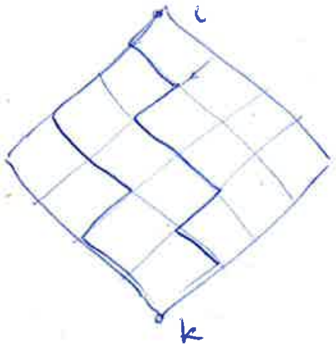
Asymptotic decay of localized wavefunctions

DPRM mapping of the locator expansion: also useful in finite dimensions!

Idea: wavefunction amplitude at large distance is dominated by the shortest hopping paths! $\Rightarrow$ Directed paths! "Forward scattering approximation"

center of the wavefunction



$$G_{ik}(E) = t^{\mathrm{dist}(i,k)} \cdot \underbrace{\#}_{\text{sum over shortest paths}} + o(t^{\mathrm{dist}})$$

$\Downarrow$

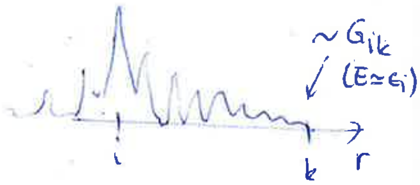
Interference!

Each path γ contributes: $\gamma = \{\overset{i}{\overset{\shortparallel}{p_1}}, \dots, p_\ell = k\}$:

$$G_{ik} = \sum_{\substack{\gamma \\ \text{shortest paths}}} G_{ik}^{(\gamma)} + o(t^{d(i,k)})$$

$\Leftarrow$ measure for wavefunction amplitude at k

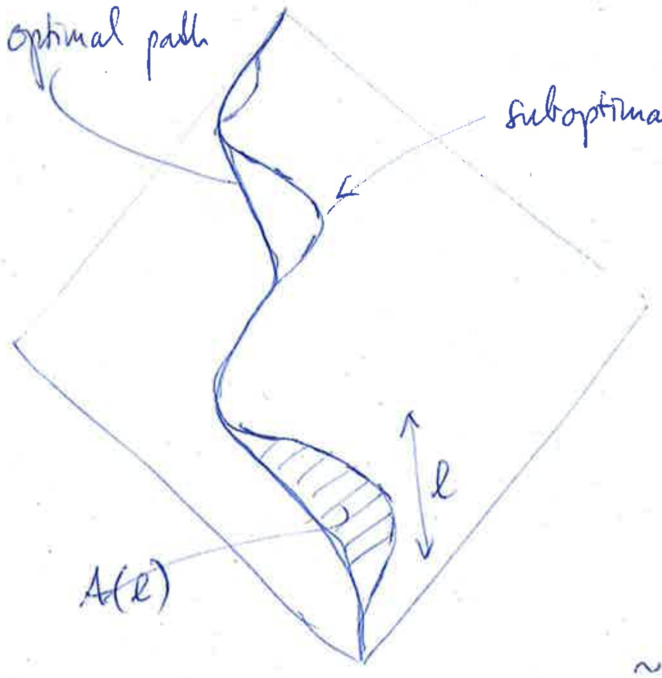
$$G_{ik}^{(\gamma)}(E) = \left[\prod_{m=1}^{\ell-1} \left(\underbrace{G_{0,p_m p_m}}_{\frac{1}{E-\epsilon_{p_m}}} t_{p_m p_{m+1}}\right)\right] G_{0 p_\ell p_\ell}$$



$\Rightarrow$ Can be also generalized to interacting problems, e.g. hardcore bosons $\Rightarrow$ interesting statistics effect: for bosons, at $E=0$ all amplitudes $G_{ik}^{(\gamma)}$ have the same sign! $\Rightarrow$ bosons delocalize more easily! $\Rightarrow$ Reason why Bose superfluids can exist in 2d (no general localization of bosons; need threshold disorder!)

Sum Random disordered, directed paths $\Rightarrow$ KPZ scaling?! Relevance?

$\Rightarrow$ Can be probed by magnetoresistance: Wavefunction decay in magnetic field?



Significant interference if partial weight along dominant and subdominant path are comparable!

DPRM: $\dfrac{\text{weight}_{\text{sub dom}}}{\text{weight}_{\text{dom}}} \simeq e^{-\beta \overbrace{\Delta F(\ell)}^{\sim \ell^{\theta}}}$ $\quad \theta = 1/3 \quad$ (KPZ)!

