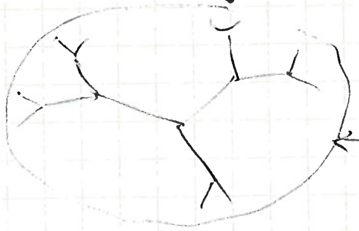
Cavity method

Originally for the SK model (→ interpret the replica solution without replicas!) ⟺ understanding the physics of the hierarchical block structure.

Cavity approach: much more direct, easier to grasp;
applicable to finite connectivity graphs ⇒ approximate scheme for finite d glasses!

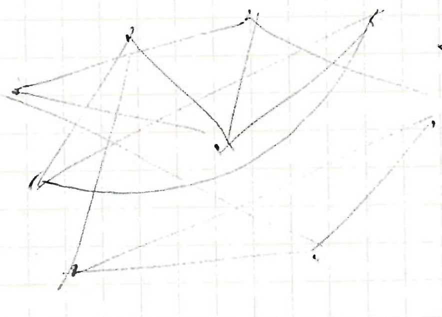
Key idea: 1. Graph = tree :



⟺ exactly solvable by iteration from outside in!

→ first step below!

2. Random graphs: no (free) boundary ⇒ frustration (possibly)
But: locally looks tree like:



← random graph of fixed connectivity $c = 3$

Important property: loops are large, typically $\sim \log(N_{\text{sites}})$

⇒ Portions of diameter $\ll \log(N_{\text{sites}})$ are typically loop-free
→ look like a tree!

Notations: i : sites ∂i : all neighbors linked with i
$\langle ij \rangle$: link from i to j $\partial i \setminus j$: ∂i without the neighbor j

Solution on a tree

Problems considered: We restrict here to two-body interactions (like spin glasses, or Potts models, etc.). Generalization to p-spin interactions are possible see notes on MF glasses.

Spins: $H = \underbrace{-\sum_{\langle ij \rangle} J_{ij} s_i s_j}_{\text{two body term}} \quad \big(\underbrace{+ \sum \varphi_i s_i}_{\text{sometimes added: one body term}}\big)$

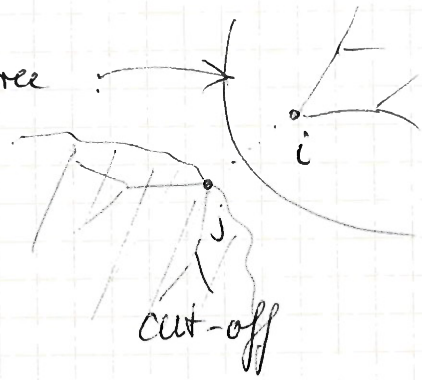
But: more generally:

$$\sum_{\langle ij \rangle} E(\sigma_i, \sigma_j)$$

σ_i: discrete variable with r different states.

e.g.: $E(\sigma_i,\sigma_j) = \delta(\sigma_i,\sigma_j) \Leftrightarrow$ Potts' model

Consider subtree:



Partition function restricted to subtree, with σ_i fixed.

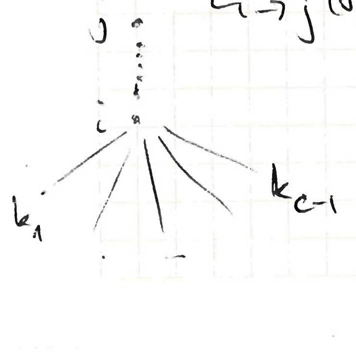
$$\equiv Z_{\underbrace{i \to j}_{\langle ij \rangle \text{ is eliminated}}}(\sigma_i)$$

$\Rightarrow$ Probability to see σ_i on the sub-tree: $\eta_{i \to j}(\sigma_i) \equiv \dfrac{Z_{i \to j}(\sigma_i)}{\sum_{\sigma_i'} Z_{i \to j}(\sigma_i')}$

on full tree: $\eta_i(\sigma_i) = \dfrac{Z_i(\sigma_i)}{\sum_{\sigma_i'} Z_i(\sigma_i)}$

Solution of the tree: via recursion for $Z_{i \to j}(\sigma_i)$:

$$Z_{i \to j}(\sigma_i) = \left(e^{\varphi(\sigma_i)}\right) \prod_{k \in \partial i \setminus j} \Big(\sum_{\sigma_k} Z_{k \to i}(\sigma_k) \underbrace{e^{-\beta E(\sigma_i, \sigma_k)}}_{\equiv \psi_{ik}(\sigma_i, \sigma_k)} \Big)$$



with initial conditions on the boundary $Z_{\substack{k \to j \\ k \in B}}(\varphi_k)$.

