Thermodynamics of quasi-two dimensional bosons: Kosterlitz-Thouless transition

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I. INFINITE UNIFORM BOSE GAS IN TWO DIMENSIONS: KOSTERLITZ-THOULESS TRANSITION

Although there have been indications of a break-down of the low-temperature superfluid phase, our phonon Hamiltonian is not sufficient to study the transition to the normal phase. Vortices are the key ingredient missing in the phonon description, and Kosterlitz and Thouless have developed the theoretical description of the superfluid transition essentially driven by vortices.

A. Single Vortex

Let us consider a single charged vortex, centered at the origin, characterized by a change of 2π of the phase for any closed path going around the origin,

$$\Psi_v(\mathbf{r}) = n^{1/2}\phi_v(r)e^{i\theta}, \quad \mathbf{r} \equiv (x, y) = r(\cos\theta, \sin\theta), \quad \theta = \arctan y/x \tag{1}$$

Inserting this ansatz in the (continuum) Gross-Pitaevskii equation, we can study the density distribution $|\phi_v(r)|^2$ of the vortex, which is well approximated by $\phi_v(r) \simeq r^2/(r^2 + \xi^2)$ with the usual healing length $\xi = (2m\mu/\hbar^2)^{-1/2}$. The density vanishes inside a core radius $\sim \xi$ and is approximately constant at larger distances.

Having a classical field distribution $\Psi(\mathbf{r})$, its energy is given by

$$E[\Psi(\mathbf{r})] = \int d\mathbf{r} \left[\frac{\hbar^2}{2m}|\nabla\Psi(\mathbf{r})|^2 - \mu|\Psi(\mathbf{r})|^2 + \frac{g}{2}|\Psi(\mathbf{r})|^4\right] \tag{2}$$

The variation of the energy due to the vortex is dominated by the kinetic energy contribution outside the vortex core, $r \gtrsim \xi$, where we have

$$E_v \equiv \frac{\hbar^2}{2m}\int_0^L d^2\mathbf{r}|\nabla\Psi_v(\mathbf{r})|^2 = E_c + T_v \tag{3}$$

$$T_v = \frac{\hbar^2}{2m}\int_\xi^L d^2\mathbf{r}|\nabla\Psi_v(\mathbf{r})|^2 = \frac{\hbar^2 n}{2m}\int_\xi^L d^2\mathbf{r}(\nabla\theta)^2 = \frac{\hbar^2 n}{2m}\int_\xi^L 2\pi r dr\left(\frac{1}{r}\right)^2 = \frac{\hbar^2 n}{2m}2\pi\log L/\xi \tag{4}$$

with $E_c \sim gn\xi^2$. (Note that we used $d/dx \arctan x = 1/(1 + x^2)$ or $\nabla f(r, \theta) = (\partial f/\partial r)\mathbf{e}_r + (1/r\partial f/\partial\theta)\mathbf{e}_\theta$). We see that a single vortex is energetically forbidden in large systems, $L \to \infty$, and the energy cost is essentially determined by the topological charge (here $+1$). Let us calculate the partition function of having a single vortex of charge ± 1 anywhere in the system

$$Z_1 \approx 2\int\frac{d^2\mathbf{r}_1}{\xi^2}e^{-\beta(T_v+E_c)} = 2\frac{L^2}{\xi^2}e^{-\beta(T_v+Ec)} = e^{-\beta(T_v+E_c)+S_v} \tag{5}$$

$$F_1 = -k_BT\log Z_1 = \left(\frac{\hbar^2 2\pi n}{2m} - 2k_BT\right)\log L/\xi + \ldots \tag{6}$$

The corresponding free energy, F_1, develops a non-analyticity at $n\lambda^2 = 4$, indicating the phase transition to the normal state. We should not trust this heuristic calculation, but more elaborate considerations show that the normal to superfluid phase transition indeed occurs at higher densities (lower temperatures), $n_s\lambda^2 = 4$, simply replace the density by the superfluid density, $n_s \leq n$. Indeed, as long as one remains within the phonon Hamiltonian, we have seen that the superfluid density coincides with the total one.

