Statistical Physics of Computation - Exercises

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Replica computation for the storage problem

We want to compute

$$\phi := \lim_{N\to\infty} \mathbb{E}_\xi \Omega(\{\xi^\mu\}_{\mu=1}^p), \tag{1}$$

where

$$\Omega(\{\xi^\mu\}_{\mu=1}^p) := \int d\mu(J) \Pi_{\mu=1}^p \theta\left(\frac{\sigma^\mu}{\sqrt{N}} J^T \xi^\mu - k\right), \tag{2}$$

and $d\mu(J)$ is the (unnormalised) uniform measure on the N-dimensional sphere of radius $\sqrt{N}$

$$d\mu(J) = dJ_1 \dots dJ_d \, \delta\left(||J||^2 - N\right) \tag{3}$$

The ξ_i^μ are i.i.d. standard Gaussian random variables. We use the replica method

$$\phi = \lim_{N\to\infty} \lim_{n\to\infty} \frac{1}{N} \frac{\mathbb{E}_\xi \Omega(\xi)^n - 1}{n}. \tag{4}$$

Introducing the gap variables. Fix $n \in \mathbf{N}$, then

$$\begin{aligned}
\mathbb{E}_\xi \Omega(\xi)^n &= \mathbb{E}_\xi \left[\int d\mu(J) \Pi_{\mu=1}^p \theta\left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_i \xi_i^\mu - k\right)\right]^n \\
&= \mathbb{E}_\xi \Pi_{a=1}^n \left[\int d\mu(J^a) \Pi_{\mu=1}^p \theta\left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_i^a \xi_i^\mu - k\right)\right] \\
&= \mathbb{E}_\xi \Pi_{a=1}^n \left[\int d\mu(J^a) \Pi_{\mu=1}^p d\Delta^{\mu a} \theta(\Delta^{\mu a} - k) \delta\left(\Delta^{\mu a} - \frac{1}{\sqrt{N}} \sum_{i=1}^N J_i^a \xi_i^\mu\right)\right].
\end{aligned} \tag{5}$$

where in the last passage we introduced a delta function to enforce the definition of the gap variables $\Delta^{\mu a}$. This is done as we want to separate the θ function enforcing the constraints from the specific way the data enters the problem, so that we can treat these two components as separately as possible.

Averaging over the data (disorder). Now we notice that $\frac{1}{\sqrt{N}} \sum_{i=1}^N J_i \xi_i^\mu$ is a Gaussian variable, as it is a linear combination of the Gaussians ξ_i^μ, with zero mean (as the ξ have zero mean) and covariance

$$
\begin{aligned}
\mathbb{E}_\xi \left(\frac{1}{\sqrt{N}} \sum_{i=1}^N J_i^a \xi_i^\mu \right) \left(\frac{1}{\sqrt{N}} \sum_{j=1}^N J_j^b \xi_j^\nu \right) &= \frac{1}{N} \sum_{i,j=1}^N J_i^a J_i^b \mathbb{E}_\xi \xi_i^\mu \xi_j^\nu \\
&= \frac{1}{N} \sum_{i,j=1}^N J_i^a J_i^b \delta_{\mu,\nu} \delta_{i,j} \\
&= \delta_{\mu,\nu} \frac{1}{N} \sum_{i=1}^N J_i^a J_i^b .
\end{aligned} \tag{6}
$$

Then we have

$$
\mathbb{E}_\xi \left[\Pi_{a=1}^n \Pi_{\mu=1}^p \delta \left(\Delta^{\mu a} - \frac{1}{\sqrt{N}} \sum_{i=1}^N J_i^a \xi_i^\mu \right) \right] = \Pi_\mu \mathcal{N} \Big(\{\Delta^{\mu a}\}_{a=1}^n \Big| 0, \frac{1}{N} \sum_{i=1}^N J_i^a J_i^b \Big), \tag{7}
$$

as the expression on the right is just the probability density function for the gap variable Δ (a random variable), whose randomness is induced by the randomness of ξ. Notice the factorisation over μ, as the covariance has a term $\delta_{\mu,\nu}$ telling us that the gap variables for each sample μ are uncorrelated. Here, $\mathcal{N}(x|\mu, \Sigma)$ is the Gaussian p.d.f. with mean μ and covariance Σ evaluated at x. Notice also that we managed to average over the disorder, i.e. the data ξ.

