Thermodynamics of quasi-two dimensional bosons

Markus Holzmann
LPMMC, CNRS and UGA, Grenoble
markus.markus@grenoble.cnrs.fr
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Motivation:

- dimensionality enters essentially in possible phases and phase diagram, in particular the genuine 2D phase transition of Kosterlitz and Thouless is more complicated than in three dimensions already at the lowest order
- enhanced correlation effects expected, visualization of correlations in general much easier in 2D
- important phenomena of condensed matter systems involving 2D physics are not (well) understood, e.g. high Tc superconductors, 2D metal-insulater transition (disorder)
- 2D-3D cross-over
- *quantitative* understanding of 2D-physics should be possible using atomic gases

I. INFINITE UNIFORM BOSE GAS IN TWO DIMENSIONS

A. Ideal Bose gas

1. Phase space density

Let us consider an ideal Bose gas in a two dimensional box of linear extension L and volume $V = L^2$. Using periodic boundary conditions the energy eigenstates are plane waves of energy

$$\varepsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m}, \quad \mathbf{k} = \frac{2\pi}{L}\,(n_x, n_y)\,, \quad n_x, n_y = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where m is the particle mass. The occupation number of each mode $\mathbf{k}$ of the gas at temperature T and chemical potential μ is given by the Bose-Einstein distribution

$$N_{\mathbf{k}} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1} \tag{2}$$

where $\beta = 1/k_B T$. The density, $n = N/V$, where N is the total number of bosons, writes

$$\begin{aligned} n &= \frac{1}{V}\sum_{\mathbf{k}} N_{\mathbf{k}} && (3)\\ &= \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, \frac{1}{e^{\beta(\hbar^2 k^2/2m - \mu)} - 1} && (4)\\ &= \frac{\pi}{4\pi^2}\frac{2mT}{\hbar^2}\int_0^\infty dx \frac{e^{-x+\beta\mu}}{1 - e^{-x+\beta\mu}} && (5)\\ &= -\frac{mT}{2\pi\hbar^2}\log\left[1 - e^{\beta\mu}\right] && (6) \end{aligned}$$

Introducing the thermal wavelength, $\lambda = \sqrt{2\pi\hbar^2/mT}$, we explicitly obtain the phase space density as a function of $\beta\mu$

$$n\lambda^2 = -\log\left[1 - e^{\beta\mu}\right] \tag{7}$$

Note that there is no Bose condensation in the infinite system, as the chemical potential is a regular function of temperature and density

$$\beta\mu = \log\left[1 - e^{-n\lambda^2}\right] \tag{8}$$

At large phase space density, the number of bosons in the ground state is large

$$N_0 \simeq \frac{T}{|\mu|} \simeq e^{n\lambda^2} \tag{9}$$

but not macroscopic, $N_0/V \to 0$.

2. Correlation function

The absence of long range order (BEC) is also seen in the first-order correlation function

$$g_1(r) = \langle \Psi^\dagger(r)\Psi(0)\rangle \tag{10}$$

where $\Psi(\mathbf{r})$ is the annihilation operator of a particle at position $\mathbf{r}$. Expanding in Fourier modes $\Psi(\mathbf{r}) = V^{-1/2}\sum_{\mathbf{k}} a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}$, and using $N_k = \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle$, we have

$$g_1(r) = \frac{1}{V}\sum_{\mathbf{k}} N_k e^{i\mathbf{k}\cdot\mathbf{r}} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{e^{\beta(\hbar^2k^2-\mu)}-1} \tag{11}$$

Let us discuss the low and high temperature limits.

At high temperature, in the classical regime, $n\lambda^2 \ll 1$, we have $-\beta\mu \approx \log\frac{1}{n\lambda^2} \gg 1$, and the modes are gaussian distributed

$$N_{\mathbf{k}} \;\approx\; e^{\beta(\mu-\varepsilon_{\mathbf{k}})} \approx n\lambda^2 e^{-(k\lambda)^2/4\pi} \ll 1 \tag{12}$$

leading to gaussian decay with length scale λ in the correlation function

$$g_1(r) \;\approx\; n e^{-\pi(r/\lambda)^2} \tag{13}$$

At low temperature, in the degenerate regime, $n\lambda^2 \gg 1$, we have $|\beta\mu| \approx e^{-n\lambda^2} \ll 1$. Now, only the high energy modes, $\beta\varepsilon_{\mathbf{k}} \gg 1$, are still gaussian distributed, whereas low-energy states are distributed as

$$N_{\mathbf{k}} \approx \frac{1}{\beta(\varepsilon_{\mathbf{k}}-\mu)} = \frac{2mk_BT}{\hbar^2}\frac{1}{k^2+\xi^{-2}}, \quad \text{for } k^2 \ll 2mk_BT/\hbar^2 \text{ and } \beta\mu \ll 1 \tag{14}$$

and $\xi = (2m|\mu|/\hbar^2)^{-1/2}$ becomes the relevant length scale which determines the long-range behavior of the correlation function. The Lorentzian form can be integrated