B. Vortex dipoles

1. velocity field

Whereas single vortices are absent at low temperatures, pairs of vortices with opposite charge will always be present, as their energy remains finite. Let us write the velocity field $\nabla\theta_\pm$ of a single vortex of charge ± 1 at positions $\mathbf{r}_\pm$, far away from the core in the following form

$$\begin{aligned}\nabla\theta_\pm &= \pm\ \nabla \arctan(y-y_\pm)/(x-x_\pm) = \pm\left(-\frac{y-y_\pm}{|\mathbf{r}-\mathbf{r}_\pm|^2}, \frac{x-x_\pm}{|\mathbf{r}-\mathbf{r}_\pm|^2}, 0\right) = \pm\frac{1}{|\mathbf{r}-\mathbf{r}_\pm|}\left(-\frac{y-y_\pm}{|\mathbf{r}-\mathbf{r}_\pm|}, \frac{x-x_\pm}{|\mathbf{r}-\mathbf{r}_\pm|}, 0\right) &(7)\\ &= \mp\nabla\times\hat{z}\log|\mathbf{r}-\mathbf{r}_\pm| &(8)\end{aligned}$$

Let us calculate now the kinetic energy of the dipole with separation $\mathbf{d}=\mathbf{r}_+-\mathbf{r}_-$ centered around the origin

$$\begin{aligned}T_d &= \frac{\hbar^2 n}{2m}\int_\xi^L d^2\mathbf{r}\left[(\nabla\theta_+)^2+(\nabla\theta_-)^2+2(\nabla\theta_+)(\nabla\theta_-\right)^2 &(9)\\ &\simeq 2T_v - 2\frac{\hbar^2 n}{2m}\int_\xi^L d^2\mathbf{r}\frac{1}{|\mathbf{r}-\mathbf{r}_+||\mathbf{r}-\mathbf{r}_-|} = 2T_v - 2\frac{\hbar^2 n}{2m}\int_\xi^L 2\pi r dr\frac{1}{\mathbf{r}^2-d^2/4} &(10)\\ &\simeq 2\frac{2\pi\hbar^2 n}{2m}\log L/\xi - 2\frac{2\pi\hbar^2 n}{2m}\log L/d = \frac{4\pi\hbar^2 n}{2m}\log d/\xi &(11)\end{aligned}$$

and the total energy is approximately $E_d \simeq T_d + 2E_c$.

Up to now, we have considered a static dipole of dipole length d, centered at the origin $\mathbf{R}=0$ whose energy gives us the probability distribution of finding such a vortex pair. Let us calculate the mean-square of the dipole length in the 1-dipole sector

$$\langle d^2\rangle_d \equiv \frac{\int d^2(d/\xi)d^2 e^{-\beta T_d}}{\int d^2(d/\xi)e^{-\beta T_d}} = \frac{\xi^2\int_1^\infty du u^{3-n\lambda^2}}{\int_1^\infty du u^{1-n\lambda^2}} = \xi^2\frac{2-n\lambda^2}{4-n\lambda^2} = \xi^2\frac{n\lambda^2-2}{n\lambda^2-4} \qquad (12)$$

Starting from low temperatures/ high densities, $n\lambda^2 \gg 1$, where vortices should be unimportant, we see that, lowering $n\lambda^2$, the mean-square value of the dipole-separation first diverges at $n\lambda^2=4$. In the limit of very large core energy, $\beta E_c\to\infty$, we can restrict the (vortex) partition function to $1+Z_d$

$$Z_d = \int\frac{d^2\mathbf{r}_+}{\xi^2}\int\frac{d^2\mathbf{r}_-}{\xi^2}e^{-\beta(T_v-2E_c)} \qquad (13)$$

and the dipole density, n_d, writes

$$n_d \simeq \frac{1}{V}\frac{Z_d}{1+Z_d} \simeq \frac{Z_d}{V} = z^2\xi^{-2}\int_1^\infty du u^{1-n\lambda^2} = -\frac{z^2\xi^{-2}}{2-n\lambda^2}, \quad z=e^{-\beta E_c}\to 0 \qquad (14)$$

assuming $n\lambda^2 > 4$ everywhere, or

$$n_d\langle d^2\rangle \simeq \frac{z^2}{n\lambda^2-4} = z^2\int_1^\infty du\, u^{3-n\lambda^2} \qquad (15)$$

to leading order in z.

