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Self-consistent theory of localization: II. Localization near the band edges

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Abstract. The solution of the integral equation which arises in the self-consistent theory of localization has been explored for a Cauchy distribution of site energies, for a uniform distribution, and for a binary alloy, with particular attention to the behaviour of the localization edge in the case of weak disorder, when it is close to the unperturbed band edge. The solution could be calculated because an exact solution is found in the limit of zero disorder, and because the Fredholm expansion truncated after two terms gives a good approximation in many cases. For the rectangular distribution at the band edge the results are similar to those obtained by other workers, but for the Cauchy distribution and the binary alloy localization occurs less easily than other theories predict.

1. Introduction

In a recent paper by Abou-Chacra *et al* (1973), which we refer to as I, a new approach to the problem of localization of electrons in disordered systems was presented. This approach was based on a self-consistent approximation, and was examined in some detail for the centre of the band, where it was shown to give results very close to those of Anderson (1958). It was pointed out that although the method gives only a somewhat crude approximation for a real lattice, it is exact for the infinite Cayley tree (Bethe lattice). In this paper we apply the same theory to the edge of the band, which was not considered in Anderson's (1958) work.

The method is based on a study of the equation for the self-energy

$$S_i = \sum_j \frac{|V_{ij}|^2}{(E - \epsilon_j - S_j)} \tag{1.1}$$

where the site energies ϵ_j are distributed randomly with some probability density $p(\epsilon)$. The matrix elements V_{ij} are taken to be equal to a constant V between neighbouring sites and zero otherwise, but they could also be taken to be random without difficulty, and the sum goes over K neighbouring sites, where K is a little less than one less than the number of neighbours. The self-consistency problem is to find a probability distribution for S_j which, when used on the right of this equation, will generate the same probability distribution for S_i. For localized states the imaginary part of S_i tends to zero as the imaginary part of E tends to zero, and we found that the limit of stability of the localized states is given by the existence of a solution $A(x)$ of the homogeneous integral equation

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$$A(y) = KV\int Q\left(y + \frac{V^2}{x}\right)\frac{A(x)}{|x|}\,\mathrm{d}x \tag{1.2}$$

where $Q(x)$ is the probability density of the quantity

$$x = E - \epsilon_i - \sum_j{}' \frac{V^2}{(E - \epsilon_j - E_j)} \tag{1.3}$$

where E is real and E_j is the real part of S_j; the sum in this expression goes over only $K - 1$ terms. The eigenfunction $A(x)$ is everywhere positive. If the probability density $p(\epsilon)$ depends on some parameter, W say, we can plot the value of W which gives a solution of equation (1.2) as a function of E, and we will get a plot of the mobility edge which separates localized from nonlocalized states.

It is an important feature of equation (1.2) that it can have a square integrable solution $A(x)$ only if K is greater than or equal to unity. With a change of variable we have from equation (1.2)

$$\begin{aligned}\int A(y)^2\,\mathrm{d}y &= K^2V^2\iiint Q(u)\,A\left(\frac{V^2}{u-y}\right)Q(v)A\left(\frac{V^2}{v-y}\right)\frac{\mathrm{d}u\,\mathrm{d}v}{|u-y|\,|v-y|}\,\mathrm{d}y\\ &\leqslant K^2V^2\iint Q(u)\,Q(v)\left\{\int A\left(\frac{V^2}{u-y}\right)^2\frac{\mathrm{d}y}{(u-y)^2}\int A\left(\frac{V^2}{v-z}\right)^2\frac{\mathrm{d}z}{(v-z)^2}\right\}^{1/2}\mathrm{d}u\,\mathrm{d}v\\ &= K^2\int A(y)^2\,\mathrm{d}y.\end{aligned} \tag{1.4}$$

since the integral of Q is unity. The Schwartz inequality gives equality only if Q is a δ-function so that there is no disorder. In this case the solution $A(x)$ is proportional to $(x^2 - Ex + V^2)^{-1/2}$ for $|E|$ less than $2V$. This shows once again that the mobility edge for the linear chain, which has $K = 1$, is at zero disorder.