Interference: Prob $(\beta \Delta F(\ell) = O(1)) \sim \dfrac{1}{\ell^{\theta}}$

Effect of magnetic field: Disturbs interference!

Significant effect only if flux through path-bubble is $O(\phi_0)$ (ϕ_0 = flux quantum)

$\Longleftrightarrow A(\ell) \gtrsim \phi_0/B \qquad A(\ell) = ? \overset{KPZ}{\sim} \ell \cdot \ell^{\zeta} = \ell^{5/3}$! $\quad$ ← rough paths!

Interference: wavefunction decay affected by factor $O(1-2)$ by all bubbles with $A(\ell) \gtrsim \phi_0/B$, where interference is substantial!

$\hookrightarrow \ell_B \sim B^{-\frac{1}{1+\zeta}}$

Change in wavefunction decay: $\Delta\left(\dfrac{R}{\xi}\right) \propto \dfrac{R}{\ell_B} \cdot \dfrac{1}{\ell_B^{\theta}}$ ← probability to see one.

per ℓ_B length look for potential bubble

↑ loc. length

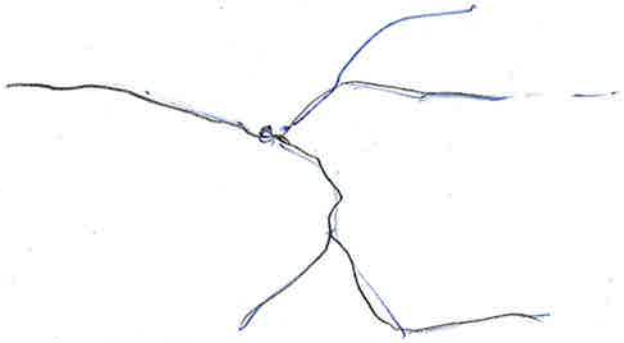
$\sim B^{\frac{1+\theta}{1+\zeta}} = {}^{4/5}$ ← non-trivial KPZ exponent!

Fermions: most relevant bubbles: have negative interference! Killed by flux $\Rightarrow \xi \uparrow$

Bosons: positive interference is killed by $B \to \xi$ shortened! $\xi \downarrow$

Multifractality of critical wavefunctions

localized wavefunctions are following optimal, finger-like paths



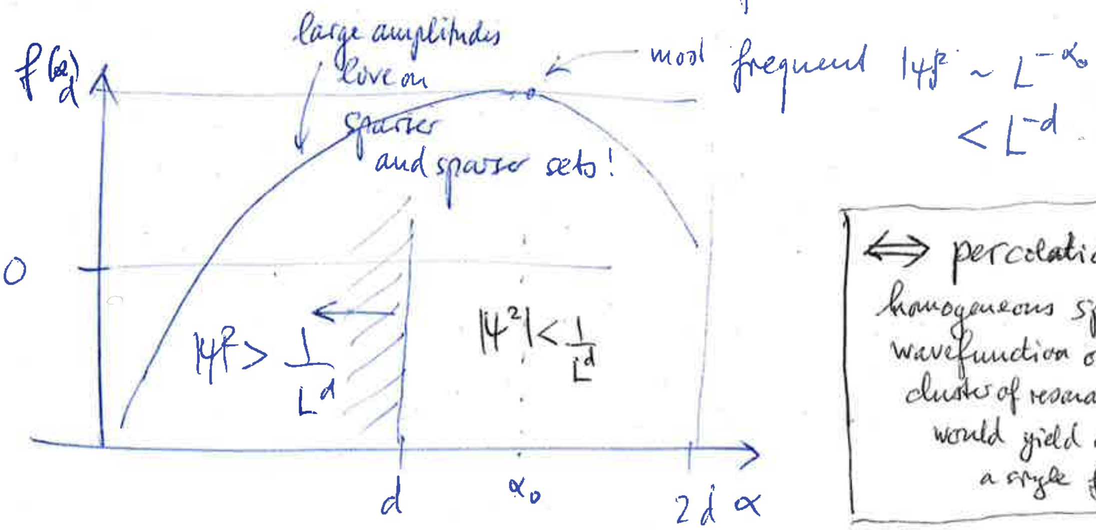
What about critical wavefunctions (at $W = W_c$) ?