→ Partition function of full tree, restricted to σ_i on site i:

$$Z_i(\sigma_i) = \prod_{k\in\partial i}\left(\sum_{\sigma_k} Z_{k\to i}(\sigma_k)\,\psi_{ik}(\sigma_i,\sigma_k)\right)$$

↑ all now!

⇒ Recursion for probabilities:

$$\eta_{i\to j}(\sigma_i) = \frac{\left(e^{\varphi(\sigma_i)}\right)}{z_{i\to j}} \prod_{k\in\partial i\setminus j}\underbrace{\left(\sum_{\sigma_k}\eta_{k\to i}(\sigma_k)\,\psi_{ik}(\sigma_i,\sigma_k)\right)}$$

normalization: $z_{i\to j} = \sum_{\sigma_i}\left(e^{\varphi(\sigma_i)}\right)\prod_{k\in\partial i\setminus j}\left(\;''\;\right)$

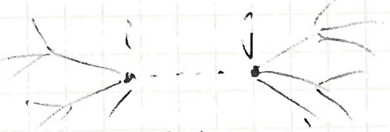
Full tree: $$\eta_i(\sigma_i) = \frac{Z_i(\sigma_i)}{\sum_{\sigma_i'} Z_i(\sigma_i')} = \frac{1}{z_i}\prod_{k\in\partial i}\underbrace{\left(\sum_{\sigma_k}\eta_{k\to i}(\sigma_k)\,\psi_{ik}(\sigma_i,\sigma_k)\right)}$$

with norm. $z_i = \sum_{\sigma_i}\left[\quad\right]$

Note: $\eta_i(\sigma_i)$ is the Gibbs probability to see σ_i

$$\Rightarrow \quad \langle\sigma_i\rangle = \sum_{\sigma_i}\sigma_i\,\eta_i(\sigma_i) \quad !$$

Free energy?



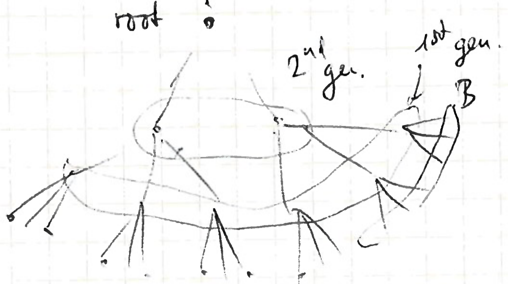
Define "link free energy" (energy gained upon linking together i and j)

$$\text{"}e^{-\beta f_{ij}}\text{"} = z_{ij} := \sum_{\sigma_i,\sigma_j}\eta_{j\to i}(\sigma_j)\,\eta_{i\to j}(\sigma_i)\,\psi_{ij}(\sigma_i,\sigma_j)$$

$$= \sum_{\sigma_i,\sigma_j} z_{j\to i}^{-1}\prod_{l\in\partial j\setminus i}\left(\sum_{\sigma_l}\eta_{l\to j}(\sigma_j)\,\psi_{lj}(\sigma_l,\sigma_j)\right)\eta_{i\to j}(\sigma_i)\,\psi_{ij}(\sigma_i,\sigma_j)$$

$$= \frac{1}{z_{j\to i}}\cdot z_j \quad \text{and similarly:} \quad = \frac{z_i}{z_{i\to j}}$$

$$Z_{\text{tree, tot}} = \Big(\prod_{k\in \text{Boundary } B} Z_{k\to j}\Big)_{\in 1} \cdot \prod_{j\in 1^{\text{st}}\text{ generation}} z_{j\to j'} \underset{\uparrow\ \in 2^{\text{nd}}\text{ gen.}}{} \prod_{j\in 2^{\text{nd}}\text{ gen.}} z_{j\to j'} \underset{\uparrow\ \in 3^{\text{rd}}\text{ gen.}}{} \cdots$$



$$\times \cdots \times \cdots \times\ z_{\text{root}\equiv 0}$$

$$= z_0 \cdot \prod_{j\in\partial 0} \underbrace{z_{j\to 0}}_{=\frac{z_j}{z_{j0}}} \prod_{k\in\partial j\setminus 0} z_{k\to j} \cdots \prod_{k\in \text{boundary}} \overset{z_{k\to k'}}{\underset{\equiv}{Z_k}}$$