In §2 we show how an exact solution of this problem can be found in the limit of vanishing disorder; the mobility edge should coincide with the band edge in this limit. In §3 the use of a truncated Fredholm expansion for solving equation (1.2) is demonstrated. In §4 the problem of the Cauchy distribution of site energies is solved, and in §5 a narrow rectangular distribution of site energies is discussed; these two distributions are examples of two different types of behaviour that are possible at the band edge. In §6 the binary alloy type of disorder is considered.

2. Limit of zero disorder

The degree of disorder of the system is determined by the width of the probability density $p(\epsilon)$, and an exact condition for a solution of equation (1.2) can be found in the limit of zero disorder, in which $p(\epsilon)$ is a delta function. In this limit the value of S_i determined by equation (1.1) is, for positive E

$$\mu = \tfrac{1}{2}E - \tfrac{1}{2}(E^2 - 4KV^2)^{1/2}; \tag{2.1}$$

at the bottom of the band the negative sign must be taken for the square root. We therefore have, from equation (1.3),

$$\begin{aligned}&Q(x) = \delta(x - a),\\ &a = E - (1 - K^{-1})\mu.\end{aligned} \tag{2.2}$$

Equation (1.2) therefore gives

$$A(y) = \frac{KV}{|a-y|} A\left(\frac{V^2}{a-y}\right). \tag{2.3}$$

This equation has a delta function type of solution, of the form

$$A(y) = \delta(y - y_0) = \frac{KV}{|a-y|}\delta\left(\frac{V^2}{a-y} - y_0\right) \tag{2.4}$$

$$= \frac{K|a-y|}{V}\delta\left(y - a + \frac{V^2}{y_0}\right),$$

which is satisfied by

$$y_0 = a - V^2/y_0,$$

$$KV = |y_0|. \tag{2.5}$$

Combining this with equations (2.2) and (2.1) we get

$$E = \pm (K+1)V. \tag{2.6}$$

Since $K+1$ is a little less than the coordination number Z this is a reasonably good approximation for the limits of the band of the unperturbed system at $\pm ZV$, which is the expected result for the limit of the mobility edge as the disorder goes to zero.

Although this is a reasonable approximation for a real lattice, it is a surprising result for the Bethe lattice, for which the band edges of the regular system are at $\pm 2\sqrt{KV}$ where the solutions of equation (1.1) with $\epsilon_i = 0$, $S_i = S_j$, become real, so that the limiting value of the mobility edge is outside the unperturbed band. In fact the derivation of equation (1.2) given in I is valid only if $Q(0)$ is nonzero, so this result holds only if there is always a tail of site energies spreading out as far as $\pm KV$ until the δ-function limit is reached. It does not matter how low the concentration of these states is. A state localized on one site of energy $\pm KV$ has fallen off in amplitude by a factor K^{-n} at a site n steps away, but there are K^n sites at this distance, so the exponential fall-off in amplitude just balances the exponential increase with distance of the number of sites, and this is sufficient to delocalize the state, according to the original arguments of Anderson (1958).

3. Fredholm method

For fixed Q, equation (1.2) can be regarded as a homogeneous integral equation with eigenvalue K^{-1}, and Fredholm theory can be applied (see Riesz and Nagy 1956). It appears that under many conditions in which we are interested, for reasonably large K, the kernel of equation (1.2) is close to being the sum of two separable terms; comparison with numerical integration of the equation bears this out. If this is the case we can truncate the Fredholm series after two terms, and this gives the equation

$$1 = KV\int \frac{Q(x+V^2/x)}{|x|}\,dx - \tfrac{1}{2}K^2V^2\left(\int \frac{Q(x+V^2/x)}{|x|}\,dx\right)^2$$

$$+ \tfrac{1}{2}K^2V^2\iint \frac{Q(x+V^2/y)\,Q(y+V^2/x)}{|xy|}\,dx\,dy. \tag{3.1}$$

If we make the further approximation of replacing x by zero in the argument of Q when $|x| < V$ and V^2/x by zero when $|x| > V$ we get

$$\left(1 - KV\int_{|x|>V} \frac{Q(x)}{|x|}\,dx\right)^2 = K^2V^2Q(0)\int_{|x|>V}\int_{|y|>V} \frac{Q(x+y)}{|xy|}\,dx\,dy. \tag{3.2}$$

which is equation (7.5) of I. This approximation is valid if Q varies slowly over a range of width V. In later sections, equations (3.1) and (3.2) are applied to a number of examples and compared with the results obtained by other methods.