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+\xi^{-2}} = K_0(r/\xi)/(2\pi) \tag{15}$$

and leads to an exponential decay of the correlation function for large distances

$$g_1(r) = \frac{2}{\lambda^2}K_0(r/\xi) \sim \frac{2}{\lambda^2}\frac{1}{\sqrt{r/\xi}}e^{-r/\xi}, \quad \text{with } \xi = \left(2mk_BT/\hbar^2\right)^{-1/2} e^{n\lambda^2/2} \gg \lambda \tag{16}$$

Since the correlation length ξ increases exponentially with $n\lambda^2$ ($\sim 1/T$ at fixed density), it may become of the order of the system size, L, for any finite box at low enough temperature. At this point, the number of particles in the $k = 0$ mode is macroscopic

$$\frac{N_0}{N} \sim \frac{1}{n\lambda^2}\left(\frac{\xi}{L}\right)^2 \tag{17}$$

and a cross-over to the Bose condensed phase occurs around $\xi \sim L$. At zero temperature, all particles are in the condensate.

3. Classical field approximation

Notice that at high degeneracy, $n\lambda^2$, most of the bosons are occupying the low-energy modes

$$n_<\lambda^2 \equiv \lambda^2 \int_{\beta\varepsilon_k<1} \frac{d^2\mathbf{k}}{(2\pi)^2} N_k = \log \frac{1-e^{\beta\mu-1}}{1-e^{\beta\mu}} = n\lambda^2 + \log\left(1-e^{\beta\mu-1}\right) \approx n\lambda^2 - \log\frac{e}{e-1} \tag{18}$$

and we may also replace the Bose occupation by the occupation of a classical field

$$N_{\mathbf{k}} \approx N_{\mathbf{k}}^{cf} \equiv \frac{1}{\beta(\varepsilon_{\mathbf{k}}-\mu)} \tag{19}$$

which gives

$$\lambda^2 \int_{\beta\varepsilon_{\mathbf{k}}<1} \frac{d^2\mathbf{k}}{(2\pi)^2} N_{\mathbf{k}}^{cl} = \log\frac{1+|\beta\mu|}{|\beta\mu|} \approx n\lambda^2 + |\beta\mu| \tag{20}$$

A classical field theory with a simple cut-off $\Lambda = (2mk_BT/\hbar^2)^{1/2}$ for high energy modes therefore describes quantitatively the highly degenerate regime using

$$H_{cl} - \mu N = \sum_{k<\Lambda} (\varepsilon_k - \mu)\, \alpha_{\mathbf{k}}^* \alpha_{\mathbf{k}} \approx \sum_{\mathbf{r}} \left[-\frac{\hbar^2}{2m} |\nabla\psi(\mathbf{r})|^2 - \mu|\psi(\mathbf{r})|^2 \right] \tag{21}$$

where $\alpha_{\mathbf{k}}$ ($\psi(\mathbf{r})$) are complex numbers which describe the classical field, the real space is discrete with minimum distance $\sim \Lambda^{-1}$. The probability distribution for a given field configuration, $\psi(\mathbf{r})$, is then

$$p[\psi(\mathbf{r})] = Z_{cl}^{-1} e^{-\beta(H_{cl}-\mu N)} \tag{22}$$

where $Z_{cl} = \int D\psi(r) e^{-\beta H_{cl}}$ is the partition function.

B. Interactions in quasi2D

The microscopic interaction potential between two atoms is given by $v(|\vec{r}|)$, where $\vec{r} = (\mathbf{r}, z)$ denotes the distance vector in three dimension and the interaction part of the full 3d Hamiltonian writes

$$V_{3d} = \frac{1}{2} \int d^3\vec{r} \int d^3\vec{r'} \Psi^\dagger(\vec{r}) \Psi^\dagger(\vec{r'}) v(|\vec{r}-\vec{r'}|) \Psi(\vec{r'}) \Psi(\vec{r}) \tag{23}$$

Let us now assume that the atoms are tightly confined by a very strong external potential, $u(z)$. Without interactions, the single particles eigenfunctions write $\psi_{\mathbf{k}}(\mathbf{r})\varphi_\nu(z)$, where $\psi_{\mathbf{k}}(\mathbf{r})$ are plane waves describing the in-plane motion, and $\varphi_\nu(z)$ is the bound state wavefunction in z with energy E_ν^z. The excitation energy to the first excited state in z, $\hbar\omega_z \equiv E_1^z - E_0^z$, introduces the energy scale which characterizes the quasi-two-dimensional behavior. At low temperatures, $T \ll \hbar\omega_z$, and low chemical potential $\mu \ll \hbar\omega_z$, we expect that the z-motion is frozen: all particles occupy the ground state in the confined direction with undisturbed density distribution $\varphi_0^2(z)$, of typical extension, a_z, which we define $l_z^{-1} = \sqrt{2\pi}\int dz \varphi_0^4(z)$ (in analogue to gaussian/harmonic oscillator eigenfunctions). We can integrate out the motion in z to obtain the effective 2D Hamiltonian, and for the quasi-two-dimensional interaction potential we expect

$$v_{q2d}(\mathbf{r},\mathbf{r}') = \int dz \int dz' |\varphi_0(z)|^2 |\varphi_0(z')|^2 v(\sqrt{(\mathbf{r}-\mathbf{r}')^2 + (z-z')^2}) \tag{24}$$

