

Statistical Physics of Computation - Exercises

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Week 2

2.1 Saddle point method: the general version

The saddle-point method in full generality adapts to integrals of complex variables. As we did before for real valued integrals, we search for an asymptotic formula for integrals of the type:

$$I(\lambda) = \int_{\gamma} dz h(z) e^{\lambda f(z)} \quad (1)$$

where $\gamma : [a, b] \rightarrow \mathbb{C}$ is a curve in the complex plane $\mathbb{C}$ and $\lambda > 0$ is a real positive number which we will take to be large. If the complex function f is holomorphic on a connected open set $\Omega \subset \mathbb{C}$, the integral $I(\lambda)$ is independent of the curve γ . The goal is therefore to choose γ wisely.

2.1.1 Geometrical properties of holomorphic functions

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function, and let $z = x + iy \in \mathbb{C}$ for real $x, y \in \mathbb{R}$. Without loss of generality, we can write $f(z) = u(x, y) + iv(x, y)$ for $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ real-valued functions. The goal of this exercise is to study the properties of f around a critical point $f'(z_0) = 0$ for $z_0 \in \mathbb{C}$.

1. Show that at a critical point, the gradients of u and v are zero.

First, note that since f is holomorphic at $z_0 = (x_0, y_0)$, we have:

$$\frac{df}{d\bar{z}} = \frac{1}{2} (\partial_x + i\partial_y) (u(x_0, y_0) + iv(x_0, y_0)) \stackrel{!}{=} 0 \quad (2)$$

On the other hand, since z_0 is a critical point, we have:

$$\frac{df}{dz} = \frac{1}{2} (\partial_x - i\partial_y) (u(x_0, y_0) + iv(x_0, y_0)) \stackrel{!}{=} 0 \quad (3)$$

Taking the sum of these two equations lead to:

$$\partial_x u(x_0, y_0) + i\partial_x v(x_0, y_0) = 0 \quad \Rightarrow \quad \partial_x u(x_0, y_0) = \partial_x v(x_0, y_0) = 0 \quad (4)$$

Similarly, taking the difference between the equations:

$$\partial_y u(x_0, y_0) + i\partial_y v(x_0, y_0) = 0 \quad \Rightarrow \quad \partial_y u(x_0, y_0) = \partial_y v(x_0, y_0) = 0 \quad (5)$$

2. Using the Cauchy integral formula, show that for all $z_0 = x_0 + iy_0$ in an open convex set Ω where f is holomorphic we have:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta u(x_0 + r \cos \theta, y_0 + r \sin \theta) \quad (6)$$

$$v(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta v(x_0 + r \cos \theta, y_0 + r \sin \theta) \quad (7)$$

for all circles of radius $r > 0$ centred at z_0 contained inside Ω . This result is known as the Mean Value Theorem in complex analysis. Let $D(r, z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$ be a disc with radius $r > 0$ centred at z_0 , and denote γ its boundary (i.e. the circle with radius r centered at z_0). Since f is holomorphic in $D(r, z_0)$, for any $a \in D(r, z_0)$ Cauchy integral formula holds:

$$f(a) = \int_{\gamma} \frac{dz}{2\pi i} \frac{f(z)}{z - a} \quad (8)$$

Note that in particular this is true for $a = z_0$. Parametrising γ as $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi)$, we have:

$$\begin{aligned} f(z_0) &= \int_0^{2\pi} \frac{d\theta}{2\pi i} i r e^{i\theta} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} f(z_0 + re^{i\theta}) \end{aligned} \quad (9)$$

Letting $z = (x_0, y_0)$ and $f(x, y) = u(x, y) + iv(x, y)$ and taking the real and imaginary values at each side of the equality yield the expected result.

3. Conclude that neither u or v can have a local extremum (maximum or minimum) inside Ω . Therefore, all critical points $z_0 \in \Omega$ are necessarily saddle points of u and v . The result above tells us that the value of a holomorphic function at the center of a circle equals to its arithmetic mean along the circle. Intuitively, it is easy to see that this implies the function cannot have a maximum or a minimum inside D . Let's show it mathematically. Suppose that $f'(z_0) = 0$ and that f has a maximum z_0 . This implies that for any point on a circle around z_0 with small enough radius $r > 0$ we would have $|f(z_0)| \geq |f(z_0 + re^{i\theta})|$. However, taking the modulus of eq. (9):

$$|f(z_0)| = \left| \int_0^{2\pi} \frac{d\theta}{2\pi} f(z_0 + re^{i\theta}) \right| \leq \int_0^{2\pi} \frac{d\theta}{2\pi} |f(z_0 + re^{i\theta})| \quad (10)$$

which contradicts the fact that z_0 is a maximum unless f is constant. One can show that f cannot have a minimum (unless it is constant) by applying the same argument to $1/f$. Therefore, all critical points of non-constant holomorphic functions are saddles.