$\Rightarrow$ Very sparse still; fractal support (like percolation clusters, but yet more intricate!)

Let ψ_i be a critical wavefunction, and take $V \to \infty$. $V = L^d$

Ask where $|\psi_i|^2 \sim L^{-\alpha}$ $\Rightarrow$ # of points: $N_\alpha \propto L^{f(\alpha)}$ $f(\alpha) \leq d$ fractal dimension

$f(\alpha)$ depends on α and is a convex function!



$\Leftrightarrow$ percolation: homogeneous spreading of wavefunction on percolating clusters of resonances would yield a single α and a single $f = \alpha$!

Often used to compute: $\sum_i |\psi_i|^{2q} \sim L^{-\tau(q)} \sim \int d\alpha \; L^{f(\alpha) - \alpha q}$

$$\left.\begin{array}{l} \Rightarrow -\tau(q) = f(\alpha) - \alpha q \quad \text{and} \quad f'(\alpha) = q \\ \Leftrightarrow f(\alpha) = -\tau(q) + \alpha q \quad \text{and} \quad \tau'(q) = \alpha \end{array}\right\}$$ $f(\alpha)$ and $\tau(q)$ are Legendre transform pair

Localization in the presence of interactions?

⇔ many-body localization? (Fleischmann + Anderson 1982,
MBL Basko-Aleiner-Altshuler 2006
Gornyi-Mirlin-Polyakov 2005
Onsager Prize '22 Aleiner-Altshuler-Huse)

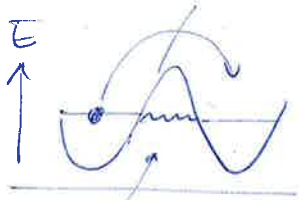
Counterintuitive: many-body systems are usually supposed to thermalize

MBL ⇔ breakdown of ergodicity! How can this be?

Reason we believe in ergodicity?

routes to non-ergodicity:

A) few particles, classical



barriers block ergodicity

↳ B) overcome by quantum tunneling

↳ C) Does not work if Anderson localization sets in: energy mismatch > tunnel amplitude

⇒ no resonance ⇒ no hybridization ⇒ localization

⇒ ubiquitous for single particle problems

but with many particles:

1. D): finite barriers can be jumped by borrowing energy from other particles (except spont. symmetry breaking: barrier→∞)

2. E): tunneling: energy mismatch can be bridged by giving/taking energy from "environment" (= other particles)

BUT: Loophole in D) and E): What if the system itself is not an environment like a thermal bath, but is localized itself?

⇒ energy cannot be extracted in continuous quanta from the environment ⇒ energy mismatch may persist!

only discrete energy available!
⇒ No good bath!

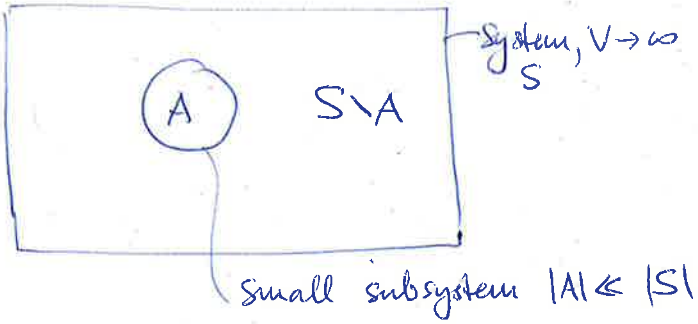
⚠ Usually: phonon system couples to spins or electrons. Phonons cannot localize (Goldstone modes!) $\Rightarrow$ MBL can only exist for an idealized system <u>with no phonons</u>. (Pure spin models; electronic degrees of freedom weakly coupled to the atomic lattice, cold atoms in <u>optical</u> lattices)

MBL $\Leftrightarrow$ the system fails to be its own bath $\Rightarrow$

transport, diffusion, spin flip processes etc. cannot take place because they <u>cannot conserve energy</u>!

MB-Delocalization: Interactions become strong enough to restore decay processes, ergodicity etc. $\Rightarrow$ <u>system becomes its own bath</u>, with a continuous spectrum of excitations.