Atomic gases are characterized by a range of the interaction potential, r_0, which is much smaller than the typical distance between atoms, $nr_0^3 \ll 1$, so that for typical thermodynamic properties the interaction can be (heuristically) described by a contact potential

$$v(\vec{r}) = \frac{4\pi\hbar^2 a_s}{m} \delta^3(\vec{r}) \tag{25}$$

where a_s is the 3d scattering length. We thus get

$$v_{q2d}(\mathbf{r}) = \frac{\hbar^2}{m}\tilde{g}\delta^2(\mathbf{r}), \quad \tilde{g} = \sqrt{8\pi}\frac{a_s}{l_z} \tag{26}$$

introducing the dimensional coupling constant $\tilde{g}$ which characterizes the strength of the interaction. Notice, that $\tilde{g} \ll 1$ only requires $a_s \ll l_z$, and the coupling constant is independent of the actual 2D density.

Actually, the above derivation can only give the leading order term in the limit $a_s/l_z \to 0$, since the scattering wave function of two particles must become isotropic for small distances and higher excitations in z get necessarily involved. Solving the two-body problem in the quasi-2D geometry, the s-wave scattering amplitude can be derived. For harmonic confinement the effective 2D scattering amplitude writes

$$f(\epsilon) = \frac{\sqrt{8\pi}a_s/l_z}{1 + a_s/l_z[\log(B\hbar\omega_z/\pi\epsilon) + i\pi]/\sqrt{2\pi}}, \quad B \simeq 0.915 \tag{27}$$

The logarithmic depends on the energy $\epsilon = \hbar^2 q^2/m$ of the colliding particles with relative momentum $\mathbf{q}$ is typical for two dimensional scattering. We can now use any two-dimensional (short range) interaction $v_{q2d}(r)$ which reproduces the scattering length above.

For a dilute gas, the typical energy is $\epsilon = k_B T$, whereas at zero temperature it is given by the chemical potential, $\epsilon = 2\mu$. In the limit $\epsilon \ll \hbar\omega_z e^{\sqrt{2\pi}(a_s/l_z)^{-1}}$ we can further neglect the imaginary part and use a contact potential with coupling constant $\tilde{g} = f(\epsilon)$.

C. Interacting Bose gas: Density and phase fluctuations

In second quantization the Hamiltonian of the interacting system reads

$$H = \int d^2\mathbf{r}\Psi^\dagger(\mathbf{r})\left[-\frac{\hbar^2}{2m}\Delta + \frac{g}{2}\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\right]\Psi(\mathbf{r}) \tag{28}$$

where the coupling constant is chosen to reproduce the effective 2D scattering properties and the Heisenberg equation of motion gives

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = [\Psi(\mathbf{r},t), H] = \left[-\frac{\hbar^2}{2m}\Delta + g\Psi^\dagger(\mathbf{r},t)\Psi(\mathbf{r},t)\right]\Psi(\mathbf{r},t) \tag{29}$$

For the ideal gas, we have already seen that at low temperatures (high degeneracy), the wave character of the atoms dominate and the system is well described by classical fields. Similar, we expect that the classical field theory is accurate for the weakly interacting case ($\tilde{g} \to 0$)

$$H_{cl} = \int_\Lambda d^2\mathbf{r}\left\{\frac{\hbar^2}{2m}|\nabla\Psi(\mathbf{r})|^2 + \frac{g}{2}|\Psi(\mathbf{r})|^4\right\} \tag{30}$$

where $\Lambda \sim (mk_BT/\hbar^2)^{-1/2}$ indicates that the integrals must be regularized at small distances, e.g. by a lattice (we will drop Λ in the following).

At low temperatures, we expect that the potential energy $\sim gn^2/2$ dominates over the kinetic energy $\lesssim \hbar^2 n/2m\Lambda^2$. In this region ($n\lambda^2 \gg \pi/\tilde{g}$), the density is approximately gaussian distributed

$$p[\psi(\mathbf{r})] \sim \exp[-\beta(H_{cl} - \mu N)] \sim \exp\left[-\beta\int d^2\mathbf{r}\frac{g}{2}\left(|\Psi(\mathbf{r})|^2 - \mu/g\right)^2\right] \tag{31}$$

The fluctuations of the density around its mean value, $n \equiv \langle\langle n(\mathbf{r})\rangle = |\psi(\mathbf{r})|^2\rangle = \mu/g$, are gaussian distributed,

$$\Delta n^2 \equiv \langle n^2(r)\rangle - n^2 \sim \frac{nk_BT}{g}, \quad n\lambda^2 \gg \pi/\tilde{g} \tag{32}$$

and therefore highly suppressed, $\Delta n^2/n^2 \sim [gn\lambda^2]^{-1} \simeq 0$. This is in striking contrast to the ideal gas case, where the kinetic energy can never be neglected at any finite temperature and the Fourier modes are gaussian distributed and we have $\langle n^2\rangle = 2n^2$ with large density fluctuations as in any gas at high temperature. In particular, due to the interactions, the density fluctuations smoothly vanish with temperature and rejoin $\Delta n^2 = 0$ for the condensate at $T = 0$ (in leading order).