4. Cauchy distribution of the site energies

In this section we consider a Cauchy distribution of the site energies

$$p(\epsilon) = \frac{\gamma/\pi}{\epsilon^2 + \gamma^2}. \tag{4.1}$$

In this case the probability distribution $Q(x)$ can be calculated exactly. From equation (1.1) we find that E_j, the real part of S_j, has a Cauchy distribution peaked at $E_j = \mu$ of width Γ where

$$\mu + i\Gamma = \frac{KV^2}{E - i\gamma - \mu - i\Gamma}. \tag{4.2}$$

Consequently, from equation (1.3), we find that $Q(x)$ is also a Cauchy distribution of the form

$$Q(x) = \frac{b/\pi}{(x-a)^2 + b^2} \tag{4.3}$$

where a and b are given by

$$a + ib = E - (1 - K^{-1})\mu + i(\gamma + (1 - K^{-1})\Gamma). \tag{4.4}$$

We have an exact solution of equation (1.2) in the limit $\gamma = 0$, and so a perturbative method of solution in this neighbourhood is appropriate. For E close to $+(K+1)V$ we expect $A(x)$ to be small for negative x, and so a reasonable approximation might be found by replacing $|x|$ by x in equation (1.2).

This gives the equation

$$K_{n-1}^{-1}A_n(y) = V\int Q\left(y + \frac{V^2}{x}\right)A_n(x)x^{-1}\,dx \tag{4.5}$$

and its adjoint

$$K_{n-1}^{-1}a_n(x) = V^{-1}x\int a_n(y)\,Q\left(y + \frac{V^2}{x}\right)dy. \tag{4.6}$$

These equations have solutions

$$A_n(y) = \frac{(y - V/K_0)^{n-1}}{(y - VK_0)^n},$$

$$a_n(x) = \frac{(x - VK_0)^{n-1}}{(x - V/K_0)^n}\frac{V(K_0 - K_0^{-1})}{2i\pi}, \tag{4.7}$$

for positive integer n with eigenvalue K_0^{1-2n} where K_0 is the larger root of the equation

$$VK_0 + VK_0^{-1} = a + ib. \tag{4.8}$$

The complex conjugates of these solutions are also solutions with eigenvalue $(K_0^*)^{1-2n}$. In the limit of $\gamma = 0$ we get the δ-function solution

$$(2i\pi)^{-1}[A_1(y) - A_1^*(y)]. \tag{4.9}$$

When γ is nonzero we must include the perturbation due to the difference between $|x|^{-1}$ and x^{-1}, and so we use degenerate perturbation theory and look for a solution which is a linear combination of the eigenfunctions corresponding to K_0^{-1} and K_0^{*-1}

$$A(x) = \alpha_0 A_1(x) + \alpha_1 A_1^*(x). \tag{4.10}$$

Perturbation theory gives the conditions

$$\begin{aligned} K^{-1}\int a_1(x)\,A(x)\,dx &= V\iint a_1(y)\,Q\left(\frac{y+V^2}{x}\right)A(x)\,|x|^{-1}\,dx\,dy, \\ K^{-1}\int a_1^*(x)\,A(x)\,dx &= V\iint a_1^*(y)\,Q\left(\frac{y+V^2}{x}\right)A(x)\,|x|^{-1}\,dx\,dy. \end{aligned} \tag{4.11}$$

If we write

$$K_0 = R\,e^{i\theta} \tag{4.12}$$

we get from equations (4.6), (4.7), (4.10) and (4.11) the quadratic equation

$$\frac{R^2}{K^2} - \frac{2R}{K}\left(\cos\theta - \frac{2\theta}{\pi}\cos\theta + \frac{2\sin\theta}{\pi}\ln R\right) - \left(\frac{4}{\pi}\frac{\sin\theta\ln R}{R-R^{-1}}\right)^2 + \left(1-\frac{2\theta}{\pi}\right)^2 = 0. \tag{4.13}$$

From equations (4.12), (4.8), (4.4) and (4.2) we can get the relation

$$\begin{aligned} K^{-1}(E+i\gamma)^2 - (1+K^{-1})(E+i\gamma)(R\,e^{i\theta} + R^{-1}e^{-i\theta}) + R^2 e^{2i\theta} + K \\ + K^{-1} + R^{-2}e^{-2i\theta} = 0 \end{aligned} \tag{4.14}$$

so small γ corresponds to small θ. For small θ equation (4.13) reduces to

$$\left(\frac{R}{K} - 1\right)^2 = \frac{4R}{K\pi}\theta(\ln R - 1) + \frac{4\theta}{\pi} \tag{4.15}$$

and so the mobility edge is tangent to the axis at $\gamma = 0$, moving outwards as γ is increased.