Strong belief $\Rightarrow$ generically this leads to <u>thermalization</u>:



under unitary dynamics $\psi \to e^{-iHt}\psi$; $\rho = |\psi\rangle\langle\psi| \to \rho(t)$

Look at subsystem: $\rho_A(t) := \mathrm{Tr}_{S\setminus A}\, \rho(t)$

<u>Thermalization:</u> $\rho_A(t) \xrightarrow[t\to\infty]{} \rho_A^{eq}(T) = \mathrm{Tr}_{S\setminus A}\left(\frac{e^{-H/k_BT}}{Z}\right)$

~~Eigen~~ Independently of the initial state!

What if the system is prepared in an eigenstate? $H\psi_E = E\psi_E \to \rho(t) = \rho(0)$ is constant!

$\Rightarrow$ Eigenstate thermalization hypothesis:

(ETH): $\mathrm{Tr}_{S\setminus A}\left(|\psi_E\rangle\langle\psi_E|\right) = \mathrm{Tr}_{S\setminus A}\left(\frac{e^{-H/k_BT_E}}{Z}\right)$ — determined by initial energy; holds for generic eigenstates and H!

ETH $\Leftrightarrow$ $\langle \psi_\alpha | \hat{O} | \psi_\alpha \rangle$ only depends on E_α, but not on eigenstate ψ_α!

Reason: Every eigenstate hybridizes all possible micro configuration of the same total energy.

$\Leftrightarrow$ MBL: This does not hold anymore $\langle \psi_\alpha | \hat{O} | \psi_\alpha \rangle$ does depend on α.

Thermalization is halted.

How does this happen??

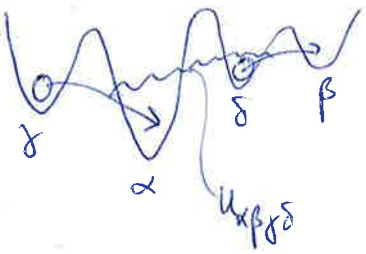
Essence: Anderson's decay to infinity breaks down! Here: perturbation theory in interactions!

Consider Fermi's Golden rule: (F.G.R.)

e.g. weakly interacting electrons

$$H = \sum_\alpha \epsilon_\alpha n_\alpha + \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta} c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta$$

basis of localized single particle states



transition matrix element: $U_{\alpha\beta\gamma\delta}$

$\Leftrightarrow$ energy mismatch:

$\Delta E = \epsilon_\alpha - \epsilon_\gamma + \epsilon_\beta - \epsilon_\delta$ $\Leftarrow$ spacing between reachable final states

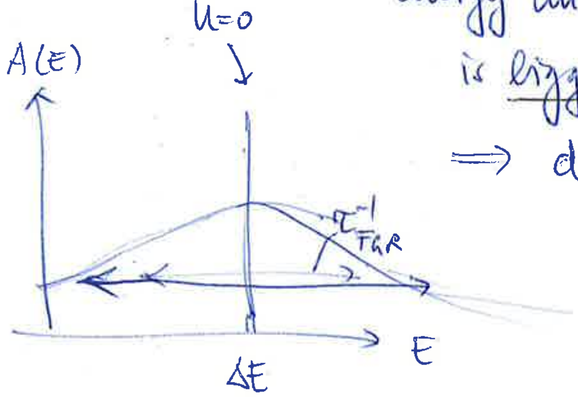
Resonance $\Leftrightarrow$ $U_{\alpha\beta\gamma\delta} > \Delta E$ $\Rightarrow$ hybridization is strong

Fermi-Golden rule $\quad \tau_{FGR}^{-1} \sim U_{\alpha\beta\gamma\delta}^2 \cdot \underbrace{\rho_{DOS}}_{\sim \frac{1}{\Delta E}} \sim \frac{U_{\alpha\beta\gamma\delta}^2}{\Delta E} \overset{!}{>} \Delta E$

$\Rightarrow$ energy uncertainty due to (assumed) decay processes is bigger than initial discreteness at the level of ΔE

$\Rightarrow$ decay produces a good continuum.

$\Rightarrow$ decay is self-consistent!



$\Leftrightarrow$ Weak interactions / strong disorder:

τ^{-1}_{FGR} falls below typical ΔE $\Rightarrow$ F.G.R. is not self-consistent and breaks down!

$\Rightarrow$ no decay, only virtual admixtures due to non-resonant U's!