1. Phonon Hamiltonian

To include better thermal fluctations at low temperature, we separate amplitude and phase

$$\Psi(\mathbf{r}) = |A(\mathbf{r})|e^{i\varphi(\mathbf{r})} \tag{33}$$

assuming small density fluctuations, $A(\mathbf{r}) = \sqrt{n[1+2\alpha(\mathbf{r})]}$, with $|\alpha(\mathbf{r})| \ll 1$ and $\int d^2\mathbf{r}\,\alpha(\mathbf{r}) = 0$. Up to quadratic terms, the Hamiltonian writes

$$H_{cl} - \mu N \simeq \int d^2\mathbf{r}\frac{\hbar^2 n}{2m}\left[\nabla\varphi(\mathbf{r})\right]^2 + \int d^2\mathbf{r}\left\{\frac{\hbar^2 n}{2m}\left[\nabla\alpha(\mathbf{r})\right]^2 + 2gn^2\alpha^2(\mathbf{r})\right\} + \left[\frac{gn^2}{2} - \mu n\right]V \tag{34}$$

Minimization with respect to n, fixes the chemical potential $\mu = gn$.

The Hamiltonian is diagonal in Fourier space

$$\varphi(\mathbf{r}) = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\varphi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \alpha(\mathbf{r}) = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\alpha_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{35}$$

with $\varphi^*_{\mathbf{k}} = \varphi_{-\mathbf{k}}$ and $\alpha^*_{\mathbf{k}} = \alpha_{-\mathbf{k}}$ since both fields, $\varphi(\mathbf{r})$ and $\alpha(\mathbf{r})$, are real. We get (dropping an overall constant term)

$$H_{cl} - \mu N = \sum_{\mathbf{k}}\frac{\hbar^2 k^2}{2m}|\varphi_{\mathbf{k}}|^2 + \sum_{\mathbf{k}}\left(\frac{\hbar^2 k^2}{2m} + 2gn\right)|\alpha_{\mathbf{k}}|^2 \tag{36}$$

and we can read of the strengths of the fields at thermal equilibrium

$$\langle\varphi_{-\mathbf{k}}\varphi_{\mathbf{k}}\rangle = \frac{1}{2}\frac{1}{\beta\varepsilon_k}, \quad \langle\alpha_{-\mathbf{k}}\alpha_{\mathbf{k}}\rangle = \frac{1}{2}\frac{1}{\beta(\varepsilon_k + 2gn)} \tag{37}$$

where the factor $1/2$ originates from the fact that we integrate over a real field (for each mode $\mathbf{k} > 0$, we split $\varphi_{\mathbf{k}} = \varphi'_{\mathbf{k}} + i\varphi''_{\mathbf{k}}$ in real and imaginary part and get gaussian integrations for $\varphi'_{\mathbf{k}}$ and $\varphi''_{\mathbf{k}}$; the factor $1/2$ arises, since two terms, $\mathbf{k}$ and $-\mathbf{k}$, of the summation in the Hamiltonian contribute).

2. Density fluctuations

We see that density fluctuations are gapped, $\langle|\alpha_{\mathbf{k}}|^2\rangle \to [2n\beta 2gn]^{-1}$, for $\mathbf{k} \to 0$, whereas phase fluctuations are gapless, $\lim_{k\to 0}\langle|\varphi_{\mathbf{k}}|^2\rangle \to \infty$. As a consequence, phase fluctuations will develop long-range correlations, and density fluctuations remain short-ranged

$$\frac{\langle n(\mathbf{r})n(0)\rangle - n^2}{n^2} = 4\langle\alpha(\mathbf{r})\alpha(0)\rangle \tag{38}$$

$$\simeq 4\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{2n\beta(\varepsilon_k + 2gn)} = \frac{4}{n\lambda^2}K_0(r/\xi) \sim \frac{4}{n\lambda^2}\frac{1}{r/\xi}e^{-r/\xi}, \quad \text{for } r \gg \xi \tag{39}$$

where $\xi = \sqrt{\hbar^2/2mgn}$ is the healing length, $(\xi/\lambda)^2 = (2\tilde{g}n\lambda^2)^{-1}$. Density correlations (at equal distances) are in general suppressed

$$\frac{\Delta n^2}{n^2} \simeq 4\langle\alpha^2(0)\rangle = 4\int_{k<\Lambda}\frac{d^2\mathbf{k}}{(2\pi)^2}\langle|\alpha_{\mathbf{k}}|^2\rangle = \frac{8m}{8\pi n\hbar^2\beta}\int_0^1\frac{dx}{x + 2gn\beta} \simeq \frac{2}{n\lambda^2}\log\frac{\pi}{\tilde{g}n\lambda^2} = \frac{2}{n\lambda^2}\log[2\pi(\xi/\lambda)^2] \tag{40}$$

The same result is obtained from the real-space calculation of the density fluctuations in the limit $r \sim \Lambda^{-1} \approx 0$ where $K_0(r/\xi) \simeq -\log(r/\xi)$ (note that the classical theory is not defined for strictly vanishing distances!).