2. superfluid density

How do vortex dipole affect the superfluid density? Let us look at its contribution to the momentum correlation function

$$\chi^v_{ij}(\mathbf{r},\mathbf{r}') = \beta\hbar^2 n^2\nabla_i\nabla'_j\langle\theta_d(\mathbf{r})\theta_d(\mathbf{r}')\rangle \qquad (16)$$

and insert the far-field of the velocity from a vortex dipole of separation $\mathbf{d}$

$$\nabla\theta_d = \nabla\theta_+ + \nabla_- = -\nabla\times\hat{z}\log\frac{|\mathbf{r}-\mathbf{r}_+|}{|\mathbf{r}-\mathbf{r}_-|} \simeq -\nabla\times\hat{z}\frac{\mathbf{d}\cdot\mathbf{r}}{\mathbf{r}^2}, \quad r\gg d \qquad (17)$$

For its Fourier transform, we obtain

$$\chi^v_{ij}(\mathbf{q}) = \frac{1}{V}\int d^2\mathbf{r}\int d^2\mathbf{r}'\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\chi^v_{ij}(\mathbf{r},\mathbf{r}') = \left[q^2\delta_{ij} - q_i q_j\right]\beta\hbar^2 n^2\frac{1}{V}\langle\langle D(\mathbf{q})D(-\mathbf{q})\rangle \tag{18}$$

We see that the momentum correlation function of a vortex dipole is purely transverse (This is easily checked explicitly by looking at χ_{xx} and χ_{xy}), and

$$D(\mathbf{q}) = \int d^2\mathbf{r}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{\mathbf{d}\cdot\mathbf{r}}{r^2} = \int d^2\mathbf{r}e^{i\mathbf{q}\cdot\mathbf{r}}[\mathbf{d}\cdot\nabla_\mathbf{r}]\log|\mathbf{r}| = -i\mathbf{d}\cdot\mathbf{q}\int d^2\mathbf{r}e^{i\mathbf{q}\cdot\mathbf{r}}\log|\mathbf{r}| = i2\pi\frac{\mathbf{d}\cdot\mathbf{q}}{q^2} \tag{19}$$

Assuming isotropically distributed dipoles of density n_d, we get

$$\chi^v_{ij}(q) = \left[\delta_{ij} - \frac{q_i q_j}{q^2}\right] 2\pi^2 n_d\langle d^2\rangle_d\beta\hbar^2 n^2 \tag{20}$$

Since χ^v_{ij} is purely transversal, we get read off the normal mass density. Concerning the longitudinal part, it is determined by the phonon Hamiltonian/ spin-wave approximation in the vortex free sector, which we discussed previously. For the superfluid density we get

$$n_s = n\left(1 - \pi n_d\langle d^2\rangle n\lambda^2\right) \simeq \frac{n}{1+\pi n_d\langle d^2\rangle_d n\lambda^2} \tag{21}$$

Consistent with our approximation, one would conclude that the transition from superfluid to normal state takes place at $n\lambda^2 = 4$, where $n_d\langle d^2\rangle$ diverges.

3. *Algebraic decay*

How do vortex dipole affect the algebraic decay of the first order correlation function? Let us calculate the phase correlations at two distances, $\mathbf{r}$ and $\mathbf{r}'$, at large separation from the dipole situated at $\mathbf{R}$, and $\langle\ldots\rangle$ indicates the averaging over $\mathbf{R}$,