Introducing the overlap order parameter. Now we introduce the definition of the covariance matrix using a delta function, and realise that the product over μ jsut gives us a power P as all terms of the product are the same (in other words, μ is a dummy index), so we get

$$
\begin{aligned}
\mathbb{E}_\xi \Omega(\xi)^n = \int \Pi_{a<b} dq^{ab} & \left[\int \Pi_a d\Delta^a \mathcal{N} \Big(\{\Delta^a\}_{a=1}^n \Big| 0, q_{ab} \Big) \theta(\Delta^a - k) \right]^P \\
& \left[\int \Pi_a d\mu(J^a) \Pi_{a<b} \delta (N q^{ab} - \sum_{i=1}^N J_i^a J_i^b) \right]
\end{aligned} \tag{8}
$$

where we overloaded a bit our notations, as the quantity q^{ab} we introduced is technically defined only for $a < b$, but we take the notational convention that it is symmetric, and equals one on the diagonal. The order parameter we introduced, i.e. the overlap

$$
q^{ab} = \frac{1}{N} \sum_{i=1}^N J_i^a J_i^b \,, \tag{9}
$$

is a central object of replica computations. It measures the similarity of the configurations of two replicas a and b, it equals 1 if $J_a = J_b$, equals -1 is $J_a = -J_b$, and equals zero if J_a and J_b are orthogonal.

Decomposition in energetic and entropic contributions. Conventionally, the term in the first square bracket of Eq. (8) is called "energetic term", and in this problem measures the probability that n correlated classifiers J^a (vectors on the N-dim sphere) all classify correctly P independent points with margin at least κ. Notice that this probability is just defined through the gap variables! It is called "energetic" because in some sense gives a score to each overlap configuration q^{ab}, in this case the score being how likely a given covariance structure q^{ab} fits P points. Notice that this term is an integral over n dimension to the power P, so if we ask ourselves naively what will the order of this term be we would expect it to be $O(\exp(nP))$.

The term in the second square bracket is conventionally called "entropic term", as it measures the volume spanned by n vectors on the sphere, constrained to have a certain overlap structure q^{ab}. As it is a volume for n vectors in N dimensions, we expect this term to scale $O(\exp(nN))$.

Thus, we see that we arrived at an expression which has an integral over an order parameter q^{ab} of two terms, both depending only on q^{ab}, that compete: the energetic term will favor values of q^{ab} with large scores, the entropic term will favor values of q^{ab} with large associated volumes, and the overall thermodynamic behaviour will stem from the tradeoff between these two terms, much like in usual thermodynamics energy and entropy interact, and dominate the behaviour based on the temperature.

Thus, if we define

$$s_{\text{entropy}}(q^{ab}) = \frac{1}{N} \log \left[\int \Pi_a d\mu(J^a) \Pi_{a<b} \delta(N q^{ab} - \sum_{i=1}^{N} J_i^a J_i^b) \right] \tag{10}$$

and

$$s_{\text{energy}}(q^{ab}) = \log \int \Pi_a d\Delta^a \mathcal{N}\Big(\{\Delta^a\}_{a=1}^n \Big| 0, q_{ab} \Big) \theta(\Delta^a - k) \tag{11}$$

we have

$$\mathbb{E}_\xi \Omega(\xi)^n = \int \Pi_{a<b} dq^{ab} \exp \left[P s_{\text{energy}}(q^{ab}) + N s_{\text{entropy}}(q^{ab}) \right] \tag{12}$$

which for $N, P \to \infty$ is amenable to the saddle-point method.

Identification of the critical scaling regime. We remark here that the computation now also provides us with the critical scaling for P. We need to choose $P = \alpha N$ for a given constant α in order for both the energetic and entropic term to contribute to the saddle-point. If $P \ll N$ then just the entropic term survives, and telling us that we can disregard the fitting part of our problem (enough parameters to fit all P points). If $P \gg N$ instead just the energetic term survives, telling us that the volume associated to our parameters does not matter (not enough parameters to fit all P points while spanning a non-zero volume). Thus, we need to compute the two terms, and then take the saddle point. Notice crucially that this is the place where we are exchanging the $N \to \infty$ and $n \to 0$ limits, as we are taking first the saddle-point and then worrying about the $n \to 0$.