$$= z_0 \prod_{j\in\partial 0} \frac{z_j}{z_{j0}} \prod_{k\in\partial j\setminus 0}\left(\frac{z_k}{z_{kj}}\right) \prod \cdots$$

$$\Rightarrow \boxed{Z_{\text{tree, tot}} = \prod_{i \text{ sites}} z_i \Big/ \prod_{\text{links } \langle ij\rangle} z_{ij}}$$

$$\Rightarrow F_{\text{tree}} = -T\log Z_{\text{tree, tot}} = \sum_i \underset{-T\log z_i}{\underset{\equiv}{f_i}} - \sum_{\langle ij\rangle} \underset{-T\log z_{ij}}{\underset{\equiv}{f_{ij}}}$$

Simplest case: • No disorder in H

• graph is regular (equal connectivities)

$\Rightarrow$ all sites are statistically equivalent $\Rightarrow$ expect $\eta_{i\to j}(\sigma_i)$ independent of i!

But more precisely: Graph: - take tree like neighborhood of site i;

- "leaves" of that tree are connected to the rest of the ~~tree~~ graph.

Assume a homogeneous $\eta(\sigma_k)$ on those leaves k.

$\Rightarrow$ Recursion from generation g to $g+1$: here: indep of link!

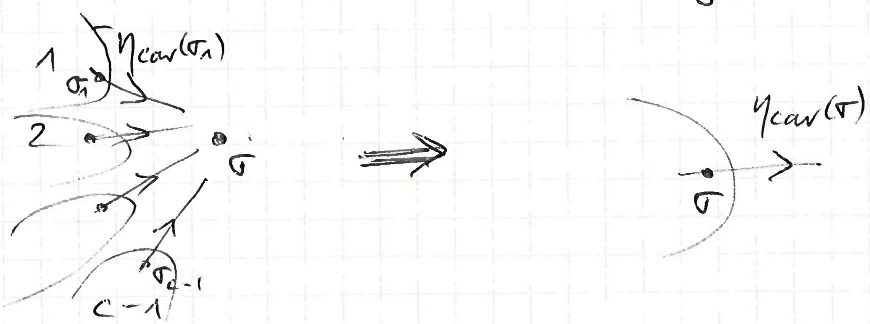
$$\eta_{g+1}(\sigma) = \frac{1}{z_g}\sum_{\sigma_1\ldots\sigma_{c-1}} \eta_g(\sigma_1)\cdots\eta_g(\sigma_{c-1})\,\psi(\sigma,\sigma_1)\cdots\psi(\sigma,\sigma_{c-1})$$

Iteration $\Rightarrow$ fixed point $\eta_g(\sigma) \xrightarrow[g\to\infty]{} \eta_{cav}(\sigma)$

where η_{cav} solves:

$$\eta_{cav}(\sigma) = \frac{1}{Z_{cav}} \sum_{\sigma_1,\dots,\sigma_{c-1}} \eta_{cav}(\sigma_1)\,\psi(\sigma,\sigma_1)\;\eta_{cav}(\sigma_2)\,\psi(\sigma,\sigma_2)\cdots \eta_{cav}(\sigma_{c-1})\,\psi(\sigma,\sigma_{c-1})$$

$$= \underbrace{\frac{1}{Z_{cav}}}_{\text{normalization ensuring } \sum_\sigma \eta_{cav}(\sigma)=1} \left(\sum_{\sigma'} \eta_{cav}(\sigma')\,\psi(\sigma,\sigma')\right)^{c-1}$$



messages / probabilities η on neighbors $\sigma_{1,\dots,c-1}$ propagate forward to σ

$\Rightarrow$ free energy ? Homogeneity $\Rightarrow$ $z_i = z_{site}$ for all i

$z_{ij} = z_{link}$ for all $\langle ij \rangle$

$$z_{site} = \sum_\sigma \left(\sum_{\sigma'} \eta_{cav}(\sigma')\,\psi(\sigma,\sigma')\right)^{c}$$

$$z_{link} = \sum_{\sigma_1\sigma_2} \eta_{cav}(\sigma_1)\,\eta_{cav}(\sigma_2)\,\psi(\sigma_1,\sigma_2)$$

