(E_n^2 + E_m^2 - 2E_nE_m\right)}_{(E_n-E_m)^2} + \sum_n dE_n^2$$

$$\Rightarrow |\det J|^2 = \det D \propto \prod_{n>m} (E_n - E_m)^{2\beta}$$

$\beta = 1$ if H real $\Leftrightarrow df_{nm}$ real

$\beta = 2$ if H complex $\Leftrightarrow$ " complex

$\Rightarrow$ Joint p.d.f.:

$$\boxed{dH \; P(H) = \{df_{n>m}\}\{dE_n\} \cdot |\Delta|^{\beta} \; \exp\left[-\sum_{n=1}^{N} V(E_n)\right]}$$

$\beta = 1$ H real
$\beta = 2$ H complex

where
$$\Delta = \Delta_N = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ E_1 & E_2 & E_3 & & E_N \\ E_1^2 & E_2^2 & E_3^2 & & E_N^2 \\ \vdots & & & & \\ E_1^{N-1} & E_2^{N-1} & E_3^{N-1} & & E_N^{N-1} \end{vmatrix} = \prod_{n>m} (E_n - E_m)$$

is the Vandermond determinant

<u>Remarkable property</u>: 1) $p(E, U) \overset{!}{=} p(E)$ only! Only depends on E, not on eigenvectors U!

$p(E) \to 0$ if $E_n \to E_m$: Two levels are unlikely to be equal!

$\Leftrightarrow$ <u>level repulsion</u>! Why is it stronger for complex matrices ($\beta = 2$)?

Simple but important calculation: 2×2 matrix of two levels that happen to be close together: $|E_1 - E_2| \ll \delta$ = typical level separation

$\Rightarrow$ only consider matrix elements in the subspace spanned by ψ_1, ψ_2:

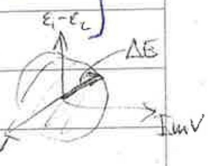
$\to H = \begin{pmatrix} \varepsilon_1 & V \\ V^* & \varepsilon_2 \end{pmatrix} \Rightarrow$ eigenenergies $E_{1,2} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + |V|^2}$

$$\Delta E = |E_1 - E_2| = \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + |V|^2}$$

$$\boxed{p(\Delta E)} = \int d\varepsilon_1 d\varepsilon_2 dV \; \delta\left(\Delta E - \sqrt{(\varepsilon_1-\varepsilon_2)^2 + |V|^2}\right) \; P(\varepsilon_1, \varepsilon_2, V)$$

$$= \frac{d}{d\Delta E} \int d\tfrac{(\varepsilon_1+\varepsilon_2)}{2} \, d(\varepsilon_1 - \varepsilon_2) \, dV \, P(\varepsilon_1, \varepsilon_2, V) \; \underbrace{\Theta\left(\Delta E - \sqrt{(\varepsilon_1-\varepsilon_2)^2 + |V|^2}\right)}_{\Theta\left(\Delta E^2 - ((\varepsilon_1-\varepsilon_2)^2 + |V|^2)\right)}$$

$$= \frac{d}{d\Delta E}\left[(\Delta E^2 \text{ or } \Delta E^3) \cdot \text{const.} \right]$$



$$\sim \begin{cases} \Delta E & \leftarrow \text{if } V \text{ has only real part} \\ (\Delta E)^2 & \leftarrow \text{if } V \text{ has real } \underline{\text{and}} \text{ imaginary part} \end{cases}$$

$$\boxed{\equiv \Delta E^{\beta}}$$

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To obtain a small level spacing one needs:

* $\varepsilon_1 - \varepsilon_2$ must be small
* all allowed off-diagonal matrix elements must be small

the more degrees of freedom one has ($\beta = 1, 2$) the stronger the level repulsion $(E_1 - E_2)^\beta$