$$\begin{aligned} g_d(\mathbf{r},\mathbf{r}') &= \left\langle \exp i\mathbf{d}\cdot\left[\frac{\mathbf{r}-\mathbf{R}}{(\mathbf{r}-\mathbf{R})^2} - \frac{\mathbf{r}'-\mathbf{R}}{(\mathbf{r}'-\mathbf{R})^2}\right]\right\rangle \simeq 1 - \frac{1}{2}\left\langle\left\{\mathbf{d}\cdot\left[\frac{\mathbf{r}-\mathbf{R}}{(\mathbf{r}-\mathbf{R})^2} - \frac{\mathbf{r}'-\mathbf{R}}{(\mathbf{r}'-\mathbf{R})^2}\right]\right\}^2\right\rangle & (22)\\ &= 1 - \frac{1}{2}\left\langle\frac{d^2}{2}\frac{1}{(\mathbf{r}-\mathbf{R})^2} + \frac{d^2}{2}\frac{1}{(\mathbf{r}'-\mathbf{R})^2} - 2\frac{[\mathbf{d}\cdot(\mathbf{r}-\mathbf{R})][\mathbf{d}\cdot(\mathbf{r}'-\mathbf{R})]}{(\mathbf{r}-\mathbf{R})^2(\mathbf{r}'-\mathbf{R})^2}\right\rangle & (23)\\ &= 1 - \left\langle\frac{d^2}{2}\frac{1}{R^2} - \frac{[\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}'-\mathbf{R})][\mathbf{d}\cdot\mathbf{R}]}{(\mathbf{r}-\mathbf{r}'-\mathbf{R})^2R^2}\right\rangle & (24) \end{aligned}$$

For the last equality, we have used that the averaging over $\mathbf{R}$ first allows us to use translational invariance, such that $g_d(\mathbf{r},\mathbf{r}')$ is only a function of $\mathbf{r}-\mathbf{r}'$. Similarly, averaging over the directions of $\mathbf{R}$ fully restores isotropy, e.g. the term linear in $\mathbf{d}\cdot\mathbf{R}$ vanishes. We then get

$$\begin{aligned} g_d(|\mathbf{r}-\mathbf{r}'|) &= -1\frac{1}{2}n_d\langle d^2\rangle_d\left[2\pi\log\frac{L}{\xi} - 2\pi\log\frac{L}{|\mathbf{r}-\mathbf{r}'|}\right] = 1 - \pi n_d\langle d^2\rangle_d\log\frac{|\mathbf{r}-\mathbf{r}'|}{\xi} & (25)\\ &\simeq e^{-\pi n_d\langle d^2\rangle_d\log\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}} & (26) \end{aligned}$$

Combining with our phase fluctuation results for the phonon Hamiltonian (spin-wave approxiamtion), $g_{sw}(r)$, we have

$$g(r) = g_{sw}(r)g_d(r) \simeq n\left(\frac{\xi}{r}\right)^{\frac{1}{n_s\lambda^2}} \tag{27}$$

with

$$\frac{1}{n_s\lambda^2} = \frac{1}{n\lambda^2} + \pi n_d\langle d^2\rangle_d \tag{28}$$

using Eq. (21). The vortex corrections therefore modifies the algrebraic decay of the phonon phase fluctuations in a way that the density n in the exponent $1/n\lambda^2$ is simply replaced by the superfluid density, n_s leading to an algrebraic decay with $1/n_s\lambda^2$ in the exponent, at least in our dipole approximation.

C. General situation, scaling

The dipole approximation above is clearly not satisfying. As we approach our "critical temperature", the dipole distance grows, and so also does the concentration of dipoles. The system will contain more and more vortex pairs, by analogy with electromagnetism, we expect that the effective interaction between vortices will be screened.

1. Dipole screening of one vortex

Let us warm up by considering the situation the (velocity) far-field of a single vortex at the origin screened by a vortex dipole at position $\mathbf{R}$ and strength $\mathbf{d}$, e.g. $\mathbf{r}_+ = \mathbf{R} + \mathbf{d}/2$ and $\mathbf{r}_- = \mathbf{R} - \mathbf{d}/2$. At distances $|\mathbf{r}| \gg |\mathbf{R}| \gg |\mathbf{d}| \gg \xi$, the total phase will be $\theta(\mathbf{r}) = \theta_v(\mathbf{r}) + \theta_d(\mathbf{r}; \mathbf{R}, \mathbf{d})$, where $\theta_v(\mathbf{r})$ and $\theta_d(\mathbf{r}; \mathbf{R}, \mathbf{d})$ are the respective far-field phases of the vortex and dipole defects. The total kinetic energy of this situation is then given by

$$T = \frac{\hbar^2 n}{2m} \int_\xi^L d^2\mathbf{r} [\nabla\theta(\mathbf{r})]^2 \tag{29}$$

$$= T_v + T_d + \delta T \tag{30}$$

$$\delta T = 2\frac{\hbar^2 n}{2m} \int_\xi^L d^2\mathbf{r} [\nabla\theta_v(\mathbf{r})][\nabla\theta_d(\mathbf{r})] \tag{31}$$

with $\nabla\theta_v(\mathbf{r}) = -\nabla \times \hat{z} \log|\mathbf{r}|$ and $\nabla\theta_d(\mathbf{r}; \mathbf{R}, \mathbf{d}) = -\nabla \times \hat{z} \frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}$.

