Quantum Information and Quantum Computing, Problem set 1

Assistants : sara.alvesdossantos@epfl.ch, clemens.giuliani@epfl.ch

The quantum mechanics needed to understand and practice quantum computing requires good skills in linear algebra. This is not a big deal, as it all adds up to rather simple concepts and mathematical formalism. In this problem set you will be asked to solve simple exercises involving these notions. If you notice that most of the content of this problem set is obscure to you, then it is maybe the moment to grab a book and catch up with linear algebra. If it happens, then come talk to us.

On purpose, I will avoid here any quantum mechanics jargon and stick only to the mathematics one. Quantum mechanics will enter the game starting from the next problem set.

Problem 1 : Matrix-vector and matrix-matrix products

Compute explicitly the following matrix-vector and matrix-matrix products

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&1&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\times\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\end{pmatrix}\times\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}$$

Problem 2 : Eigenvalues and eigenvectors

Find the eigenvalues and eigenvectors (if they exist) of the matrix:

$$M=\begin{pmatrix}1&i\\2&1\end{pmatrix}.$$

Problem 3 : Pauli matrices

Pauli matrices are defined as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove the following relations

- $X^2 = Y^2 = Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $AB + BA = 0$, with A, $B = X, Y, Z$ and $A \neq B$
- $XY - YX = 2iZ$, $XZ - ZX = -2iY$, $YZ - ZY = 2iX$

Problem 4 : Exponential of a matrix

We will see later that in quantum mechanics it is important to know how to compute the exponential of a matrix. This is defined in terms of the matrix-matrix product and of the Taylor expansion of the exponential function

$$e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

In order to familiarize with this notion, let's compute

- $\exp(i\alpha Z)$
- $\exp(i\alpha Y)$

Let us now briefly recall the *Kroenecker or tensor product.* If two vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are spanned by basis vectors $\{|0\rangle_1, |1\rangle_1\}$, then we can define the tensor-product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which is spanned by the (mutually orthogonal) basis vectors $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$. For brevity of notation we rewrite these vectors as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where the position inside the symbol $|xy\rangle$ denotes whether the label refers to space 1 or 2. Given two vectors $|\psi\rangle = \alpha|0\rangle_1 + \beta|1\rangle_1$ and $|\phi\rangle = \gamma|0\rangle_2 + \delta|1\rangle_2$, respectively in $\mathcal{H}_1$ and $\mathcal{H}_2$, we can define a vector in $\mathcal{H}$ as

$$|\psi\rangle \otimes |\phi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \,.$$

Tensor products of matrices naturally follow from the above definition. If A is a $m \times n$ matrix and B a $p \times q$ matrix, then $A \otimes B$ is a $mp \times nq$ matrix defined as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \, , \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B & \cdots \\ a_{21}B & a_{22}B & a_{23}B & \cdots \\ a_{31}B & a_{32}B & a_{33}B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

<u>Problem 5 :</u> Tensor product

- Compute
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
- Show, using one example, that in general $A \otimes B \neq B \otimes A$
- Show that
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
- Prove that
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
cannot be expressed as a single tensor product of two matrices as $A \otimes B$
- The same holds for vectors. Prove that $|\psi\rangle = |00\rangle + |11\rangle$ cannot be written as a simple tensor product, i.e. as $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$.