3. Phase fluctuations

Now, phase fluctuations are more delicate. Let us first look at the phase correlations at zero distance

$$\langle\varphi^2(\mathbf{r})\rangle = \frac{1}{V}\sum_{\mathbf{k}\neq 0}\langle|\varphi_{\mathbf{k}}|^2\rangle \approx \frac{1}{2n\lambda^2}\int_{\pi^2/L^2}^{\Lambda^2}d(k^2)\frac{1}{k^2} \simeq \frac{1}{2n\lambda^2}\int_0^{\Lambda^2}d(k^2)\frac{1}{k^2 + \pi^2/L^2} \simeq \frac{1}{n\lambda^2}\log L\Lambda/\pi \tag{41}$$

which diverges $\sim \log L/\lambda$ for large systems. In addition, correlations are present, even at large distances

$$\langle[\varphi(\mathbf{r})-\varphi(0)]\varphi(0)\rangle = \frac{1}{V}\sum_{\mathbf{k}\neq 0}\langle|\varphi_{\mathbf{k}}|^2\rangle\left[e^{i\mathbf{k}\cdot\mathbf{r}}-1\right] = \frac{2\pi}{n\lambda^2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}-1}{k^2+\pi^2/L^2} \tag{42}$$

$$\simeq \frac{1}{n\lambda^2}\left[K_0(\pi r/L)-K_0(\pi/(L\Lambda))\right] \simeq -\frac{1}{n\lambda^2}\log r\Lambda \tag{43}$$

which grow logarithmically with r.

Let us note, that

$$G(r) = \lim_{L\to\infty}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1-e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+\pi^2/L^2} \simeq \frac{1}{2\pi}\log r\Lambda \tag{44}$$

satisfies also the 2D-Poisson equation

$$\nabla^2 G(r) = \delta^2(\mathbf{r}) \tag{45}$$

and therefore corresponds to the Green's function of the 2D-Coulomb potential.

D. Low T: Algebraic ordered phase (Berenzinskii)

Let us now use the results derived in the last section to calculate the first order correlation function of an interacting Bose gas at low temperatures. Within the classical field approximation, separating amplitude from phase fluctuations, we get

$$g_1(\mathbf{r}) = \langle\Psi^*(\mathbf{r})\Psi(0)\rangle = n\langle\sqrt{1+2\alpha(\mathbf{r})}\sqrt{1+2\alpha(0)}e^{-i[\varphi(\mathbf{r})-\varphi(0)]}\rangle \tag{46}$$

Using our effective Hamiltonian, Eq.(36), amplitude and phase fluctuations are independent and we get

$$g_1(r) \simeq n\left\{1+\langle[\alpha(\mathbf{r})-\alpha(0)]\alpha(0)\rangle\right\}\langle e^{-i[\varphi(\mathbf{r})-\varphi(0)]}\rangle \tag{47}$$

where we have expanded the root, $\sqrt{1+2x}\simeq 1+x-x^2/2$, restricting up to quadratic terms in α. We can directly use our previous results, Eq. (39), to obtain the contributions of the density fluctuations

$$\langle|\Psi^*(\mathbf{r})\Psi(0)|\rangle \simeq n\left\{1+\frac{1}{n\lambda^2}\left[K_0(r/\xi)-K_0(1/(\xi\Lambda))\right]\right\} \to n\left[1-\frac{\Delta n^2}{4n^2}+\frac{1}{n\lambda^2}\frac{e^{-r/\xi}}{r/\xi}\right], \quad \text{for } r\gg\xi=\lambda/\sqrt{2\tilde{g}n\lambda^2} \tag{48}$$

which approaches a constant at large distances. Since we expect that the density correlations grow monotonously from $\Delta n^2=0$ at $T=0$ to $\Delta n^2=n^2$ at high temperatures, the amplitude fluctuations never vanishes at large distances within our approximations.

The term due to phase fluctuations is also rapidly evaluated using that $\langle e^{iu}\rangle = \exp[-\langle u^2\rangle/2]$ assuming that u is gaussian distributed. We have $\langle[\varphi(\mathbf{r})-\varphi(0)]^2\rangle = -2\langle[\varphi(\mathbf{r})-\varphi(0)]\varphi(0)\rangle$ and we can use the phase correlation function calculated above, Eq. (43), to obtain

$$\langle e^{-i[\varphi(\mathbf{r})-\varphi(0)]}\rangle = \exp\left[\langle[\varphi(\mathbf{r})-\varphi(0)]\varphi(0)\rangle\right] = \exp\left[-\frac{2\pi}{n\lambda^2}G(r)\right] \approx \left(\frac{\Lambda}{r}\right)^{1/n\lambda^2} \tag{49}$$

which determines the algebraic decay of the first order correlation function

$$g_1(r) \sim n\left[1-\frac{\Delta n^2}{4n^2}\right]\left(\frac{\Lambda}{r}\right)^{1/n\lambda^2}, \quad r\gg\xi \tag{50}$$

In the thermodynamic limit (infinite large systems), Bose condensation is therefore absent at any finite temperature, as g_1 approaches zero macroscopic distances. The absence of long range order in two dimensions is a general feature (theorem of Hohenberg, Mermin and Wagner, and others). However, the algebraic decay is very slow at low temperature and gives rise to a new (superfluid) phase, distinct from the usual high-temperature disordered phase, as pointed out by Berezinskii.