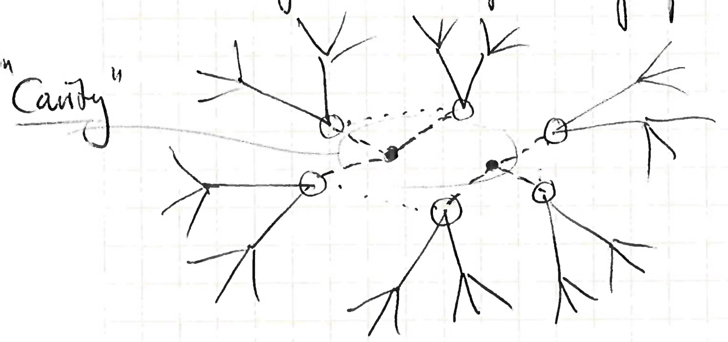
Free energy per site : $f = \frac{F}{N} = -T\left(\log z_{site} - \underbrace{\frac{\#\text{links}}{N}}_{\frac{c}{2}} z_{link}\right)$

"Cavity method" $\Leftrightarrow$ solve for $\eta_{cav}(\sigma)$ fixed point!

$\Rightarrow$ Compute f

Reason why $f = f_{site} - \frac{c}{2} f_{link}$?

Imagine incomplete graph:



$2 \cdot c$ sites ○ have only $c-1$ neighbors

$\Rightarrow$ Two ways to complete the graph:

I) add two sites • and connect them each to c of these sites (⋯⋯)

II) Connect the 2c sites pairwise

$\Rightarrow$ The difference in gained free energy is

$\Delta f^{(I)} = 2 \cdot f^{(site)}$ $\rightarrow$ ends up with two more sites in the graph $\Rightarrow F = F_{N+2} = F_{before} + \Delta f^{I}$

$\Delta f^{(II)} = c \cdot f^{(link)}$ $\Rightarrow$ same number of sites $\Rightarrow F = F_N = F_{befor} + \Delta f^{II}$

$\Rightarrow$ For large graphs: $f = \frac{F_N}{N} \rightarrow f = \frac{\Delta f^{I} - \Delta f^{II}}{2} = f^{site} - \frac{c}{2} f^{link}$

⚠ Assumption made:

1) The 2c sites are uncorrelated in the $N \to \infty$ limit

$\Leftrightarrow$ requires clustering property $\Leftrightarrow$ system must be in a pure state (e.g. high-T Gibbs state!)

2) The pure states of the system remain pure upon addition:

$\rightarrow$ local perturbations (from coupling an extra spin) should not propagate to infinity! That's reasonable

Disordered Hamiltonians, fluctuating connectivities etc

$\Rightarrow$ η_i will depend on i (non-equivalence of sites)

To compute disorder-averaged free energy: suffices to know the distribution of η_i (as one varies the disorder)

$$\Rightarrow \mathcal{P}[\eta_{cav}(\sigma)] \equiv \text{Probability}[\eta_{i\to j}(\sigma_i) = \eta_{cav}(\sigma_i)]$$

Self-consistency:

$$\eta_0(\sigma_0) = \frac{1}{Z_{cavity}} \sum_{\sigma_1 \cdots \sigma_{c-1}} \eta_{1\to 0}^{new}(\sigma_1) \cdots \eta_{c-1\to 0}(\sigma_{c-1})\, \psi_1(\sigma_0,\sigma_1) \cdots \psi_{c-1}(\sigma_0,\sigma_{c-1})$$

$$\equiv \mathcal{F}_J[\eta_{1\to 0}, \ldots, \eta_{c-1\to 0}]$$

disorder dependent functional

$$\to \mathcal{P}[\eta_{cav}] = \overline{\int d\mathcal{P}[\eta_{1\to 0}] \cdots d\mathcal{P}[\eta_{c-1\to 0}]\, \delta(\eta_{cav} - \mathcal{F}_J[\eta_{1\to 0}, \ldots, \eta_{c-1\to 0}])}^{\text{disorder}}$$

Example: SK model in a field! $\Rightarrow$ effective local fields depend on site i. $h_i = \sum_{j\neq i} u_{j\to i} + h_{ext}$.

$$\underbrace{\mathcal{P}[\eta_{cav}(s)]}_{\sim \exp(\beta h s)} \Leftrightarrow P(h)\ :$$ distribution over disorder and/or sites!

