Quantum Information and Quantum Computing, Problem set 3

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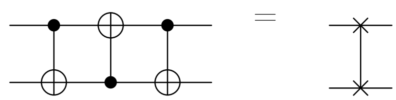
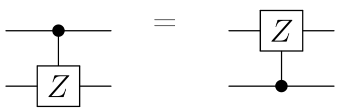
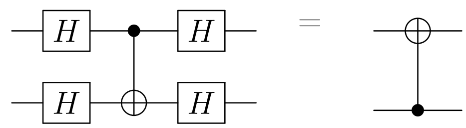
Problem 1 : Elementary quantum circuits

1. Show that this circuit is a quantum swap, i.e. it swaps the $|0\rangle$ and $|1\rangle$ states.



2. Prove the following equalities





3. For $U = V^2$, with V a single-qubit unitary, construct a $C^5(U)$ gate without using ancilla qubits. You may use multiply (up to 4 control qubits) controlled-V and $V^\dagger$ gates in addition to the universal set of gates H, S, T, CNOT.

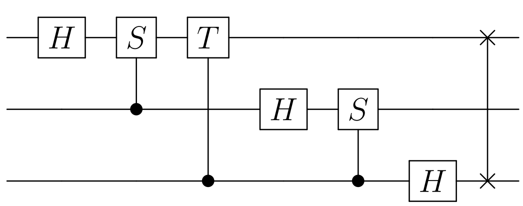
4. The discrete Fourier transform from a set of complex numbers $x_0, \ldots, x_{N-1}$ to a set of complex numbers $y_0, \ldots, y_{N-1}$ is defined as

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

The quantum Fourier transform is defined in a Hilbert space of dimension N as the unitary transformation

$$\sum_{j=0}^{N-1} x_j |j\rangle \rightarrow \sum_{k=0}^{N-1} y_k |k\rangle$$

where $|j\rangle$ and $|k\rangle$ are vectors of the computational basis. Show that the following circuit performs the quantum Fourier transform for $N = 2^3$. Write the corresponding unitary matrix (define $\omega = e^{2\pi i/8}$).



5. Find the quantum circuits that, starting from the state $|00\rangle$ as an input, generate the four Bell states $(|00\rangle + |11\rangle)/\sqrt{2}$, $(|00\rangle - |11\rangle)/\sqrt{2}$, $(|01\rangle + |10\rangle)/\sqrt{2}$, $(|01\rangle - |10\rangle)/\sqrt{2}$.

Problem 2 : Quantum teleportation

The quantum teleportation protocol enables the transfer the quantum state of a system (on the side of Alice) to another remote system (on the side of Bob). Notice that an important result of quantum mechanics, the *no-cloning theorem* plays a fundamental role here. The no-cloning theorem states that the arbitrary (i.e. unknown) state of a quantum system cannot be copied onto another system of the same kind. If we could clone an arbitrary state, then we could achieve quantum state teleportation in a straightforward way by simply cloning the state. The peculiarity of this protocol is that the state is transferred – not cloned – without any knowledge of the state itself.

In this protocol, Alice has one qubit – which we denote as qubit 1 – initially set in the arbitrary state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1}$$

Alice wishes to transfer to Bob the quantum information stored in qubit 1 – namely the complex coefficients α and β – even if she doesn't know herself the values of these two coefficients. To this purpose, Alice and Bob each possess one qubit of a pair of qubits – denoted here qubit 2 and 3 – that are set to the entangled state

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{2}$$

The state of this pair of qubits has been initially set by Alice, Bob, or by a third actor, and the two qubits have subsequently been distributed to Alice and Bob. The initial state of the whole system is therefore

$$|\Psi_0\rangle = |\psi\rangle \otimes |\beta_{00}\rangle = (\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \frac{|0\rangle_2 \otimes |0\rangle_3 + |1\rangle_2 \otimes |1\rangle_3}{\sqrt{2}}, \tag{3}$$

where the indices are here only used to recall the order in which the three qubits enter into the tensor products. This notation can be dropped in what follows, knowing that qubit 3 is the one owned by Bob.

1. Alice applies to her two qubits a Controlled-NOT gate.[1] Compute the total state $|\Psi_1\rangle$ of the three qubits after this first operation.

2. Alice then applies to qubit 1 a Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4}$$

What is the total state $|\Psi_2\rangle$ after this step?

[1] A C-NOT gate is defined by its action on the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. If qubit 1 is set to zero then both qubits are untouched. If qubit 1 is set to one, then qubit 2 is flipped. The operator can be written as $\mathrm{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X$, and we have computed the corresponding 4x4 matrix in Problem Set 1.

3. Alice then measures qubits 1 and 2 "in the computational basis". This wording means that she measures an observable which is diagonal and non degenerate in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ – which is usually called the computational basis. The result of this measurement answers the question "0 or 1?" for each of the two qubits on which the measurement is performed. Let's call $(x, y) \in \{0, 1\}^2$ the outcome of this measurement. What are the states of Alice's pair of qubits, and of Bob's qubit, after the measurement, for each of the 4 possible outcomes (x, y)? Alice sends the classical information (x, y) to Bob through a classical communication channel (e.g. e-mail). Show that, depending on the value of (x, y), Bob can choose to apply gates X and Z to his qubit so to set it in the state $|\psi\rangle$ that initially characterized qubit 1 on Alice's side.

In this way, the state $|\psi\rangle$ has been "teleported" from Alice to Bob. Think about the following points:

4. Was there teleportation of matter from Alice to Bob?
5. Was the state $|\psi\rangle$ cloned?
6. Was there any superluminal transmission of information?