Let us discuss the consequences of the algebraic decay for the momentum distribution, obtained by Fourier transform of $g_1(r)$

$$N_k = \int d^2\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} g_1(r) \tag{51}$$

Dimensional analysis immediately shows that

$$N_k \sim \frac{n\lambda^2}{(k\lambda)^{2-1/n\lambda^2}}, \quad k \to 0 \tag{52}$$

By continuation to $k = 0 \sim \mathcal{O}(L^{-2})$, the number of particles in the condensate scales as

$$N_0 \sim L^{2-1/n\lambda^2} \tag{53}$$

Although the condensate can be quite large, it is never macroscopic $N_0/V \to 0$ for $V \to \infty$.

Within our approximations, the algebraic order remains up to arbitrary large temperature, which is clearly an artifact. At high temperature, we expect $N_0 \sim N_k \gtrsim 1$, so that our low temperature approximations are certainly invalid for $n\lambda^2 \lesssim 1/2$. Actually, the transition occurs earlier, as we will discuss later.

Appendix: explicit calculation of $\langle \exp[i(\varphi(r) - \varphi(0))]\rangle$

Let us explicitly calculate $\langle \exp[i(\varphi(r) - \varphi(0))]\rangle$ for the Hamiltonian Eq. (34). Using Fourier space we have

$$\langle \exp[i(\varphi(r) - \varphi(0))]\rangle = \frac{\int D\varphi(\mathbf{r}) e^{i[\varphi(r)-\varphi(0)]} e^{-\frac{\hbar^2 n}{2m}\int d^2\mathbf{r}[\nabla\varphi(\mathbf{r})]^2}}{\int D\varphi(\mathbf{r}) e^{-\frac{\hbar^2 n\beta}{2m}\int d^2\mathbf{r}[\nabla\varphi(\mathbf{r})]^2}} \tag{54}$$

$$= \prod_{\mathbf{k}>0} \frac{\int d\varphi_{\mathbf{k}} d\varphi^*_{\mathbf{k}} \exp\left[i\varphi_{\mathbf{k}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} - 1\right]/V^{1/2} - \beta n\hbar^2 k^2 |\varphi_{\mathbf{k}}|^2/2m + c.c.(\text{term with } -\mathbf{k})\right]}{\int d\varphi_{\mathbf{k}} d\varphi^*_{\mathbf{k}} \exp\left[-2\beta n\hbar^2 k^2 |\varphi_{\mathbf{k}}|^2/2m\right]} \tag{55}$$

$$= \prod_{\mathbf{k}>0} \frac{\int da db \exp\left[(2ia\left[\cos\mathbf{k}\cdot\mathbf{r} - 1\right]/V^{1/2} - 2ib\left[\sin\mathbf{k}\cdot\mathbf{r} - 1\right]/V^{1/2} - \beta n\hbar^2 k^2(a^2+b^2)/m\right]}{\int da db \exp\left[-\beta n\hbar^2 k^2(a^2+b^2)/m\right]}$$

$$= \prod_{\mathbf{k}>0} \exp\left[-\frac{2}{V}\frac{1-\cos\mathbf{k}\cdot\mathbf{r})}{2\beta n\hbar^2/2m}\right] = \exp\left[\frac{1}{V}\sum_{\mathbf{k}\neq 0}\langle|\varphi_{\mathbf{k}}|^2\rangle(\cos\mathbf{k}\cdot\mathbf{r} - 1)\right] \tag{56}$$

$$= \exp\left[\langle[\varphi(\mathbf{r}) - \varphi(0)]\varphi(0)\rangle\right] = \exp\left[-\frac{1}{2}\langle[\varphi(\mathbf{r}) - \varphi(0)]^2\rangle\right] \tag{57}$$

E. Superfluidity

Although Bose condensation and superfluidity frequently occur together (in three dimensions), they are distinct phenomena. For (ideal and interacting) Bose gases, condensation is absent in the thermodynamic limit at any finite temperature as shown above. In the following we discuss the description of superfluidity and show that it persists at low temperatures for interacting gases.

The Hess-Fairbank effect shows that superfluidity is a thermodynamic property. Rotating very slowly the container of the system, only the normal part of the fluid will rotate with the container in the stationary regime, the superfluid part will remain at rest. To calculate the superfluid response, we therefore consider first the response to a perturbation coupling to the momentum in general.