Computeation of entropic term. In Homework 4 you will compute the entropic term, and see that at leading order for large N and fixed n

$$s_{\text{entropy}}(q^{ab}) = \frac{1}{N} \log \left[\int \Pi_a d\mu(J^a) \Pi_{a<b} \delta(N q^{ab} - \sum_{i=1}^{N} J_i^a J_i^b) \right] = \frac{1}{2} \log \det q^{ab} + 2n + \frac{n}{2} \log(2\pi). \tag{13}$$

A comment on this formula: it's typical in the physics literature to drop the last two pieces because they are independent of q^{ab}, i.e. they will not alter the state equations. We also see here that if we normalised the partition function by the total volume of the sphere, that term will give a q-independent contribution that we would drop here. If we were interested in the precise value of the Gardner's volume, and not just about the state equation, we would need to keep all such terms!

The fact that we can compute the entropic term for all matrices q^{ab} in such a compact form is special of the fact that our weights J are spherical. If that was not the case, we would have not managed to compute the entropic term in such generic conditions. This is precisely the problem we have for the energetic term: it is a complicated Gaussian integral in n dimensions of a product of step functions, and it has no closed form solution for arbitrary q^{ab}.

RS ansatz. Thus, we need to make some assumption on q^{ab} to proceed. This is the place where, by choosing an ansatz for q, we are also implicitly choosing an analytic continuation of our expression to small n, as we will see in a second. The most natural ansatz is the so-called Replica Symmetric (RS) ansatz. We argue that in our original expression all replicas are equivalent, and they satisfy a permutation symmetry, so it is natural to seek an order parameter satisfying this symmetry (unless symmetry is spontaneously broken. . .). Thus, we choose

$$q^{ab} = \delta^{ab} + (1 - \delta^{ab})q, \tag{14}$$

i.e. entries on the diagonal equal to one and entries out of the diagonal equal to q.

Computation of the entropic term in the RS ansatz. This allows to further simplify the entropic term (as you will see in Homework 4) to

$$s_{\text{entropy}} = n \left(\frac{1}{2} \log(1 - q) + \frac{q}{2(1 - q)} \right) + o(n) \tag{15}$$

where we are already considering the small n limit, as we will need to take the $n \to 0$ limit at some point.

Computation of the energetic term in the RS ansatz. Tha RS ansatz also allows us to compute the energetic term.

$$s_{\text{energy}} = \log \mathbb{E}_{\Delta \sim N(0,q)} \left[\prod_{a=1}^{n} \theta \left(\Delta^a - \kappa \right) \right] \tag{16}$$

Here Δ^a are Gaussian variables with mean zero and covariance q^{ab}.

$$\mathbb{E}[\Delta^a] = 0 \,, \qquad \mathbb{E}[\Delta^a \Delta^b] = q^{ab} \tag{17}$$

We introduce the Fourier transform $\hat{\Delta}^a$ of Δ^a to write the identity

$$s_{\text{energy}} = \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^{n} \theta \left(\Delta^a - \kappa \right) \exp \left\{ -\frac{1}{2} \sum_{a,b} \hat{\Delta}^a \hat{\Delta}^b q^{ab} + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{18}$$

where the $\hat{\Delta}^a$ variables are integrate on the imaginary line. This step comes from the definition of Fourier transform of the Gaussian:

$$\frac{1}{\sqrt{\det Q}} e^{-\frac{\mathbf{x}^\top Q^{-1} \mathbf{x}}{2}} = \int_{\mathbb{R}^n} \mathrm{d}\hat{\mathbf{x}}\; e^{-\frac{\hat{\mathbf{x}}^\top Q \hat{\mathbf{x}}}{2} + i \hat{\mathbf{x}}^\top \mathbf{x}} \tag{19}$$

and has the main advantage of not needing to invert q^{ab}.

We then impose the RS ansatz with 1 on the diagonal and q outside to write

$$s_{\text{energy}} = \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1}{2}\sum_{a,b} \hat{\Delta}^a \hat{\Delta}^b q^{ab} + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{20}$$

$$= \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1}{2}\sum_a (\hat{\Delta}^a)^2 q^{aa} - \frac{1}{2}\sum_{a\neq b} \hat{\Delta}^a \hat{\Delta}^b q^{ab} + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{21}$$

$$= \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1}{2}\sum_a (\hat{\Delta}^a)^2 - \frac{q}{2}\sum_{a\neq b} \hat{\Delta}^a \hat{\Delta}^b + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{22}$$

$$= \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1}{2}\sum_a (\hat{\Delta}^a)^2 + \frac{q}{2}\sum_a (\hat{\Delta}^a)^2 - \frac{q}{2}\sum_{a,b} \hat{\Delta}^a \hat{\Delta}^b + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{23}$$

$$= \log \int \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1-q}{2}\sum_a (\hat{\Delta}^a)^2 - \frac{q}{2}\left(\sum_a \hat{\Delta}^a\right)^2 + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{24}$$

