MATHEMATICAL METHODS IN QUANTUM PHYSICS

I. GENERALISED FUNCTIONS

In physics one encounters objects like $\delta(x)$. How to give a precise definition of such objects?

§1.1. Schwartz space and Schwartz functions

Recall the

Thm of Riesz & Fréchet. The anti-linear map

$$H \ni x \longmapsto \langle x, \ \rangle =: J(x) \in H'$$

is an isometric isomorphism between the Hilbert space H and its topological dual H'

proof: exercice $\square$

Duality permits therefore to identify a continuous functional on a Hilbert space to some function (in the case $H = L^2(\mathbb{R}^N, \mu_L)$.

Lebesgue measure

The idea is now to take a smaller space than $L^2(\mathbb{R}^N, \mu_L)$, so that its dual may have more functions.
We might start with $C_c^\infty(\mathbb{R}^N)$, the space of smooth functions on $\mathbb{R}^N$ with compact support. The problem is now that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \mu_L)$ and we have the

Thm: Let $D \subset H$ be dense and let $\xi : D \to B$ be a continuous map to some Banach space B. Then there is a unique and isometric extension $\Xi : H \to B$ $(\Xi|_D = \xi)$

proof: exercice ☐

As a consequence, $C_c(\mathbb{R}^N)' \cong H'$ and we have gained nothing.
To get a larger topological dual, one must therefore change the topology as well:
The more severe the notion of convergence on $C_c(\mathbb{R}^N)$, the larger its dual may be.

Before we continue let's introduce some

Notations:

For $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we write

- $x \cdot y := \sum_{k=1}^{N} x_k \cdot y_k$, • $x+y := (x_1+y_1, \dots, x_N+y_N)$
- $xy := (x_1 y_1, \dots, x_N y_N)$
- $x \leq y \iff \forall k = 1, \dots, N : x_k \leq y_k$.

A <u>multi-index</u> α is an n-tuple $(\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$. The <u>order</u> of α is $|\alpha| := \alpha_1 + \dots + \alpha_N \in \mathbb{N}$

One writes

- $\alpha! := (\alpha_1!) \times (\alpha_2!) \times \dots \times (\alpha_N!)$,
- $\binom{\alpha}{\beta} := \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$ if $\alpha \geq \beta$.

Specific multi-indices are $\bar{0} := (0, \dots, 0)$ and $\bar{1} := (1, \dots, 1)$.

For $f \in C^\infty(\mathbb{R}^N)$, we write

$$\partial^\alpha f := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_N)^{\alpha_N}} f(x)$$

<u>Exercise</u>: Show that $\partial^{\alpha+\beta} f = \partial^\alpha \partial^\beta f$ and

$$\partial^\alpha (fg) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \partial^\beta f \, \partial^\gamma g$$

One also writes $x^\alpha := (x_1)^{\alpha_1} \dots (x_N)^{\alpha_N}$.

<u>Exercise</u>: Show that $|\mathbb{N}^N_{\leq n}| = \binom{n+N}{n}$

where $\mathbb{N}^N_{\leq d} := \{ \alpha \in \mathbb{N}^N \mid |\alpha| \leq n \}$

<u>Def</u>: The set of <u>Schwartz functions</u> is defined as $S(\mathbb{R}^N) := \{ f \in C^\infty(\mathbb{R}^N) \mid \forall n \in \mathbb{N}, \forall \alpha \in \mathbb{N}^N, \lim_{x \cdot x \to \infty} (1 + x \cdot x)^n \partial^\alpha f(x) = 0 \}$

<u>Def</u>: For $n \in \mathbb{N}$, define the pre-Hilbert space $h_n(\mathbb{R}^N) = (S(\mathbb{R}^N), \langle \, , \, \rangle_n)$, where

$$\langle f, g \rangle_n := \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}^N} \partial^\alpha \bar{f} \, (1+x^2)^n \partial^\alpha g \, d\mu_L$$

A completion of $h_n(\mathbb{R}^N)$ into a bona fide Hilbert space $H_n(\mathbb{R}^N)$ may be obtained by considering

$$h_n(\mathbb{R}^N) \ni f \mapsto J(f) := (\partial^\alpha f)_{\alpha \in \mathbb{N}^N_{\leq n}} \in K$$

where $K = \bigoplus_{\alpha \in \mathbb{N}^N_{\leq n}} L^2(\mathbb{R}^N, (1+x^2)^n \mu_L)$

J is obviously isometric and K is a Hilbert space.