The solution of this equation for $K = 3$ is shown in figure 1 for $\gamma < V$.

It is possible to extend this method to get an approximation valid for all E if K is large, so that we can neglect all terms of order K^{-2} or R^{-2}. Equation (4.14) gives to this approximation,

$$E + i\gamma = R\,e^{i\theta} + (K/\kappa)\,e^{-i\theta}. \tag{4.16}$$

All the eigenvalues of equation (4.5) except for the first two are negligible, and it can be shown that all terms of order higher than R^{-2} are included if second-order perturbation theory is used. In place of equation (4.13) we get

$$\frac{R^2}{K^2} - \frac{2R}{K}\left[\left(1-\frac{2\theta}{\pi}\right)\cos\theta + \frac{2}{\pi}\sin\theta\ln R\right] + \left(1-\frac{2\theta}{\pi}\right)^2 - \tfrac{2}{3}\sin^2\theta = 0 \tag{4.17}$$

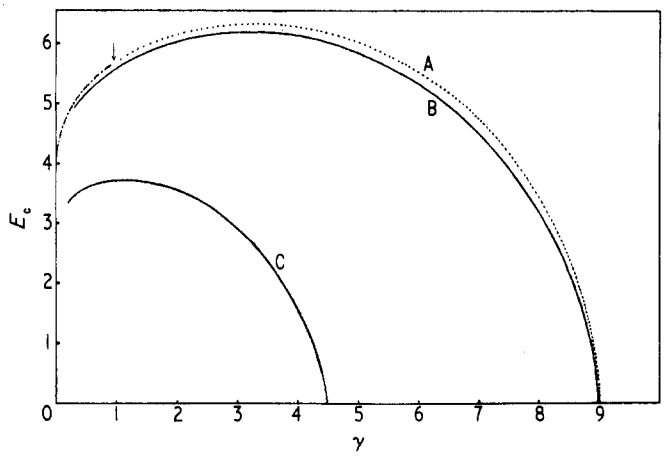

Figure 1. Mobility edge E_c as a function of γ. For $K = 3$, A represents the results of Fredholm Theory in the region $\gamma > V$, and perturbation theory in the region $\gamma < V$; B shows the results of the numerical calculations; C numerical results of E_c for $K = 2$.

where the last term is the additional one produced by the second order. In the case $\theta = \frac{1}{2}\pi$ this agrees with equation (6.17) of I.

To use equation (3.1) of the Fredholm method, it is necessary to evaluate the integrals involved. With some rather heavy work we got

$$I = V\int \frac{Q(x + V^2/x)}{|x|}\,dx = \frac{2(R - R^{-1})(\pi - 2\theta)\cos\theta + 4\ln R(R + R^{-1})\sin\theta}{\pi(R^2 + R^{-2} - 2\cos 2\theta)} \tag{4.18}$$

and

$$\begin{aligned} J \equiv V^2 \iint \frac{Q(x + V^2/y)\,Q(y + V^2/x)}{|xy|}\,dx\,dy \\ = \frac{2V}{\pi}\frac{(R - R^{-1})\cos\theta\,(4b\ln R - 4a\theta + \pi a) + (R + R^{-1})\sin\theta(4a\ln R + 4b\theta - \pi b)}{(a^2 + b^2)(R^2 + R^{-2} - 2\cos 2\theta)} \\ + 4V^2\frac{\{b\ln[(1 + a/V)^2 + b^2/V^2] - 2a\tan^{-1}[b/(a + V)]\}^2}{\pi^2(a^2 + b^2)^2} \\ + \frac{8bV^2}{a^2 + b^2}\frac{b\ln[(3 + a/V)^2 + b^2/V^2] - 2(a + 2V)\tan^{-1}[b/(a + 3V)]}{\pi^2[(a + 2V)^2 + b^2]} \end{aligned} \tag{4.19}$$

where Q, a, b, R and θ are defined by equations (4.3), (4.4), (4.12) and (4.8). The second-order Fredholm approximation is

$$1 = KI - \tfrac{1}{2}K^2(I^2 - J) \tag{4.20}$$

and solutions of this for $K = 3$ and $\gamma > V$ are shown in figure 1. It can be seen that they join up smoothly with the results obtained from perturbation theory, although we would not get a sensible answer in this way for small γ. If terms of order R^{-2} and K^{-2} are neglected we can recover equation (4.17) from equations (4.18), (4.19) and (4.20).