4. Let z_0 be a critical point of f such that $f''(z_0) \neq 0$. Using the Taylor series of f around z_0 and using the polar decomposition $f''(z_0) = \rho e^{i\alpha}$, $z - z_0 = re^{i\theta}$, find the values of $\theta \in [0, 2\pi)$ corresponding to the two directions of steepest-descent of u as a function of α in the complex plane.

The Taylor expansion of f around the critical point z_0 is given by:

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2}f''(z_0)(z - z_0)^2 + O((z - z_0)^3) \\ &= f(z_0) + \frac{1}{2}f''(z_0)(z - z_0)^2 + O(|z - z_0|^3) \end{aligned} \quad (11)$$

Letting $z = z_0 + re^{i\theta}$ and $f''(z_0) = \rho e^{i\alpha}$ for $\theta, \alpha \in [0, 2\pi)$ and $r, \rho > 0$:

$$f(z) - f(z_0) = \frac{1}{2}r^2\rho e^{i(\alpha+2\theta)} + O(|z - z_0|^3) \quad (12)$$

Therefore, the rate at which f varies around z_0 is determined by $\frac{1}{2}r^2\rho$, while the direction is set by the phase $e^{i(\alpha+2\theta)}$. The direction corresponding to the steepest-descent (i.e. faster way down) is given by

$$e^{i(\alpha+2\theta)} = -1, \quad \Leftrightarrow \quad \theta = \frac{\pi}{2} - \frac{\alpha}{2} + k\pi, \quad k \in \mathbb{Z} \quad (13)$$

2.1.2 Choosing the good curve γ

1. What are the regions of γ which dominate the integral $I(\lambda)$?

As in the real case, as $\lambda \gg 1$ the integral will be dominated by the values of the exponential that are higher. However, letting $f(z) = u(z) + iv(z)$ for real-valued $u, v \in \mathbb{R}$, we now have a contribution in the exponential given by the imaginary part of f :

$$I(\lambda) = \int_{\gamma} dz h(z) e^{\lambda u(z) + i\lambda v(z)} = \int_{\gamma} dz h(z) e^{\lambda u(z)} (\cos(v(z)) + i \sin(v(z))) \quad (14)$$

When $\lambda \gg 1$, the imaginary part will contribute with strong oscillations, and on average will cancel even if the exponential is big. Therefore, the regions contributing the most to the integral will be the ones in which f has a constant phase, i.e. $v(z) = \text{constant}$.

2. Let z_0 be a critical point $f'(z_0) = 0$. Explain why should we choose γ to pass through z_0 following the steepest-descent directions of the real part $\text{Re}[f]$?

As we have discussed above, we want to choose a path with constant phase. Indeed, as we have shown in Part I d), the path passing through z_0 through the steepest-descent direction is one such path. Moreover, as we discussed in Part I a), the critical point is a saddle, and the the steepest-descent direction precisely goes through a concave trajectory with z_0 as the maximum, allowing us to apply Laplace's method.

3. Show that such a γ , we can rewrite the integral as:

$$I(\lambda) = e^{i\lambda \text{Im}[f(z_0)]} \int_{\gamma} dz h(z) e^{\lambda \text{Re}[f(z)]} \quad (15)$$

Let γ be the curve passing through the critical point z_0 in the steepest-descent direction, and let $f(z) = u(z) + iv(z)$. As we have shown in part I d), $v(z) = \text{constant} = v(z_0)$ along γ . Therefore:

$$I(\lambda) = e^{i\lambda v(z_0)} \int_{\gamma} dz h(z) e^{\lambda u(z)}. \quad (16)$$

4. Let $\gamma(t) = x(t) + iy(t)$ for $t \in [a, b]$ be a parametrisation of the curve passing through $z_0 = \gamma(t_0)$ through the steepest-descent direction of $\text{Re}[f]$. Letting $f(t) = f(\gamma(t))$, $h(t) =$

$h(\gamma(t))$, $u(t) = \text{Re}[f(t)]$ and $v(t) = \text{Im}[f(t)]$, show that the problem boils down to the evaluation of the following integral:

$$\int_a^b dt \gamma'(t) h(t) e^{\lambda u(t)} \quad (17)$$

Let $\gamma(t) = x(t) + iy(t)$ for $t \in [a, b]$ be a parametrisation of γ , define $h(t) \equiv h(\gamma(t))$, $u \equiv u(\gamma(t))$, $v(t) \equiv v(\gamma(t))$. Evaluating the expression above at the parametrisation leads to:

$$I(\lambda) = e^{i\lambda v(t_0)} \int_a^b dt \gamma'(t) h(t) e^{\lambda u(t)}. \quad (18)$$