Therefore, $H_n(\mathbb{R}^N) \cong \overline{J(h_n(\mathbb{R}^N))}^K$.

$H_n(\mathbb{R}^N)$ may therefore be identified with a closed subspace in K. An element $\xi \in H_n(\mathbb{R}^N)'$ can be extended to an element $\Xi \in K'$ by imposing $\Xi|_{\mathcal{J}(H_n(\mathbb{R}^N))^{\perp}}=0$

Hence, any element $\Xi \in H_n(\mathbb{R}^N)'$ is of the form

$$\Xi(f) = \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}^N} g_\alpha(x)(1+x^2)^n \partial^\alpha f(x) \, d\nu_L$$

with $(g_\alpha)_{\alpha \in \mathbb{N}^N_{\leq n}} \in K$.

For $n \geq m$, there are obvious inclusion maps

$$i_n^m : H_n(\mathbb{R}^N) \hookrightarrow H_m(\mathbb{R}^N)$$

which are norm-continuous. We have hence the inclusions

$$L^2(\mathbb{R}^N, \nu_L) = H_0(\mathbb{R}^N) \supset H_1(\mathbb{R}^N) \supset \dots \supset H_n(\mathbb{R}^N) \supset \dots$$

<u>Def:</u> The <u>Schwartz space</u> S_N is defined as

$$S_N := \bigcap_{n \geq 0} H_n(\mathbb{R}^N) \quad (\supset S(\mathbb{R}^N)).$$

The topology τ_S is the collection of all sets $U \subset S_N$ s.t.

$$\forall f \in U, \exists n \in \mathbb{N}, \exists \varepsilon > 0 \text{ s.t. } U \supset \{ g \in S_N \,/\, \|f-g\|_n < \varepsilon \}$$

<u>Exercise:</u> Show that τ_S is a topology for S_N.

(S_N is an example of a <u>nuclear space</u>, or even more generally an <u>indirect limit</u> of topological spaces).

Thm for $f \in S(\mathbb{R}^N)$ and $n \in \mathbb{N}$ set

$$|||f|||_n := \max \{ \| (1+x\cdot x)^n \partial^\alpha f \|_\infty \mid \alpha \in \mathbb{N}^N_{\le n} \}$$

1. the family of norms $\{ \| \ \|_n, n \in \mathbb{N} \}$ and $\{ ||| \ |||_n, n \in \mathbb{N} \}$ are equivalent, i.e.
$\forall n \in \mathbb{N}, \exists c, D > 0$ s.t. $\forall f \in S(\mathbb{R}^N)$

$$|||f|||_n \le C \|f\|_{2n+2N},$$

$$\|f\|_n \le D \, |||f|||_{n+N}.$$

2. $S(\mathbb{R}^N)$ is complete for τ_s

proof: $||| \ |||_n$ are clearly norms. For $f \in S(\mathbb{R}^N)$

$$\|f\|_n^2 = \sum_{|\alpha| \le n} \int_{\mathbb{R}^N} (1+x\cdot x)^n |\partial^\alpha f|^2 \, d\mu$$

$$= \sum_{|\alpha| \le n} \int_{\mathbb{R}^N} \frac{(1+x\cdot x)^n}{(1+x\cdot x)^{n+N}} (1+x\cdot x)^{n+N} |\partial^\alpha f|^2 \, d\mu$$

$$\le \sum_{|\alpha| \le n} \int_{\mathbb{R}^N} \frac{1}{(1+x\cdot x)^N} \| (1+x\cdot x)^{n+N} |\partial^\alpha f|^2 \|_\infty \, d\mu$$

$$\le \sum_{\alpha \le n} \int_{\mathbb{R}^N} \frac{1}{(1+x\cdot x)^N} |||f|||^2_{n+N}.$$

set $D = \left(\binom{N+n}{n} \int_{\mathbb{R}^N} \frac{1}{(1+x\cdot x)^N} \, d\mu_L \right)^{1/2}.$