To proceed, we will use two identities which will be generally helpful in two dimensions

$$[\nabla \times \hat{z} f(\mathbf{r})] \cdot [\nabla \times \hat{z} g(\mathbf{r})] = [\nabla f(\mathbf{r})] [\nabla g(\mathbf{r})] \tag{32}$$

for any functions $f(\mathbf{r})$ and $g(\mathbf{r})$ (shown straightforwardly by spelling out explicitly in x-y components), as well as

$$\nabla^2 \log|\mathbf{r}| = 2\pi\delta^2(\mathbf{r}) \tag{33}$$

which we have already seen before, e.g. in Fourier space, with appropriate regularization on a lattice, e.g. $\log 0 \simeq \log\xi$.

Equation (31) then writes

$$\delta T = 2\frac{\hbar^2 n}{2m} \int_\xi^L d^2\mathbf{r} \, [\nabla \log|\mathbf{r}/\xi|] \left[\nabla \frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right] \tag{34}$$

$$= -2\frac{\hbar^2 n}{2m} \int_\xi^L d^2\mathbf{r} \, [\nabla^2 \log|\mathbf{r}/\xi|] \left[\frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right] \tag{35}$$

$$= 4\pi \frac{\hbar^2 n}{2m} \frac{\mathbf{d}\cdot\mathbf{R}}{|\mathbf{R}|^2} \tag{36}$$

and the effective statistical weight of our vortex will obtain a contribution $\langle \exp[-\beta\delta T] \rangle_d$ where $\rangle \ldots \rangle_d$ denotes the averaging over the probability of a vortex dipole of strength $\mathbf{d}$ at $\mathbf{R}$, with respect to $\mathbf{d}$ and $\mathbf{R}$.

We have

$$\left\langle e^{-\beta\delta T} \right\rangle_d \simeq 1 - \beta\langle T\rangle_d + \frac{\beta^2}{2}\langle \delta T^2\rangle_d \tag{37}$$

$$= 1 - \frac{1}{2}\left(\frac{2\pi\hbar^2 n}{mT}\right)^2 \left\langle \frac{(\mathbf{d}\cdot\mathbf{R})^2}{|\mathbf{R}|^4} \right\rangle_d \tag{38}$$

where we have used that $\langle T\rangle_d$ vanishes due to the isotropy of $\mathbf{d}$. Similar, $\langle (\mathbf{d}\cdot\mathbf{R})^2\rangle_d = \langle \mathbf{d}^2\mathbf{R}^2\rangle_d/2$, and we get

$$\left\langle e^{-\beta\delta T} \right\rangle_d \simeq 1 + \frac{1}{4}(n\lambda)^2 n_d \langle d^2\rangle_d \int_\xi^L d^2R \frac{1}{R^2} \tag{39}$$

$$= 1 + \frac{\pi}{2}(n\lambda)^2 n_d \langle d^2\rangle_d \log\frac{L}{\xi} \tag{40}$$

$$\simeq e^{\frac{n\lambda^2}{2}\pi n\lambda^2 n_d \langle d^2\rangle_d \log L/\xi} \tag{41}$$

such the probability of a single vortex at the origin $\sim z\exp[-\beta T_v] = z\exp[-\frac{n\lambda^2}{2}\log L/\xi]$ is modified according to an effecive kinetic energy

$$\beta T_v^{eff} \;\simeq\; \frac{n\lambda^2}{2}\left[1-\pi n\lambda^2 n_d\langle d^2\rangle_d\right]\log L/\xi \tag{42}$$

or

$$\beta T_v^{eff} \;=\; \frac{n_s\lambda^2}{2}\log L/\xi \tag{43}$$

using Eq. (21) for the superfluid density accounting for a single vortex dipole. Following the same heuristic arguments from above, we then expect a possible phase transition (non-analyticity in the partition function) when $n_s\lambda^2 = 4$.