We can introduce a new scalar Gaussian variable $z \sim \mathcal{N}(0,1)$ to deal with the squared sum. We get that

$$s_{\text{energy}} = \log \int Dz \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{ -\frac{1-q}{2}\sum_a (\hat{\Delta}^a)^2 - iz\sqrt{q}\sum_a \hat{\Delta}^a + i \sum_a \Delta^a \hat{\Delta}^a \right\} \tag{25}$$

where we introduced the notation

$$Dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z \tag{26}$$

It's easier to justify this step a posteriori by directly integrating over z the r.h.s., and recovering the l.h.s.. Such a trick is often used to transform a square term into a linear one, and it's typically called a Hubbard-Stratonovich (HS) transform.

The HS transform is convenient because it completely decoupled the different replicas. The remaining part is just a Guassian integral.

$$\int Dz \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{-\frac{1-q}{2}\sum_a (\hat{\Delta}^a)^2 - iz\sqrt{q}\sum_a \hat{\Delta}^a + i\sum_a \Delta^a \hat{\Delta}^a\right\} \tag{27}$$

$$= \int Dz \prod_a \mathrm{d}\Delta^a \mathrm{d}\hat{\Delta}^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{-\frac{1-q}{2}\sum_a (\hat{\Delta}^a)^2 + i\sum_a (\Delta^a - z\sqrt{q})\hat{\Delta}^a\right\} \tag{28}$$

$$= \int Dz \prod_a \mathrm{d}\Delta^a \prod_{a=1}^n \theta\left(\Delta^a - \kappa\right) \exp\left\{-\sum_a \frac{(\Delta^a - z\sqrt{q})^2}{2(1-q)}\right\} \tag{29}$$

$$= \int Dz \left[\frac{1}{\sqrt{2\pi(1-q)}} \int \mathrm{d}\Delta\, \theta\left(\Delta - \kappa\right) \exp\left\{-\frac{(\Delta - z\sqrt{q})^2}{2(1-q)}\right\}\right]^n \tag{30}$$

$$= \int Dz \left[\int D\Delta'\, \theta\left(\Delta'\sqrt{1-q} + z\sqrt{q} - \kappa\right)\right]^n \tag{31}$$

$$\tag{32}$$

where in the second to last passage we changed variable $\Delta \to \Delta' = (\Delta - z\sqrt{q})/\sqrt{1-q}$. Therefore,

$$\int \Pi_a d\Delta^a \mathcal{N}\Big(\{\Delta^{\mu a}\}_{a=1}^n \Big| 0, \delta^{ab} + (1-\delta^{ab})q\Big)\theta(\Delta^{\mu a} - \kappa) = \int Dz H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)^n, \tag{33}$$

where we introduced the complementary c.d.f. of the Gaussian density

$$H(x) = \int_x^\infty dt \frac{e^{-t^2/2}}{\sqrt{2\pi}}. \tag{34}$$

Finally, we expand this to small n to get

$$\begin{aligned} s_{\text{energy}} &= \log \int Dz H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)^n \\ &= \log \int Dz \exp\left(n \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)\right) \\ &= \log \int Dz \left(1 + n \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right) + O(n^2)\right) \\ &= \log\left(1 + n\int Dz \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right) + O(n^2)\right) \\ &\approx \log\exp\left(n\int Dz \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)\right) \\ &\approx n\int Dz \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right) \end{aligned} \tag{35}$$

where the second-to-last passage is performed by matching the first terms of the expansions of the two terms.

Computation of the Gardner volume, and state equation. Finally, we obtain

$$\begin{aligned}
\phi &= \lim_{n,N\to 0,\infty} \frac{\mathbb{E}_\xi \Omega(\xi)^n - 1}{nN} \\
&= \mathrm{extr}_q \left[\frac{1}{2} \log(1-q) + \frac{1}{2}\frac{q}{1-q} + \alpha \int Dz \log H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)\right] \\
&= \mathrm{extr}_q \phi(q)\,.
\end{aligned} \tag{36}$$

Notice that we can always map $z \to -z$ without changing the Dz in the integral, so you may find this expression written with different sign in front of z.