1. *Longitudinal and transverse response*

The momentum density operator (m times the current operator), $\pi(r)$, writes

$$\pi(\mathbf{r}) = m\mathbf{j}(\mathbf{r}) = \frac{\hbar}{2i}\left\{\Psi^\dagger(\mathbf{r})[\nabla\psi(\mathbf{r})] - [\nabla\Psi^\dagger(\mathbf{r})]\Psi(\mathbf{r})\right\} \tag{58}$$

and the static response of the momentum density with respect to some perturbation coupling to the momentum is a tensor, $\chi_{ij}(\mathbf{r},\mathbf{r}')$, proportional to the current-current correlation function

$$\chi_{ij}(\mathbf{r},\mathbf{r}') = \int_0^\beta d\tau \langle \pi_i(\mathbf{r},0)\pi_j(\mathbf{r}',\tau)\rangle \tag{59}$$

where $\mathbf{g}(\tau)$ is the momentum density at imaginary time τ. For an isotropic system, the tensor is diagonal in Fourier space

$$\chi_{ij}(\mathbf{k}) = \int d^2\mathbf{r} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\chi_{ij}(\mathbf{r},\mathbf{r}') \tag{60}$$

and the only tensors we can construct are δ_{ij} and $k_i k_j$, so that its general form can always be written as

$$\chi_{ij}(\mathbf{k}) = \frac{k_i k_j}{k^2}\chi_L(k) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\chi_T(k) \tag{61}$$

introducing the longitudinal and transverse response, $\chi_L(k)$ and $\chi_T(k)$, respectively.

Imagine that we disturb our system applying our an external field in x direction, and measuring the response in the same direction, we get $\chi_{11}(\mathbf{k})$. If the external fields is uniform in the transverse (y,z) direction (it has purely longitudinal character), we have $k_y = k_z = 0$, but $k_x \neq 0$, so that $k_x k_x/k^2 = 1$, and we get the longitudinal response, $\chi_L(k)$. Alternatively we can apply a pure transverse field, uniform in x, but with spatial variation in one of the transverse directions, e.g. y. We have $k_x = 0$, but $k_y, k_z \neq 0$, so that $k_x k_x/k^2 = 0$, and we get the transverse response, $\chi_T(k)$.

2. Normal density

Now, applying an external perturbation which couples to the momentum, we can properly define superfluidity. Let us put our gas in a ring geometry, with large (eventually infinite) diameter. Moving the walls of the ring with a very small velocity, we expect the superfluid fraction to remain at rest, only the normal fluid will move with the walls at equilibrium. The movement of the walls in the ring with infinite diameter corresponds to a transverse perturbation, so that $\chi_T(k)$ contains essentially the response of the normal part of the fluid. To eliminate the bulk response, we better take $k \to 0$ so that

$$\rho_n \equiv m(n - n_s) = \chi_T(k \to 0) \tag{62}$$

where ρ_n defines the normal mass density. The longitudinal response is different. For the ring it introduces perturbations along the ring. At equilibrium, the whole fluid must react to a longitudinal perturbation to reach a stationary state and we have

$$\rho = mn = \chi_L(k \to 0) \tag{63}$$

This relation also follows from the continuity equation of the density and is called f-sum rule.

Note that for classical systems, momenta are always gaussian distributed (assuming velocity independent forces), so that $\chi_i j(\mathbf{k}) \propto \delta_{ij}$ and $\rho_n = \rho$. Superfluidity is a pure quantum phenomena.

3. Results for Phonon Hamiltonian/ Spin-wave approximation

Using density and phase variables, the momentum density of our classical fields writes

$$\pi_i(\mathbf{r}) = \hbar \mathrm{Im}\Psi^*(\mathbf{r})\nabla_i\Psi(\mathbf{r}) = \hbar|\Psi(\mathbf{r})|^2\nabla\varphi(\mathbf{r}) \tag{64}$$

Since the classical fields do not depend on imaginary time, we get for the momentum response

$$\chi_{ij}(\mathbf{r},0) = \beta n^2\hbar^2\left[1 + 4\langle\alpha(\mathbf{r})\alpha(0)\rangle\right]\langle\nabla_i\varphi(\mathbf{r})\nabla_j\varphi(0)\rangle \tag{65}$$

where we have already used our low temperature expansion of the Hamiltonian, quadratic in phase and density fluctuations. Since density correlations are short ranged, we can neglect the term involving $\alpha(\mathbf{r})$ in the following where we concentrate on the long-range limit. Inserting the Fourier transform, we get

$$\chi_L(k) = \beta n\hbar^2 k^2\langle|\varphi_k|^2\rangle = mn, \quad \chi_T(k) = 0, \quad \text{for } k \to 0 \tag{66}$$

Therefore our system is superfluid in the low-temperature phase and within our approximation the superfluid density, n_s, equals the total density, $n = n_s$. Since the density fluctuations do not modify the superfluid density, nor the algebraic decay, they are frequently dropped, and only the phase gradient term in the Hamiltonian is left (spin-wave approximation).

F. $T = 0$: Bogoliubov analysis

Our discussion at finite temperature indicate that Bose condensation should occur at zero temperature, e.g. $g_1(r) \sim 1$ for large r in the limit $n\lambda^2 \to \infty$. However, our classical field theory neglects quantum fluctuations which may become important in this limit. To include them, we have to quantize our excitation.