"Standard model" of MBL: random spinchain

$$H = \sum_i h_i s_i^z + \sum_{\langle i,j \rangle} J_{ij} \vec{s}_i \vec{s}_j$$ Quantum spins $s = 1/2$.

strong randomness $h_i \in [-W, W]$ $J_{ij} \equiv 1$

Expect MBL for $W \gtrsim O(10)$ in $d = 1$. – Nearly rigorous proof by J. Imbrie.
(Resonance counting, F.G.R. breaking down)

numerics: level statistics
entanglement entropy
⋮
} ⇐ ⚠ turn out to be heavily affected by finite size effects. Treacherous.

True W_c is > 20!

So far: no thorough understanding of the large W_c!
Open problem!

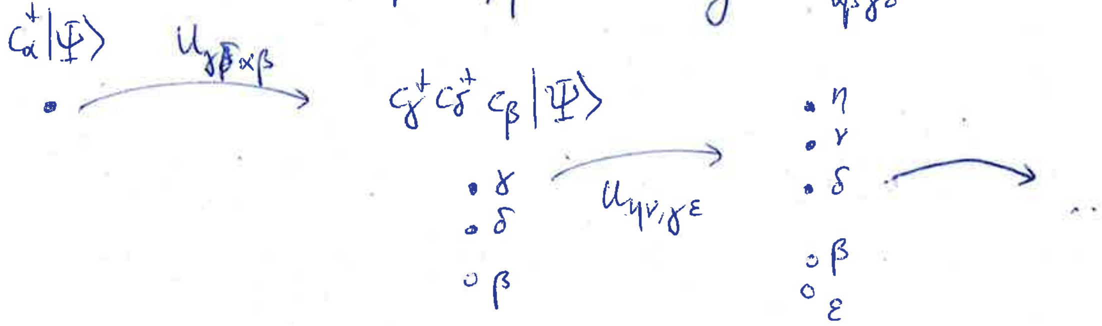
More interesting setting: Original model of weakly interacting, disordered electrons

Basko et al. 2006; Anderson-Fleischmann-Licciardello '80s:

$$H = \sum_\alpha n_\alpha \epsilon_\alpha + \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

$U_{\alpha\beta\gamma\delta}$: only non-negligible where wavefunctions $\psi_\alpha, \psi_\beta, \psi_\gamma, \psi_\delta$ overlap in space

Idea: Start from a supposed eigenstate (many body) Ψ. (highly excited eigenstate!)
Now make a local excitation $c_\alpha^\dagger |\Psi\rangle$ and consider its dynamics in Hilbert space, perturbatively in $U_{\alpha\beta\gamma\delta}$!

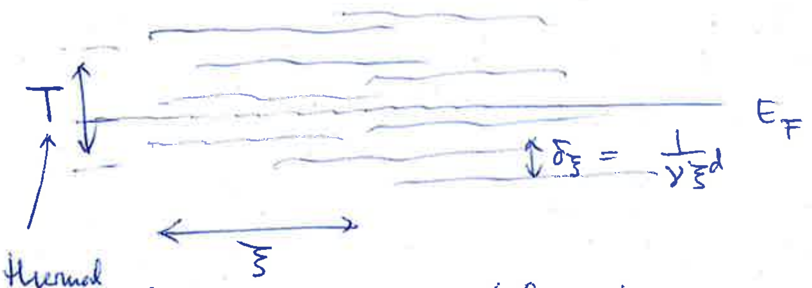


⇒ Analyze "locator expansion": hopping in phase space: $U_{\alpha\beta\gamma\delta}$
disorder: many body energies $\sum_\alpha \epsilon_\alpha n_\alpha$

Try to argue à la Anderson that perturbation theory converges for small U!

Parametrically controllable regime:

single particle states localized, with relatively large ξ!



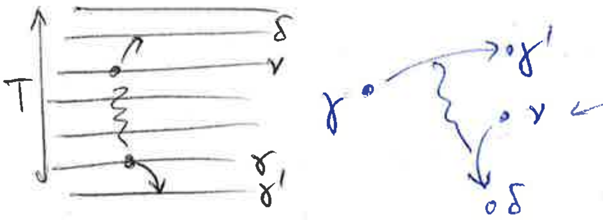
thermal window of states with $0 < n_\alpha < 1$ (if equilibrium were attained)

> Q: $\Rightarrow$ Where in the space of U, ξ does one expect a MBL transition (or crossover)?

$\Rightarrow$ Within a localization volume: $\boxed{N_T = T/\delta_\xi}$ are available for particle-hole scattering.