Below we will see matrices where the entries are "real quaternions"

$$\tau_0 = 1\!\!1 \qquad \tau_{1,2,3} \equiv -i \cdot \sigma_{1,2,3} \leftarrow \text{Pauli matrices.}$$

$$\tau_i \tau_j = \epsilon_{ijk} \tau_k \qquad \tau_i^2 = -1$$

There $V = \sum_{i=0}^{3} \xi_i \tau_i$

ξ_i real! $\Rightarrow$ 4 degrees of freedom $\Rightarrow \beta = 4$

$$\rho(\Delta E) \sim \Delta E^4.$$

The role of time-reversal symmetry: Wigner Dyson symmetry classes

Generic systems have no symmetries, except perhaps time reversal symmetry

[If spin or parity is conserved $\Rightarrow$ only look at levels in one sector! RMT only applies within such a sector; no level repulsion between sectors!)

$\Rightarrow$ Understand how time reversal symmetry restricts the matrix H_{nm}!

($\Leftrightarrow$ meaning of β!)

Time reversal operator $\mathcal{T}$ is antiunitary ($\mathcal{T}[e^{-iEt}\psi_0] \to e^{iEt}\mathcal{T}\psi_0$)

$$\mathcal{T}(a\psi) = a^* \mathcal{T}\psi$$

and $\mathcal{T}^2 = \alpha \cdot 1\!\!1$ with $|\alpha| = 1$.

(reversing time twice should get one back to the same state (up to a phase)

$\Rightarrow$ anti-unitarity implies $\mathcal{T} = K \cdot C$

K: unitary matrix $K^{-1} = K^+$ $\quad$ C: complex conjugation

$$KK^+ = 1 \Leftrightarrow K^* K^T = 1\!\!1$$

$$\mathcal{T}^2 = \alpha 1\!\!1 \Leftrightarrow KCKC = \boxed{KK^* = \alpha 1\!\!1}$$

Multiply by $K^T \to K = \alpha K^T = \alpha(\alpha K^T)^T = \alpha^2 K \Rightarrow \alpha = \pm 1$!

Recall: spin 1/2 particle: $\mathcal{T} = \underbrace{i\sigma_2}_{\begin{pmatrix}0&1\\-1&0\end{pmatrix}} \cdot C$ $\quad \mathcal{T}^2 = -1$

general spin with angular momentum $J_{1,2,3}$: Choose representation J_1, J_3 real $K = e^{i\pi J_2}$ $\quad \mathcal{T}^2 = e^{2\pi i J_2} = \begin{cases} -1 & \text{half integer } J \\ +1 & \text{integer } J \end{cases}$

$\alpha = 1 \leftrightarrow$ integer spin (or spinless); $\alpha = -1$: half integer spin (but necessary to break rotation invariance $\Rightarrow$ spin-orbit coupling)

<u>TRS with $\alpha = 1$</u>

$\Rightarrow KK^* = 1 \Rightarrow$ one can always choose a basis for which $K = 1$

Namely: Construct a basis as follows:

- Start with some state ψ. If ψ and $\mathcal{T}\psi$ are linearly independent
- $\psi + \mathcal{T}\psi$ and $i(\psi - \mathcal{T}\psi)$ are both eigenvectors of $\mathcal{T}$

(else: $\psi = \pm \mathcal{T}\psi$ is an eigenvector of $\mathcal{T}$)

$\Rightarrow$ Go to the orthogonal complement and proceed. $\Rightarrow$ construct a basis of states for which $\mathcal{T}\psi_n = \psi_n$! $\Rightarrow$ In this basis $K = \mathbb{1}$.

Basis changes that preserve this property are real <u>orthogonal</u> matrices, $RR^T = \mathbb{1}$.

<u>TRS with $\alpha = -1$</u>

Here the above is <u>not</u> possible: in no basis one will have $K = \mathbb{1}$.