Therefore, field and density fluctuations have to satisfy the commutation relation

$$2n[\alpha(\mathbf{r}), \varphi(\mathbf{r}')] = i\delta^2(\mathbf{r} - \mathbf{r}') \tag{67}$$

(Note that the phase is ill defined in the continuum theory, and we should rather use a lattice version.) We obtain this commutation relation, by splitting separating real and imaginary part in the Fourier composition (only $\mathbf{k} > 0$ since it is real), $\alpha_{\mathbf{k}} = (x_{\mathbf{k}} + ix_{-\mathbf{k}})/2$, and $\varphi_{\mathbf{k}} = (p_{\mathbf{k}} + ip_{-\mathbf{k}})/\hbar$, and quantizing the x's and p's as usual: $[x_{\mathbf{k}}, p_{\mathbf{k}}] = i\hbar$. Our Hamiltonian then has the standard form of an harmonic oscillator for each mode $\mathbf{k}$

$$H = \sum_{\mathbf{k}\neq 0} \frac{p_{\mathbf{k}}^2}{2M_{\mathbf{k}}} + \frac{1}{2} M_{\mathbf{k}} \omega_{\mathbf{k}}^2 x_{\mathbf{k}}^2 \tag{68}$$

with $M_{\mathbf{k}} = \hbar^2/\varepsilon_{\mathbf{k}}$ and $M_{\mathbf{k}}\omega_{\mathbf{k}}^2/2 = (\varepsilon_{\mathbf{k}} + 2gn)/2$. Introducing the usual annihilation/ creation operator of the harmonic oscillator, $a_{\mathbf{k}} = \sqrt{m\omega/2\hbar}(x_{\mathbf{k}} + ip_{\mathbf{k}}/m\omega)$, we diagonalize the Hamiltonian

$$H = \sum_{\mathbf{k}\neq 0} \hbar\omega_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right), \quad \hbar\omega_{\mathbf{k}} = \sqrt{\varepsilon_k \left(\varepsilon_k + 2gn \right)} \tag{69}$$

We can now calculate the first order correlation function, expanding $\Psi = n^{1/2} e^{i\varphi} \sqrt{1 + 2\alpha}$ (the order is imposed to recover the usual commutation relation for Ψ) up to second order $\Psi \simeq n^{1/2}[1 + i\varphi - \varphi^2/2][1 + \alpha - \alpha^2/2]$ treating α and φ as operators

$$\begin{aligned}
g_1(r) &= \langle \Psi^\dagger(\mathbf{r}) \Psi(0) \rangle & (70)\\
&\simeq n \{ 1 + \langle [\alpha(\mathbf{r}) - \alpha(0)]\alpha(0) + [\varphi(\mathbf{r}) - \varphi(0]\varphi(0) - i\varphi(\mathbf{r})\alpha(0) + i\alpha(\mathbf{r})\varphi(0) - i\alpha(\mathbf{r})\varphi(\mathbf{r}) + i\varphi(0)\alpha(0) \rangle \} & (71)\\
&= n \{ 1 + \langle [\alpha(\mathbf{r}) - \alpha(0)]\alpha(0) + [\varphi(\mathbf{r}) - \varphi(0)]\varphi(0) \rangle \} - \delta^2(\mathbf{r})/2 + \delta^2(0)/2 & (72)
\end{aligned}$$

where we have used isotropy and the commuation relation. We then get

$$\begin{aligned}
g_1(r) &= n + \frac{1}{2V} \sum_{\mathbf{k}\neq 0} \left[\langle x_{\mathbf{k}}^2 \rangle + \langle p_{\mathbf{k}}^2 \rangle / \hbar^2 - 1 \right] \left(e^{i\mathbf{k}\cdot\mathbf{r}} - 1 \right) & (73)\\
&= n + \frac{1}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{\varepsilon_k}{2\hbar\omega_k} + \frac{\hbar\omega_k}{2\varepsilon_k} - 1 \right] \left(e^{i\mathbf{k}\cdot\mathbf{r}} - 1 \right) & (74)
\end{aligned}$$

where we have used that $m\omega^2\langle x^2 \rangle/2 = \langle p^2 \rangle/2m = \hbar\omega/4$ in the ground state of an harmonic oscilator. The integral converges both for $\mathbf{k} \to 0$ and $\mathbf{k} \to \infty$, so that its value is finite for any distance. In the limit $r \to \infty$ we approach the condensate density

$$n_0 = n - \frac{1}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{\varepsilon_k}{2\hbar\omega_k} + \frac{\hbar\omega_k}{2\varepsilon_k} - 1 \right] = n \left(1 - \frac{\tilde{g}}{4\pi} \right) \tag{75}$$

and we have

$$\begin{aligned}
g_1(r) &= n_0 + \frac{1}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{\varepsilon_k}{2\hbar\omega_k} + \frac{\hbar\omega_k}{2\varepsilon_k} - 1 \right] e^{i\mathbf{k}\cdot\mathbf{r}} & (76)\\
&\to n_0 + \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\sqrt{8}\xi k} e^{i\mathbf{k}\cdot\mathbf{r}} = n_0 + \frac{1}{\sqrt{2}4\pi\xi r}, \quad r \to \infty & (77)
\end{aligned}$$

where $\xi = (2\tilde{g}n)^{-1/2}$ is the usual healing length of the condensate.

We see, that we have true Bose condensation at zero temperature in two dimensions. We note, that the superfluid density is equal to the total density, since the phonon excitations have only longitudinal character.

From our phonon calculation, we see that for $\tilde{g} \gtrsim 4\pi$, the gas may not condense even at zero temperature and a quantum phase transition to a normal state may occur.