Steps of argument

1) $\Rightarrow$ Each particle (e.g. γ) has N_T partners to choose from:



N_T choices for ν ; $U_{\gamma'\nu\,\gamma\delta}$ favors $\gamma\gamma'$ and $\nu\delta$ to be neighbors in energy (matrix element is not cancelled by random fluctuations of $\psi_\gamma^* \psi_{\gamma'}$)

2) As long as the cascade of p-h-excitations starting from different particles do not interfere (in space), they ~~must~~ should be analyzed separately:

$\Rightarrow$ the relevant connectivity in phase space is N_T; it does not grow indefinitely with the number of excited particles! how many possible/relevant states can be reached due to one additional U?

$\Rightarrow$ approximate map to a Cayley tree in Hilbertspace:

connectivity: $K \equiv N_T$ hopping: $t = U_{\gamma\gamma', \nu\nu'} \Rightarrow U_{typ}$

Disorder = typical energy spread $\Rightarrow W \simeq \delta_\xi$

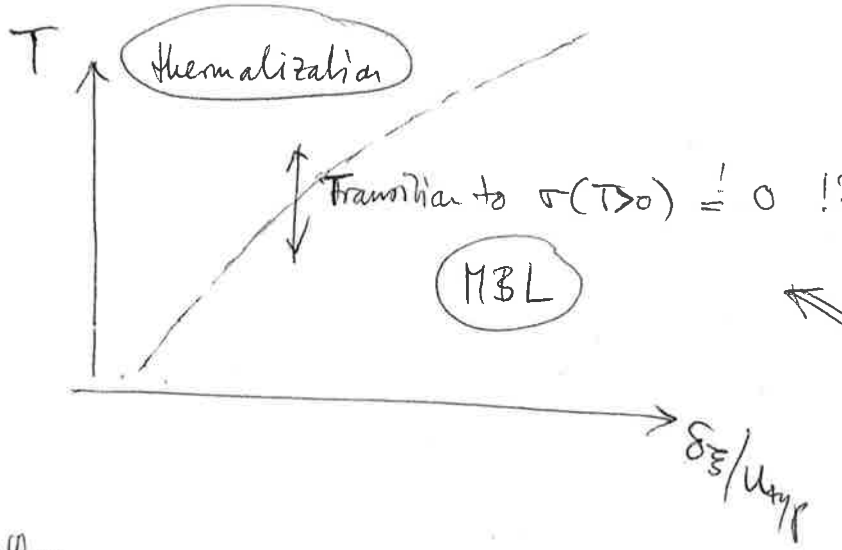
T-dependent!

Cayley tree result: No decay to infinity for $\frac{\delta_\xi}{U_{typ}} \left(= \frac{W}{t}\right) \gtrsim 2e\, K \log K = 2e N_T \log(N_T)$

→ Spectacular prediction:

Temperature tunes connectivity $K \simeq T/\delta_\xi$!

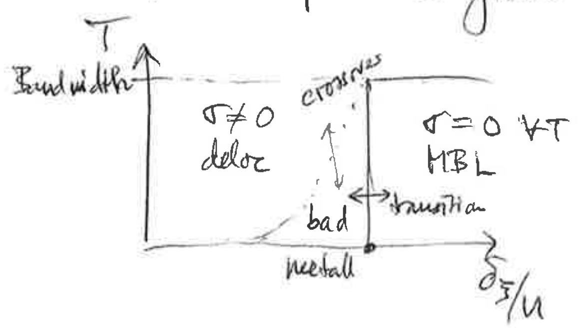
$\Rightarrow T\downarrow \Rightarrow$ MBL when $T/\delta_\xi \log T/\delta_\xi \lesssim \frac{\delta_\xi}{U_{typ}}$!?



Rather: crossover to bad metall

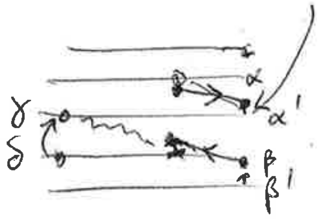
⇔

Belief: true phase diagram:



Why:
Several problems with approximation:

i) not necessary that particle γ participates in the next hop!
it suffices that the presence of γ changes the energies of other states and creates a new resonance



$\Rightarrow N_T$ should be replaced by N_T^2
(choice of two possible particles $\alpha, \beta \to \alpha', \beta'$!)

$$\Rightarrow \left(\frac{\delta_\xi}{U_{typ}}\right)_c \simeq N_T^2 \log (N_T^2)$$

→ parametrically different T_c!

ii) if perturbation theory diverges for typical diagrams at high $T > T^*$ then there are divergent subsets of diagrams at any low T as well!