The state equation, i.e. the extremisation condition w.r.t. q, gives

$$\frac{q}{1-q} = \frac{\alpha}{2\pi} \int Dz\, e^{-\frac{(\kappa-\sqrt{q}z)^2}{1-q}} \left[H\left(\frac{\kappa - \sqrt{q}z}{\sqrt{1-q}}\right)\right]^{-2} \tag{37}$$

(notice a factor 1/2 of difference w.r.t. the equation reported in the book by Engel, Chapter 6. This should be a typo on their side, as the derivation below shows).

Derivation of the state equation. To derive that notice that

$$\partial_q \frac{1}{2}\left(\log(1-q) + \frac{q}{1-q}\right) = \frac{1}{2}\left(-\frac{1}{1-q} + \frac{1}{1-q} + \frac{q}{(1-q)^2}\right) = \frac{q}{2(1-q)^2} \tag{38}$$

The integral term is more tricky. We have

$$
\begin{aligned}
\partial_q \int Dz \log H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right) &\overset{(1)}{=} -\int Dz \frac{1}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{2(1-q)}\right)}{\sqrt{2\pi}} \partial_q \left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right) \\
&= -\int dz \frac{e^{-z^2/2}}{\sqrt{2\pi}} \frac{1}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{2(1-q)}\right)}{\sqrt{2\pi}} \frac{z-\sqrt{q}\kappa}{2\sqrt{q}(1-q)^{3/2}} \\
&= -\frac{1}{4\pi\sqrt{q}(1-q)^{3/2}} \int dz \frac{\exp\left(-\frac{z^2}{2}-\frac{(\kappa-\sqrt{q}z)^2}{2(1-q)}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} (z-\sqrt{q}\kappa) \\
&\overset{(2)}{=} -\frac{e^{-\kappa^2/2}}{4\pi\sqrt{q}(1-q)^{3/2}} \int dz \frac{\exp\left(-\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} (z-\sqrt{q}\kappa) \\
&= -\frac{e^{-\kappa^2/2}}{4\pi\sqrt{q}(1-q)^{3/2}} \int dz \frac{1}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} \partial_z \left[(-(1-q)) \exp\left(-\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)}\right)\right] \\
&= \frac{e^{-\kappa^2/2}}{4\pi\sqrt{q(1-q)}} \int dz \frac{1}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} \partial_z \left[\exp\left(-\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)}\right)\right] \\
&\overset{(3)}{=} -\frac{e^{-\kappa^2/2}}{4\pi\sqrt{q(1-q)}} \int dz \exp\left(-\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)}\right) \partial_z \frac{1}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)} \\
&\overset{(4)}{=} -\frac{e^{-\kappa^2/2}}{4\pi\sqrt{q(1-q)}} \int dz \frac{\exp\left(-\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)^2} \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{2(1-q)}\right)}{\sqrt{2\pi}} \sqrt{\frac{q}{1-q}} \\
&\overset{(5)}{=} -\frac{1}{4\pi(1-q)} \int Dz \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{1-q}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)^2}
\end{aligned} \tag{39}
$$

where in (1, 4) we used that

$$
\partial_x H(x) = \partial_x \int_x^\infty dt \frac{e^{-t^2/2}}{\sqrt{2\pi}} = -\frac{e^{-x^2/2}}{\sqrt{2\pi}} \tag{40}
$$

and in (2,5) we used

$$
\begin{aligned}
-\frac{1}{2}\left(z^2+\frac{(\kappa-\sqrt{q}z)^2}{(1-q)}\right) &= -\frac{z^2-qz^2+\kappa^2+qz^2-2\sqrt{q}z\kappa}{2(1-q)} \\
&= -\frac{z^2+\kappa^2-2\sqrt{q}z\kappa}{2(1-q)} \\
&= -\frac{z^2-2\sqrt{q}z\kappa+q\kappa^2}{2(1-q)} - \frac{\kappa^2-q\kappa^2}{2(1-q)} \\
&= -\frac{(z-\sqrt{q}\kappa)^2}{2(1-q)} - \frac{\kappa^2}{2}
\end{aligned} \tag{41}
$$

and in (3) we integrated by parts, giving

$$\begin{aligned} 0 = \partial_q \phi(q) &= \frac{q}{2(1-q)^2} - \frac{\alpha}{4\pi(1-q)} \int Dz \, \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{1-q}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)^2} \\ \implies \frac{q}{1-q} &= \frac{\alpha}{2\pi} \int Dz \, \frac{\exp\left(-\frac{(\kappa-\sqrt{q}z)^2}{1-q}\right)}{H\left(\frac{\kappa-\sqrt{q}z}{\sqrt{1-q}}\right)^2} \end{aligned} \tag{42}$$