Figure 1 also shows a plot of E against γ for K equal to 2 and 3 obtained by a numerical iterative solution of equation (1.2). It can be seen that the departure of this solution from the approximate solution is quite small.

The behaviour near the band edge is qualitatively different from that found from the theory of Economou and Cohen (1972), where the geometric mean of the energy denominators is the quantity of most significance. As Bishop (1973) has shown, this theory gives a mobility edge which moves inwards as γ is increased from zero.

5. Uniform distribution of the site energies

The problem of solving the integral equation (1.2) was particularly simplified in the case of a Cauchy distribution of E_i, where the exact analytical expression of $Q(x)$ was found. The problem of finding $Q(x)$ becomes more difficult when we are dealing with other distributions. A typical short-tailed probability distribution is the uniform distribution

$$p(\epsilon) = \frac{1}{W} \qquad |\epsilon| < \tfrac{1}{2}W$$
$$= 0 \text{ otherwise,} \tag{5.1}$$

which we examine in detail in this section. In order to find $Q(x)$ in this case two kinds of approximation are considered according to the value of the energy parameter E in the band. The first is when E is near the centre of the band, which implies that the width W of $p(\epsilon)$ is large compared with V and we can ignore the sum in (1.3) so the probability distribution $Q(x)$ becomes

$$Q(x) = p(E - x). \tag{5.2}$$

The second is when E is near the edge of the band; then W is small and hence we replace the sum in (1.3) by its value at the edge which is obtained from (2.1) and $(1 - K^{-1})\mu = (1 - K^{-1})V$. Thus $Q(x)$ in this approximation is taken to be

$$Q(x) = p(E - V + K^{-1}V - x). \tag{5.3}$$

By using these two approximations the position of the mobility edge can be found either from the Fredholm method or by a direct numerical iteration of equation (1.2). Consider first the second-order Fredholm method. For W small, the approximation (5.3) is valid, and $Q(0)$ is zero for E greater than $\frac{1}{2}W + V$. We can use equation (3.2) in place of equation (3.1) without serious error, and this gives

$$1 = KV \int \frac{Q(x)}{|x|}\,\mathrm{d}x = \frac{KV}{W} \ln\left(\frac{E - V + K^{-1}V + \frac{1}{2}W}{E - V + K^{-1}V - \frac{1}{2}W}\right) \tag{5.4}$$

Near the centre of the band, the second term is small, and equation (7.8) of I, which is

$$1 = \frac{2KV}{W} \ln \frac{\frac{1}{4}W^2 - E^2}{V^2} \tag{5.5}$$

is given by the first-order term. In figure 2 is shown a curve of E_c against W obtained from equations (5.4) and (5.5) for low and high values of W respectively, joined smoothly near $W = 2E$.

Figure 2 also shows the results of a numerical iteration of equation (1.2) for Q given by equations (5.2) or (5.3); these two curves are also joined smoothly in the neighbourhood of $W = 2E$. The discrepancy between the results obtained from numerical integration and from the Fredholm method is small, and we would expect the error to get less for larger K.

The parabolic shape of the curve near $W = 0$ will be obtained for any probability distribution that has a short tail, so that $Q(0)$ is negligible, and so that equation (5.4) is valid. If we ignore the term $K^{-1}V$ in the argument of p in equation (5.3) we get the localization edge correctly at

$$E_B = (K + 1)V \tag{5.6}$$

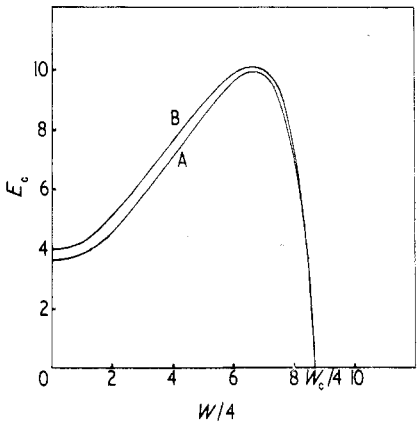

Figure 2. Mobility edge in the case of a uniform distribution of site energies for $K = 3$. A: results of Fredholm Theory; B: results of the numerical calculations.