2. *Full vortex Hamiltonian*

Extending our above discussion of dipoles to an arbitrary number of vortices at positions $\mathbf{r}_i$ and charge $q_i = \pm 1$, the gradient of the phase can be written as

$$\nabla\theta(\mathbf{r}) = -\sum_i q_i \nabla\times\hat{z}\log|\mathbf{r}-\mathbf{r}_i|/\xi \tag{44}$$

and the kinetic energy writes

$$T = \frac{\hbar^2 n}{2m}\int d^2\mathbf{r}[\nabla\theta(\mathbf{r})]^2 \tag{45}$$

Since $|\nabla\times\hat{z}f(\mathbf{r})| = |\nabla f(\mathbf{r})|$, and $\nabla^2\log|\mathbf{r}|/\xi = 2\pi\delta^2(\mathbf{r})$, we have

$$T = -\frac{2\pi\hbar^2 n}{2m}\sum_{i\neq j} q_i q_j \log\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\xi} \tag{46}$$

(The self-interaction term with $i=j$ represents the core energy and gives NE_c in the total energy).

Notice further that the phonon phase fluctuation energy phase gradient $\nabla\varphi(\mathbf{r})$ does not mix with the vortex contribution since

$$\int d^2\mathbf{r}[\nabla\varphi(\mathbf{r})]\cdot[\nabla\times\hat{z}f(\mathbf{r})] = -\int d^2\mathbf{r}\varphi(\mathbf{r})\nabla\cdot[\nabla\times\hat{z}f(\mathbf{r})] = 0 \tag{47}$$

for any function $f(\mathbf{r})$, and therfore also for our logs. Therefore our partition function simplifies

$$Z \;\propto\; Z_{sw}Z_Q \tag{48}$$

$$Z_{sw} \;=\; \int D\varphi(\mathbf{r})\exp\left[-\beta\frac{\hbar^2 n}{2m}\int d^2\mathbf{r}[\nabla\varphi(\mathbf{r})]^2\right] \tag{49}$$

$$Z_Q \;=\; \sum_{N=0}^{\infty}[1-\delta_{\sum q_i,0}]z^N\int\prod_{i=1}^{N}\frac{d^2\mathbf{r}_i}{\xi^2}\exp\left[-n\lambda^2\sum_{i<j}q_iq_j\log\frac{\mathbf{r}_i-\mathbf{r}_j}{\xi^2}\right] \tag{50}$$

In the following, although the logarithmic terms in the vortex Hamiltonian stems from the kinetic energy, we will call them interaction, since it represents an interacting Coulomb gas in 2D (where $1/r$ is replaced by the 2D-Coulomb potential $\log r$). In particular a dipole of seperation $\mathbf{r}$ leads to

$$\beta v(r) = n\lambda^2\log|\mathbf{r}|/\xi \tag{51}$$

such that we have

$$Z_Q = \sum_{N=0}^{\infty}[1-\delta_{\sum q_i,0}]z^N\int\prod_{i=1}^{N}\frac{d^2\mathbf{r}_i}{\xi^2}\exp\left[-\beta\sum_{i<j}q_iq_jv(\mathbf{r}_i-\mathbf{r}_j)\right] \tag{52}$$

3. screening of the dipole-interaction

Let us calculate the screening of the dipole interacting due to the presence of a vortex. We start with the positive charged vortex at $\mathbf{r}$ interacting with the dipole at $\mathbf{r}_\pm = \mathbf{R} \pm \mathbf{d}/2$. Let us average over the dipole distribution (holding the charges at $\mathbf{r}$ and $\mathbf{r}'$ fix) to obtain the effective interaction

$$\exp\left[-\beta v_{\text{eff}}(\mathbf{r}-\mathbf{r}')\right] = \exp\left[-n\lambda^2 \log|\mathbf{r}-\mathbf{r}'|\xi\right] \left\langle \exp\left\{-n\lambda^2 \left[\log\frac{|\mathbf{r}-\mathbf{r}_+||\mathbf{r}'-\mathbf{r}_-|}{|\mathbf{r}-\mathbf{r}_-||\mathbf{r}'-\mathbf{r}_+|}\right]\right\}\right\rangle \tag{53}$$