Indeed: ψ and $\mathcal{T}\psi$ are always linearly independent:

Proof: suppose $\psi = \beta\,\mathcal{T}\psi$ for some $\beta \in \mathbb{C}$.

$$\Rightarrow \mathcal{T}\psi = \beta^* \mathcal{T}^2 \psi = -\beta^* \psi \Rightarrow -|\beta|^2 = 1 \; \lightning$$

Therefore: Construct a basis as follows:

- Start with some state ψ.
- Add $\mathcal{T}\psi$ as further basis state. $\mathcal{T}$ acts like $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot C$ in this subspace!
- Go to orthogonal complement and continue as above

$\Rightarrow$ In such a basis $\mathcal{T} = K \cdot C$ with

$$Z \equiv K = \begin{pmatrix} \begin{pmatrix}0&-1\\1&0\end{pmatrix} & 0 & \cdots \\ 0 & \begin{pmatrix}0&-1\\1&0\end{pmatrix} & 0 \\ & & \ddots \end{pmatrix} \Big\} \; N \text{ blocks}$$

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Unitary basis changes that leave the matrix K (of $\mathcal{T} = K \cdot C$) invariant, that is $B \in U_{2N\times 2N}$ with

$$Z = BZB^T$$

$[\psi \to U\psi \;\Rightarrow\; \mathcal{T} \to U\mathcal{T}U^\dagger \to UKU^TC]$

form the N-dimensional symplectic group $Sp(N)$.

Transformation of Hamiltonian under time reversal

$\mathcal{T}$: takes a forward propagating wavefunction into a backward propagating one:

$$\psi^{TR}(t) := \mathcal{T}\psi(-t) = K\psi^*(-t)$$

What operator governs $\psi^{TR}(t)$'s evolution?

$$-i\partial_t \psi^{TR}(t) \equiv H^{TR}\psi^{TR}(t)$$

$$-i\partial_t K\psi^*(-t) = -iK\Big(\underbrace{-\partial_{t'}\psi^*(t'=-t)}_{\left[\frac{H\psi}{i}\right]^*(t'=-t)}\Big) = KH^*\psi^*(-t) = KH^*K^{-1}\psi^{TR}(t)$$

$$\Rightarrow \boxed{H^{TR} = KH^*K^{-1}}$$

Time reversal invariance $\Leftrightarrow H \overset{!}{=} H^{TR}$

$\alpha = 1$ (orthogonal ensemble): $K = \mathbb{1} \Rightarrow$ TRS requires $H^* = H$ (even spin) $\Rightarrow H$ has only real matrix elements.
$\Rightarrow \beta = 1$ degree of freedom per matrix entry.

$\alpha = -1$ (symplectic ensemble) $K = Z = \begin{pmatrix} \tau_2 & & \\ & \tau_2 & \\ & & \tau_2 \end{pmatrix}$ $[\tau_2 = -i\sigma_2]$

$\Rightarrow$ Work in this type of basis, write H in terms of $N\times N$ matrix with entries that are 2×2.

$$H = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1N} \\ & & & \\ Q_{N1} & Q_{N2} & \cdots & Q_{NN} \end{pmatrix} \qquad Q_{ij} = 2\times 2 \text{ matrix}$$

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Expand Q_{ij} in basis of quaternions: $1\!\!1, -i\sigma_{1,2,3} = \tau_{1,2,3}$
$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$$Q_{ij} = q^0_{ij} \cdot 1\!\!1 + \vec{q}_{ij} \cdot \vec{\tau}$$

$$\boxed{\begin{matrix} -\tau_2\tau_1\tau_2 = -\tau_1 & -\tau_2\tau_2\tau_2 = \tau_2 \\ -\tau_2\tau_3\tau_2 = -\tau_3 & -\tau_2\tau_2 = 1\!\!1 \end{matrix}}$$

$$Q^*_{ij} = q^{0*}_{ij}\, 1\!\!1 + q^*_{ij1}(-\tau_1) + q^*_{ij2}\,\tau_2 + q^*_{ij3}(-\tau_3)$$

$$= -\tau_2\left(q^{0*}_{ij}\, 1\!\!1 + q^*_{ij1}\tau_1 + q^*_{ij2}\,\tau_2 + q^*_{ij3}\,\tau_3\right)\tau_2$$

Condition for TRS: $Q^R_{ij} = \tau_2\, Q^*_{ij}(-\tau_2) \stackrel{!}{=} Q_{ij}$