in the limit of zero disorder. If $p(x)$ is symmetrical about the origin we get

$$1 = KV \int \frac{p(x)}{E - V - x} \, dx \approx \frac{KV}{E - V}\left(1 + \frac{\overline{\epsilon^2}}{(E - V)^2} + \ldots\right) \tag{5.7}$$

so that

$$E \approx E_B + \overline{\epsilon^2}/KV \tag{5.8}$$

where $\overline{\epsilon^2}$ is the mean square deviation of the site energy. A formula of this sort would be obtained for the band edge in the limit of small disorder from the coherent potential approximation (Soven 1967) or from a perturbative calculation of the energy of the highest Bloch state. The coefficient would be different, being

$$N^{-1} \sum_k (E - E_k)^{-1} \tag{5.9}$$

the average over all Bloch states of a denominator, rather than $(E - V)^{-1}$, but this difference is probably a result of the crudity of the fundamental approximation (1.1). For the distributions with short tails the coherent potential approximation for the band edge is probably a better approximation for the localization edge for small disorder, and the qualitative results are very similar. The use of the coherent potential approximation near the band edge was discussed by Economou and Cohen (1972).

In the case of weak disorder there are localized states, associated with regions in which the average site energy deviates from zero to a sufficient degree to form a bound state. This phenomenon, considered by Lifshitz (1964), depends strongly on the dimensionality of the lattice and is not adequately described in the self-consistent theory. The work of Abram and Edwards (1972) is devoted to a problem of this sort.

6. The binary alloy

Consider a binary alloy formed from two types of atoms A and B. Suppose that there is a concentration c of sites of type A, with energy zero, and a concentration $1 - c$ of sites B, with energy W. We know that the eigenvalues of the alloy AB lie within the bands of the pure lattice A and the pure lattice B (Thouless 1970, Kirkpatrick *et al* 1970). Then, in order to examine the localization of eigenstates, we should examine the states corresponding to the energies within the bands

$$W - ZV \leqslant E \leqslant W + ZV$$
$$-ZV \leqslant E \leqslant +ZV. \tag{6.1}$$

The probability distribution of the site energies can be written in the form

$$p(\epsilon) = c\delta(\epsilon) + (1 - c)\,\delta(\epsilon - W). \tag{6.2}$$

An approximate expression of $Q(x)$ can be found in the case of small concentration of sites A, ($c \ll 1$). In this case the real part of the self-energy

$$E_i = \sum_j \frac{V^2}{E - \epsilon_j - E_j} \tag{6.3}$$

can be approximated by taking the most probable value of the sum and solving this equation consistently. Then, we obtain a quadratic equation for the real part $R = E_i$ which gives

$$R = \tfrac{1}{2}(E - W) + \tfrac{1}{2}[(E - W)^2 - 4KV^2]^{1/2} \tag{6.4}$$

for E below the majority sub-band; the negative sign for the square root must be taken above the majority sub-band. Thus the sum in the expression (1.3) was replaced by its most probable value $(1 - K^{-1})R$, and the probability density $Q(x)$ taken to be

$$Q(x) = c\delta(a - x) + (1 - c)\,\delta(a - w - x), \tag{6.5}$$

where

$$a = E - (1 - K^{-1})R. \tag{6.6}$$

When this expression for $Q(x)$ is substituted into equation (1.2) we get the functional equation

$$A(y) = \frac{KVc}{|a - y|} A\left(\frac{V^2}{a - y}\right) + \frac{KV(1 - c)}{|a - W - y|} A\left(\frac{V^2}{a - W - y}\right). \tag{6.7}$$

The eigenvalue K was found numerically by an iterative solution of this equation for $c = 0{\cdot}1$ and varying E and by this method the value of E that gives a mobility edge for a particular value of K can be found. Figure 3 shows the regions of nonlocalized states for varying values of W. We notice that as W is increased the region of nonlocalized states divides into two bands separated by a region of localized states for W/V approximately 2·5. As W increases further the minority band of nonlocalized states narrows, but it does not disappear. This is in contrast to the result of Economou *et al* who found that the minority band of nonlocalized states ceases to exist for W/V approximately 3.3.

