

Statistical Physics of Computation - Exercises

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September 2024

Week 10

10.1 Phase diagrams of low-rank matrix denoising

This week we will take the result of the replica computation we previously did for the low-rank denoising problem and study its implications on the hardness of the problem. Our starting point is the state equation on m :

$$m = \int Dz P_0(x_0) \frac{\int P_0(x) x x_0 dx \exp \left\{ -\frac{\lambda m}{2} x^2 + \sqrt{\lambda m} x z + \lambda m x x_0 \right\}}{\int P_0(x) dx \exp \left\{ -\frac{\lambda m}{2} x^2 + \sqrt{\lambda m} x z + \lambda m x x_0 \right\}} \quad (1)$$

which is the extremisation equation of the free entropy ϕ

$$\begin{aligned} \phi &= \max_m \phi(m) \\ &= \max_m \left[-\frac{\lambda}{4} m^2 + \int Dz P_X(x_0) dx_0 \log \left(\int P_X(x) dx e^{-\frac{\lambda m}{2} x^2 + \sqrt{\lambda m} x z + \lambda m x x_0} \right) \right] \end{aligned} \quad (2)$$

where Dz denotes integration against a standard Gaussian variable.

Gaussian prior. We start with a simple case, i.e. Gaussian prior on the components of x :

$$x, x_0 \sim \mathcal{N}(0, 1). \quad (3)$$

1. Show that the state equation becomes

$$m = \frac{\lambda m}{\lambda m + 1}. \quad (4)$$

2. Compute the variational free entropy $\phi(m)$ for this prior.
3. Solve the state equation. In principle it should have two solutions. How should one choose the right one? Use this criterion to find the correct $m(\lambda)$ for each λ .

Sparse binary prior. A more interesting case is the sparse binary prior. The elements of x have a probability $0 < \rho < 1$ of being one and $1 - \rho$ of being zero

$$P_0(x_i) = \rho\delta(x_i - 1) + (1 - \rho)\delta(x_i) \quad \forall i, \quad (5)$$

where δ is the Dirac's delta.

4. Show that the state equation becomes

$$m = \int Dz \frac{\rho^2}{\rho + (1 - \rho)e^{-\frac{\lambda m}{2} + z\sqrt{\lambda m}}} \quad (6)$$

5. We would like to solve the state equation, but it would only be possible to do so numerically. Luckily we know that the state equation is the fixed point of the state evolution on m^t

$$m^{t+1} = \int Dz \frac{\rho^2}{\rho + (1 - \rho)e^{-\frac{\lambda m^t}{2} + z\sqrt{\lambda m^t}}} . \quad (7)$$

Iterate numerically the state evolution until convergence for a large value of ρ (for example $\rho = 0.8$) for several value of λ . Use $m^{t=0} = 0.01$ and $m^{t=0} = 0.99$ as initializations. What do you observe? Is the behavior of m at convergence dependent on the initialization? Are there any phase transitions?

Hint: you will need to integrate (possibly by quadrature) over z . We advise to choose a suitable integration interval (something like $[-5, 5]$ would do).

6. We want to repeat the same numerical experiment for small ρ . First, it is useful to estimate the magnitude of the SNR and of the magnetization in that limit. Show that

$$Q = Q_* = \mathbb{E}_{x \sim P_0} x^2 = \rho . \quad (8)$$

Argue that this implies $m < \rho$, so that it is useful to plot m/ρ as our order parameter for small ρ . Argue that the strength of the signal scales linearly with ρ , hence the effective signal to noise ratio of the model is $\lambda\rho^2$.

7. Show numerically that for $\rho = 0.02$ small enough there is a first order phase transition by plotting m at convergence for the state evolution as a function of λ . Hint: In order to see anything from your plot you should use the scaling in m and λ from the previous point. Remember to initialize the state equation both in $m^{t=0} = 0.01$ and $m^{t=0} = 0.99$.
8. To which value of m_* would the AMP algorithm converge in this problem?

As discussed in class, to find the precise point at which the first order phase transition happens one needs to compare the value of the free entropy at the two solutions involved in the transition. For this prior, this is not so simple to do, as computing numerically the free entropy for small ρ leads to numerical instabilities. The solution is to expand the free entropy for small ρ after rescaling the magnetization and the SNR to highlight their natural dependence on ρ , in order to obtain a numerically more manageable approximate expression. This whole procedure starts being outside of the scope of these lectures, so we do not cover it here.