$\Leftrightarrow q^0_{ij}, \vec{q}_{ij}$ must be real: $\Rightarrow$ "real quaternionic" matrix

$\Rightarrow$ 4 real degrees of freedom per offdiagonal block! $\Rightarrow \beta = 4$

<u>Kramers degeneracy</u> $(\alpha = -1)$

To understand better the level repulsion in the symplectic case let us understand first the emergence of two-fold degeneracy:

If ψ is an eigenstate of H, $H\psi = E\psi$,

then $K\psi^*$ is also an eigenstate with the <u>same</u> energy
$= \tau\psi$ but, as already shown, ψ and $\tau\psi$ are lin independent!

This follows from the fact that $\psi \cdot e^{-iEt}$ solves the Schrödinger equation and so does $(K\psi^*)\, e^{iE(-t)} = K\psi^*\, e^{-iEt}$

Or simply: $\underline{H K\psi^*} = KH^*\psi^* = K(H\psi)^* = K E\psi^* = \underline{E\, K\psi^*}$.

$\Rightarrow \alpha = -1$ implies a double degeneracy of <u>all</u> levels.

If one tries to understand level repulsion one should consider two doublets at energies E_1, E_2 with $|E_1 - E_2| \ll \delta$

To allow the two doublets to come close, all 4 parameters in the 2×2 matrix V_{12} that couples the two doublet spaces, must be small $\longrightarrow$ level repulsion by a factor $(\Delta E)^{\beta = 4}$!

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Summary:

- No TRS: Any orthogonal basis is equally good; matrix elements are complex. V_{12} has $\boxed{\beta=2}$ d.o.f.
  Needed: time reversal symmetry must be broken, e.g. by magnetic field or magnetic impurities $\Rightarrow$ basis changes: general unitary $\Rightarrow$ $\boxed{GUE}$ Gaussian unitary ensemble.
- TRS: Two cases (i) $J^2 = \mathbb{1}$ ($\alpha = 1$) : spinless particles, integer spin, (or conserved decoupled spin)
  $\Rightarrow$ Work in basis with $J\psi_n = \psi_n$ $\Rightarrow$ H is real.
  $\Rightarrow V_{12}$ has one d.o.f. $\Rightarrow \boxed{\beta=1}$.
  basis changes: orthogonal, real $\Rightarrow$ $\boxed{GOE}$ Gaussian orthogonal ensemble
  (ii) $J^2 = -\mathbb{1}$ ($\alpha = -1$) : half-integer spin that is not trivially conserved: spin-orbit coupling
  $\Rightarrow$ Work in basis of pairs $(\psi_n, J\psi_n)$
  $\Rightarrow$ H is real quaternionic
  V_{12} has $\boxed{\beta=4}$ d.o.f.
  Basis changes are symplectic $\boxed{GSE}$ Gaussian symplectic ensemble.

Even though TRS restricts the Hamiltonian (which is why $\beta=1$ is smaller!), in the case $J^2=-\mathbb{1}$ the degeneracy of the levels increases the possibilities how two levels (doublets!) can connect. $\beta=4$ is however only half of 8 which is the maximum degree of freedom a matrix Q_{12} could have.

Joint probability distribution of energies:

$$p(\{E_n\}) \propto \prod_{n>m} (E_n - E_m)^\beta \, e^{-a\sum_n E_n^2}$$

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Beyond Wigner-Dyson classification: The ten-fold way

One can extend symmetry classes by adding to Time reversal additionally particle hole symmetry (in superconductors). This requires extension to describe particles and holes simultaneously in a Bogoliubov-de-Gennes framework. Exchange of particles and holes is another anti-unitary operation C (charge conjugation), and $C^2 = \pm 1$.

One finds then 9 possibilities: TRS (no, yes $T^2 = 1$, $T^2 = -1$)

C (no, yes $C^2 = 1$, $C^2 = -1$)

But the case where C and TRS are separately no symmetries splits into two: $(C \cdot T)$, a unitary transformation, may be present or not. In total one then has 10 symmetry classes.

(Altland + Zirnbauer 1995)