The numerical iteration of equation (6.7) is difficult in this region of large W, as the function A is sharply peaked, and so these results may contain numerical errors. If the Fredholm method is applied to this problem—and it should be observed that it may

not be justified for the singular form of Q given in equation (6.5)—the only term of order c comes from the third term on the right of equation (3.1). For small c the Fredholm method gives approximately

$$\left(1 - \frac{(1-c)KV}{W-a}\right)^2 = \frac{2cK^2V^2[4V^2(W/a - 1) + (W-a)^2]^{1/2}}{4V^2(W-a) + a(W-a)^2}. \tag{6.8}$$

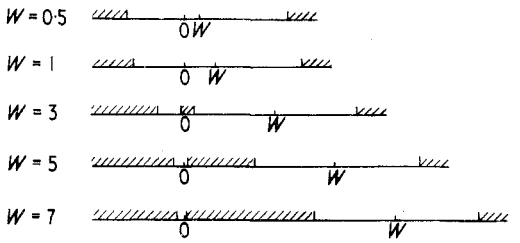

Figure 3. Localization regions (shaded) obtained numerically for a binary alloy with a concentration $c = 0{\cdot}1$ of sites A and $K = 2$.

For small c this equation has a solution with small positive a. and so the band of nonlocalized states extends from $a = 0$ up to this value of a, and so, from equations (6.6) and (6.8), the band of nonlocalized states has a width approximately equal to

$$\frac{c^2K^2V^2}{W}\left(1 - \frac{KV}{W}\right)^{-4} \tag{6.9}$$

This is considerably narrower than the band shown in figure 3.

In a binary alloy the minority carriers are coupled together by an effective coupling whose strength is of order V^n/W^{n-1}, where n is the number of steps needed to go from one minority atom to another, and this should give the bandwidth of the nonlocalized states. Since n is proportional to $c^{-1/3}$ we expect the bandwidth to decrease exponentially with the concentration, and such a behaviour is not given either in this theory or in the coherent potential approximation.

The case in which W becomes infinite, so that the sites B are completely impenetrable, is very simple. In this case we have only the first term on the right of equation (6.5), and equation (1.2) gives

$$A(y) = \frac{cKV}{|a-y|} A\left(\frac{V^2}{a-y}\right) \tag{6.10}$$

This is the same as equation (2.3) for the pure system, with K replaced by cK, and there is a solution if cK is greater than unity. This is the percolation limit for the Cayley tree (Fisher and Essam 1961), and, in so far as equation (6.5) is adequate, it says that there are nonlocalized states until the percolation limit is reached. The inequality (1.4) shows that for cK less than unity there are no nonsingular solutions of the equation.

In this case and in the other case of the dilute binary alloy the distribution of the real part of the self-energy is more complicated that was assumed in the derivation of equation (6.5), and there may be important effects that we have neglected.

7. Discussion

We have shown how satisfactory solutions of the integral equation that was derived in I for the self-consistent theory of localization can be obtained under a wide variety of circumstances by combining the Fredholm expansion with our knowledge of the limit of no disorder. We have paid particular attention to the behaviour of the mobility edge when it is close to the band edge, which occurs when the disorder is weak. For short-tailed distribution, such as the rectangular distribution or the normal distribution, the results of this theory are similar to those of the coherent potential approximation (Soven 1967) or the theory of Economou and Cohen (1972). For such distributions the right side of equation (3.2) is zero or negligible. For long-tailed distributions, such as a Cauchy distribution, the right side of equation (3.2) gives, close to the band edge,

$$\frac{4K^2V^2}{E}Q(0)\ln\left(\frac{E}{V}\right) \tag{7.1}$$

and it is this term which gives rise to the rapid increase in the width of the band of nonlocalized states shown in figure 1. This term proportional to $\ln(E/V)$ is reminiscent of the term proportional to $\ln(W/2V)$ which distinguishes the Anderson (1958) theory in the centre of the band from other theories.

The results for the binary alloy are also different from those obtained by other means, but the results and methods of calculation are not entirely satisfactory.

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