2.1.3 Back to Laplace's method

1. Suppose $h(t_0) \neq 0$, and note we can choose a parametrisation of γ such that $\gamma'(t_0) \neq 0$. Use Laplace's method to show that $I(\lambda)$ admits the following asymptotic expansion for $\lambda \gg 1$:

$$I(\lambda) \asymp h(t_0) \gamma'(t_0) \sqrt{\frac{2\pi}{-\lambda u''(t_0)}} e^{\lambda f(t_0)} \quad (19)$$

As we have discussed in part II a), in the steepest descent we go through a concave direction in the saddle. Therefore, $z_0 = \gamma(t_0)$ is the only maximum along γ . Applying the standard saddle point method with $c = t_0$ we have:

$$I(\lambda) \underset{\lambda \gg 1}{=} h(t_0) \gamma'(t_0) \sqrt{\frac{2\pi}{-u''(t_0)}} e^{\lambda(u(t_0) + iv(t_0))} = h(t_0) \gamma'(t_0) \sqrt{\frac{2\pi}{-u''(t_0)}} e^{\lambda f(t_0)} \quad (20)$$

where we have assumed $h(t_0) \neq 0$.

2. Write the second derivative of $f(t)$ with respect to t and show that at the critical point z_0 we have:

$$\frac{d^2 f(t_0)}{dt^2} = \gamma'(t_0)^2 \frac{d^2 f(z_0)}{dz^2} \quad (21)$$

Recall that $f(t) \equiv f(\gamma(t))$. Therefore, by the chain rule:

$$\frac{df}{dt} = \gamma'(t) \frac{df}{dz}, \quad \Rightarrow \quad \frac{d^2 f}{dt^2} = \gamma''(t) \frac{df}{dz} + \gamma'(t)^2 \frac{d^2 f}{dz^2} \quad (22)$$

At the critical point we have $f'(z_0) = 0$, yielding the desired result.

3. Show that the second derivative $f''(t_0)$ is necessarily real and negative. Conclude that:

$$u''(t_0) = -|f''(z_0)| |\gamma'(t_0)|^2 \quad (23)$$

Recall the result from Part I d). Along the steepest-descent solution we had $f''(z_0) = -\rho$ with $\rho = |f''(z_0)| > 0$. Therefore, letting $f(z) = u(z) + i v(z)$, inserting this in the above result gives:

$$u''(t_0) = -|f''(z_0)| |\gamma'(t_0)|^2 \quad (24)$$

4. Let θ be the angle between the curve γ and the real axis at the critical point z_0 , see figure below. Show that:

$$\gamma'(t_0) = |\gamma'(t_0)|e^{i\theta} \quad (25)$$

This follows from the fact that close to the critical point $z_0 = \gamma(t_0)$

$$z - z_0 \equiv \gamma(t) - \gamma(t_0) = \gamma'(t_0)(t - t_0) + O((t - t_0)^2) \quad (26)$$

which implies that the phase of $\gamma'(t_0)$ is the same as the phase of $(z - z_0)$ at a neighbourhood of z_0 . From Part I d), the phase of $(z - z_0)$ is precisely chosen to be the steepest-descent direction θ .

5. Letting $f''(z_0) = |f''(z_0)|e^{i\alpha}$, show that $\theta = \frac{1}{2}(\pi - \alpha)$ or $\theta = \frac{1}{2}(\pi - \alpha) + \pi$ depending on the orientation of the curve γ .

Recall your result from part I d) showing that the steepest-descent direction is given by $\theta = \frac{1}{2}(\pi - \alpha) + \pi k$ for $k \in \mathbb{Z}$. Choosing $k = 0$ or $k = 1$ fix the direction of the curve.

6. Conclude that:

$$I(\lambda) \asymp \pm h(z_0)e^{\lambda f(z_0)}e^{i\frac{\pi-\alpha}{2}}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}} = h(z_0)e^{\lambda f(z_0)}\sqrt{\frac{2\pi}{-\lambda f''(z_0)}} \quad (27)$$

where the $\pm$ is given by the orientation of the steepest-descent curve.

Using the result of part III c) and the choice $\theta = \frac{1}{2}(\pi - \alpha)$ together with eq. (20):

$$I(\lambda) \underset{\lambda \gg 1}{=} h(z_0)|\gamma'(t_0)|e^{i\frac{\pi-\alpha}{2}}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}}e^{\lambda f(z_0)} = h(z_0)\sqrt{\frac{2\pi}{-\lambda f''(z_0)}}e^{\lambda f(z_0)} \quad (28)$$

where in the last equality we have used $e^{i\frac{\pi-\alpha}{2}} = \sqrt{-1}e^{-i\alpha/2}$ and have fixed $|\gamma'(t_0)| = 1$. Note that it is always possible to fix the velocity $\gamma'(t)$ of your parametrisation at a point $t_0 \in [a, b]$ to a chosen value v . It just amounts to solving an ordinary differential equation, which given the boundary condition $v \equiv \gamma'(t_0)$ has one and only one solution, see Picard-Lindelöf's theorem.