$\Rightarrow$ a finite T transition is not possible!

BUT: a strong crossover, as estimated above, survives of course

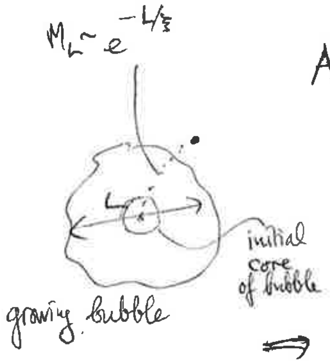
iii) role of dimension and rare events:

"Ergodic bubbles" (regions of low disorder) can destroy MBL completely: Consider a bubble of diameter L

The many-body level spacing scales as $\exp(-\# L^d) \equiv \delta_L$

Any degree of freedom outside is coupled to transitions in the bubble by matrix elements that decay at most as

$$M_L \sim \exp(-L)$$



$\Rightarrow$ in $d>1$ $\quad M_L \gg \delta_L$ for L large enough

$\Rightarrow$ an ergodic bubble hybridizes with all degrees of freedom outside and grows bigger; invading and thermalizing the whole sample. ($\Rightarrow$ "quantum avalanches")

$\Leftrightarrow$ 1d: Imbrie's proof of MBL contains the growth of such bubbles. The MBL transition is believed to be driven by the explosion of bubbles.

Local integrals of motion (LIOM's)

If nothing diffuses: are there lots of conserved quantities?

Usually only: E_{tot}, N_{tot}, maybe S_{tot}.

$\Leftrightarrow$ MBL: Extensively many conserved, local quantities! For any disorder realization!

Anderson (non-interacting): $H = \sum_i \epsilon_i c_i^+ c_i - \sum_{ij} t_{ij} c_i^+ c_j + h.c.$

$$\Rightarrow \text{Find } \psi_\alpha : \; H\psi_\alpha = E_\alpha \psi_\alpha \;\Rightarrow\; c_\alpha^+ := \sum_i \psi_{\alpha,i}\, c_i^+$$

$$\Rightarrow n_\alpha = c_\alpha^+ c_\alpha \text{ is conserved: } [H, n_\alpha] = 0$$

and local! $\quad [n_\alpha, n_\beta] = 0$

Interacting generalisation of n_α can be perturbatively constructed, or numerically found. In terms of those the Hamiltonian must take the form

$$H = \sum_\alpha \epsilon_\alpha n_\alpha + \sum_{\alpha,\beta} J_{\alpha\beta} n_\alpha n_\beta + \sum_{\alpha\beta\gamma} K_{\alpha\beta\gamma} n_\alpha n_\beta n_\gamma + \ldots$$

$\leftarrow$ n_α: like localized quasiparticles ($\Leftrightarrow$ Fermi liquid, but localized)

Summary of MBL

- A remarkable breakdown of ergodicity in a disordered many particle system

- MBL (standard version): Anderson-like localization <u>in spite of</u> weak interactions:

  The system cannot be its own bath!

  <u>Interesting question:</u> Can strong classical (commuting) interactions generate disorder intrinsically, so as to localize the dynamics?

  Answer: Presumably not fully, but crossovers as for standard MBL in $d>1$ are perfectly possible.

- Genuine transition believed to occur only at <u>all</u> T or none.
  - only in 1d avalanches can be prevented,
  - only in lattice models (In the continuum high energy excitations delocalize!)

- The only kind of model where extensively many conserved quantities exist <u>without</u> finetuning of the Hamiltonian
  ($\Leftrightarrow$ unlike integrable systems)

- <u>Use of MBL?</u> The degrees of freedom n_α that are conserved (LIOMs) do not decay $\Rightarrow$ noise (e.g. charge noise) is suppressed $\Rightarrow$ good for coherence of other objects

  but: Interactions like $J_{\alpha\beta} n_\alpha n_\beta$ still mean that they weakly entangle with each other $\Rightarrow$ An MBL system is not a good quantum memory!