Illustration: Ising model: $H = -\sum_{\langle ij\rangle} J_{ij}\, s_i s_j$

$s_i = \pm 1$

$$\Rightarrow \quad \eta_{i\to j}(s_i) \equiv \frac{e^{\beta h_{i\to j} s_i}}{2\cosh(\beta h_{i\to j})} \quad \Leftarrow \text{ parametrization of } \eta \text{ via the field } h_{i\to j}\,!$$

Check: · $\eta_{i\to j}$ is normalized

· since $\eta_{i\to j}(s_i=+1) + \eta_{i\to j}(s_i=-1) = 1$ and only two values of η need to be fixed, one parameter suffices!

namely: $h_{i\to j} = \frac{1}{2\beta} \log \frac{\eta_{i\to j}(1)}{\eta_{i\to j}(-1)}$

Note: $\cancel{\eta}\; e^{\beta h \overset{\pm 1}{\overset{\shortparallel}{s}}} = \cosh(\beta h)\cdot\left[1 + s\tanh(\beta h)\right] \quad \Leftarrow \quad e^{\pm x} = \cosh x \pm \sinh x$

$\Rightarrow$ Cavity recursion:

$$\eta_{i\to j}(s_i) = \frac{1}{z_{i\to j}} \prod_{k\in\partial i\setminus j} \Bigg(\sum_{s_k=\pm 1} \underbrace{\eta_{k\to i}(s_k)}_{\frac{1}{2}(1+s_k\tanh(\beta h_{k\to i}))} \underbrace{e^{\beta J_{ki} s_i s_k}}_{\cosh\beta J_{ki}\,(1+s_i s_k \tanh(\beta J_{ki}))} \Bigg)$$

only even powers of s_k survive

$$= \frac{1}{z_{i\to j}} \prod_{k\in\partial i\setminus j} \frac{\cosh(\beta J_{ki})}{\cancel{2}} \underbrace{\left[1 + s_i \tanh(\beta h_{k\to i})\cdot\tanh(\beta J_{ki})\right]}_{\equiv\; 1 + s_i \tanh(\beta u_{k\to j})}$$

$$\boxed{\text{with "message" } \; u_{k\to j} = \frac{1}{\beta}\,\operatorname{atanh}\left[\tanh(\beta h_{k\to i})\tanh(\beta J_{ki})\right]}$$

$$= \frac{\exp(\beta h_{i\to j} s_i)}{2\cosh(\beta h_{i\to j})} = \frac{1}{z_{i\to j}} \prod_{k\in\partial i\setminus j} \cosh(\beta J_{ki}) \frac{e^{\beta s_i u_{k\to i}}}{\cosh(\beta u_{k\to i})} \quad \Rightarrow \quad h_{i\to j} = \sum_{k\in\partial i\setminus j} u_{k\to i}$$

$Z_{i\to j}$ can be computed easily, too.

$\Rightarrow$ The "cavity field" $h_{i\to j}$ is the sum of messages $u_{k\to i}$ or "beliefs" from its neighbors down in the tree!

$\Leftrightarrow$ Belief propagation in combinatorial / Boolean optimization:

Belief $u_{i\to j}$ becomes stronger as $h_{i\to j}$ or J_{ij} increase!

and $u\to 0$ as h or $J\to 0$.

Interesting exercises:

I. Apply the homogeneous cavity method to a Ising! ferromagnet.

$\Rightarrow$ What is T_c? What asymptotics do you find for large connectivity? $c\to\infty$

II. Apply to SK model:

Derive TAP equations from the cavity equations!

Note: $m_0 = \tanh\big(\beta \sum_{j\in\partial_0} u_{j\to 0}\big)$

all neighbors

Express $u_{j\to 0}$ in terms of $h_{j\to 0}$, using that $J_{ij}\sim\frac{1}{\sqrt{N}}$ is small!

$\Rightarrow$ Express "total field" on j: $\sum_{k\in\partial j} u_{k\to j} = \underbrace{h_{j\to 0}}_{\sum_{k\in\partial j\setminus 0} u_{k\to j}} + u_{0\to j}$

$\Rightarrow$ Find TAP Equation: $m_0 = \tanh\Big(\beta\Big(\sum_{j\in\partial_0} J_{0j}\, m_j - m_0 J_{0j}^2\, \beta(1-m_j^2)\Big)\Big)$