We have

$$\log|\mathbf{r}-\mathbf{r}_+| - \log|\mathbf{r}-\mathbf{r}_-| \simeq \mathbf{d}\cdot\nabla_{\mathbf{r}}\log|\mathbf{r}-\mathbf{R}| = -\mathbf{d}\nabla_{\mathbf{R}}\log|\mathbf{r}-\mathbf{R}| \tag{54}$$

and similar (but opposite sign) for the negative charged dipole at $\mathbf{r}'$. We can then write

$$\begin{aligned}
\exp\left\{-\beta[v_{\text{eff}}(\mathbf{r}-\mathbf{r}') - v(\mathbf{r}-\mathbf{r}')]\right\} &\simeq \left\langle \exp\left\{n\lambda^2 \mathbf{d}\cdot\nabla_{\mathbf{R}}\log\frac{|\mathbf{r}-\mathbf{R}|}{|\mathbf{r}'-\mathbf{R}|}\right\}\right\rangle && (55)\\
&\simeq 1 + \frac{1}{2}(n\lambda^2)^2 n_d\langle d^2\rangle_d/2 \int d\mathbf{R}\left[\nabla_{\mathbf{R}}\log\frac{|\mathbf{r}-\mathbf{R}|}{|\mathbf{r}'-\mathbf{R}|}\right]^2 && (56)\\
&= 1 - (n\lambda^2/2)^2 n_d\langle d^2\rangle_d \int d\mathbf{R}\log\frac{|\mathbf{r}-\mathbf{R}|}{|\mathbf{r}'-\mathbf{R}|}\nabla^2_{\mathbf{R}}\log\frac{|\mathbf{r}-\mathbf{R}|}{|\mathbf{r}'-\mathbf{R}|} && (57)\\
&= 1 + 4\pi(n\lambda^2/2)^2 n_d\langle d^2\rangle_d \log\frac{|\mathbf{r}-\mathbf{r}'|}{\xi} && (58)
\end{aligned}$$

where we have used $\langle\mathbf{d}\rangle_d = 0$ in the beginning and set $\log 0 \approx \log\xi$ in the end. We see that

$$\beta v_{\text{eff}}(r) = \beta v(r) - \pi(n\lambda^2)^2 n_d\langle d^2\rangle_d \log\frac{|\mathbf{r}-\mathbf{r}'|}{\xi} \tag{59}$$

Integrating out the vortex dipoles just lead to an effectively screened interaction which we write in the form

$$\begin{aligned}
v_{\text{eff}}(r) &= n_s\lambda^2\log r/\xi && (60)\\
n_s &= n - \pi n\lambda^2 n_d\langle d^2\rangle_d \simeq \frac{n}{1+\pi n\lambda^2 n_d\langle d^2\rangle_d} && (61)
\end{aligned}$$

4. scaling - RG

Let us spell out our previous result explicitly, inserting the dipole density, n_d and mean square dipole length, $\langle d^2\rangle_d$ in the dilute vortex limit $z \to 0$

$$n_s^{-1} = n^{-1}(\xi) + \pi\lambda^2 z^2(\xi)\int_1^\infty du u^{3-n\lambda^2} \tag{62}$$

where we have inserted the dependence on ξ, to stress that we could consider our Hamiltonian dependent on the parameter ξ. However, our result above was derived under the approximation that only a very dilute gas of dipoles leads to screening. If our test-dipole is close together, this assumption should be valid. However, for larger separation, we expect that the dipoles leading to screening are screened themselves. Let us therefore only perform part of the integration of the right hand side, including separations between ξ and $\xi' = \xi(1+dt) > \xi$. We get

$$\begin{aligned}
n_s^{-1} &= n^{-1}(\xi) + \pi\lambda^2 z^2(\xi)\left[\int_1^{1+dt} du\, u^{3-n\lambda^2} + \int_{1+dt}^\infty du\, u^{3-n\lambda^2}\right] && (63)\\
&= n^{-1}(\xi) + \pi\lambda^2 z^2(\xi)\left[\frac{u^{4-n\lambda^2}}{4-n\lambda^2}\Big|_{u=1}^{u=1+dt} + (1+dt)^{4-n\lambda^2}\int_1^\infty dx\, x^{3-n\lambda^2}\right] && (64)\\
&= n^{-1}(\xi) + \pi\lambda^2 z^2(\xi)\left[dt + [1+(4-n\lambda^2)dt]\int_1^\infty du\, u^{3-n\lambda^2}\right] + \mathcal{O}(dt^2) && (65)
\end{aligned}$$

The rhs of the equation has exactly the same functional form as before if we set

$$n^{-1}(\xi(1+dt)) = n^{-1}(\xi) + \pi\lambda^2 z^2(\xi)dt \quad (66)$$

$$z^2(\xi(1+dt)) = z^2(\xi)\left[1 + (4 - n(\xi)\lambda^2)dt\right] \quad (67)$$

The value of ξ was not defined very precisely, so imagine we would have started with a different value $\xi' > \xi$, but using different bare parameter $n(\xi')$ and $z(\xi')$ we would obtain the same value of n_s. From above we derive the following set of differential equations for these parameters

$$\frac{dn^{-1}}{dt} = \pi\lambda^2 z^2 \quad (68)$$

$$\frac{dz}{dt} = (2 - n\lambda^2/2)z \quad (69)$$

These are the well known Kosterlitz-Thouless recursion relations.

Let us integrate the relations from $t = 0$ (original bare parameters) to $t = \infty$. From Eq. (62) we see that the superfluid density is given by $n_s = n(\xi = \infty)$. The point $n_s\lambda^2 = 4$ is particular. Let us assume that we start with $n\lambda^2 > 4$ and not too big z. From the second equation we see that z will decrease, and therefore also n^{-1} as can be seen from the first equation. When we reach $z = 0$ while still having $n(\xi)\lambda^2 > 4$ the parameters will not change any more and the superfluid density is finite $n_s\lambda^2 \geq 4$. If we reach values with $n\lambda^2 < 4$ and finite $z > 0$, any further integration towards larger ξ will increase z (2nd equation) and decrease n (first equation) unless will will get $n = \infty$, giving $n_s = 0$.

We see that the theory of Kosterlitz and Thouless predicts a discontinuity in the superfluid density which will jump from zero to $4/\lambda^2$ at the transition. The critical behaviour around the transition can be further analyzed linearizing the recurrence relations. We simply note that the coherence length, ζ, above the transition will rise and show an essential singularity

$$\zeta \propto \xi \exp\left[\frac{\pi^2}{4b\sqrt{(T-T_c)/T_c}}\right] \quad (70)$$

D. Summary of Kosterlitz-Thouless results, BEC vs superfluidity (finite systems)

We have seen that the Kosterlitz-Thouless transition takes place at $n_s\lambda^2 = 4$ with a jump of the superfluid density from $n_s = 0$ above T_c to $n_s = 2mk_BT/\pi\hbar^2$. However, we do not know the critical temperature in absolute values. Below T_c, the first-order correlation function has an algebraic decay, $g_1(r) = r^{-\eta(T)}$ with $\eta = 1/n_s\lambda^2$,

The algebraic decay is quite small, so for any finite system we expect a significant condensate fraction

$$n_0 = \frac{N_0}{N} = \frac{1}{N}\int_0^L d^2\mathbf{r}g_1(r) \sim \frac{L^{2-\eta}}{L^2} \sim N^{-\eta/2} \quad (71)$$

since $\eta(T_c) = 1/4$, and gets smaller below T_c. Therefore, if we have some residual condensate fraction at a given finite size, we need 10^8 more particles to suppress it by 0.1. The absence of long-range order is only guaranteed for extremely large, macroscopic samples. For mesoscopic samples, e.g. atomic gases with $N \sim 10^{4-5}$, it is difficult to distinguish algebraic order from long range order.

Appendix: some useful results

The vortex dipole of separation $\mathbf{d}$ centered around 0 will then give the following velocity far-field

$$\nabla\theta_d = \nabla\theta_+ + \nabla_- = -\nabla\times\hat{z}\log\frac{|\mathbf{r}-\mathbf{r}_+|}{|\mathbf{r}-\mathbf{r}_-|} \simeq -\nabla\times\hat{z}\frac{\mathbf{d}\cdot\mathbf{r}}{\mathbf{r}^2}, \quad r \gg d \quad (72)$$

References