This proves the second inequality.
For the first, if $x \in \mathbb{R}^N$, $\alpha \in \mathbb{N}^N$,

$$\left|(1+x\cdot x)^n \partial^\alpha f(x)\right| = \left| \int_{y\le x} \partial^{I}\left[(1+y\cdot y)^n \partial^\alpha f(y)\right] d\mu_L \right|$$

$$= \left| \sum_{\beta+\gamma=I} \int_{y\le x} \partial^\beta (1+y\cdot y)^n \partial^{\alpha+\gamma} f(y)\, d\mu_L \right|$$

$$\le \sum_{\beta+\gamma=I} \int_{y\le x} |\partial^\beta(1+y\cdot y)^n| |\partial^{\alpha+\gamma} f(y)|\, d\mu_L$$

$$\le \sum_{\beta+\gamma=I} \int_{\mathbb{R}^N} n!(1+y\cdot y)^n \frac{1}{(1+y\cdot y)^{n+N}} \left|(1+y\cdot y)^{n+N} \partial^{\gamma+\alpha} f(y)\right| d\mu_L$$

$$\le \left\| \frac{n!}{(1+y\cdot y)^N} \right\|_{L_2} \left(\sum_{\beta+\gamma=I} \left\|(1+y\cdot y)^{N+n} \partial^{\alpha+\gamma} f\right\|_{L_2} \right)$$

$$\le \underbrace{\left\| \frac{n!}{(1+y\cdot y)^N} \right\|_{L}}_{C} \cdot \| f \|_{2n+2N}$$

Taking supremums gives

$$|||f|||_m \le \underbrace{C}_{n} \cdot \|f\|_{2n+2N}$$

2. Let $(f_k)_{k\in\mathbb{N}}$ be Cauchy in $S(\mathbb{R}^N)$ for τ_S.

This means $\forall\, U \in \tau_S$, if $0 \in U$ $\exists\, N_U$ s.t. $k, l \ge N_U$ implies $f_k - f_l \in U$

This means $\forall\, n \in \mathbb{N}$, $\forall \varepsilon > 0$, $\exists\, N_{n,\varepsilon}$

$k, l \ge N_{n,\varepsilon}$ implies $\|f_k - f_l\|_n < \varepsilon$

This means that for any $n \in \mathbb{N}$, and and any $\alpha \in \mathbb{N}^N_{\leq n}$,
$((1+x\cdot x)^n \partial^\alpha f_k)_{k\in\mathbb{N}}$ is Cauchy for $\| \ \|_\infty$. Show then, that $f_k \to f \in S(\mathbb{R}^N)$ for τ_S.

$\square$

Corollary: S_N is the set of Schwartz functions and is complete for τ_S

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proof: Let $f \in S_N$. Then, $\forall n \in \mathbb{N}$, $f \in H_n(\mathbb{R}^N)$.
By density of $S(\mathbb{R}^N)$ in $H_n(\mathbb{R}^N)$ there is an $f_n \in S(\mathbb{R}^N)$ s.t. $\|f - f_n\|_n < \frac{1}{n}$.
The sequence $(f_n)_{n\in\mathbb{N}}$ is obviously Cauchy for τ_S. Indeed,

$$\|f_k - f_\ell\|_n \leq \|f_k - f\|_n + \|f_\ell - f\|_n$$
$$\leq \|f_k - f\|_{\max\{k,\ell,n\}} + \|f_\ell - f\|_{\max\{k,\ell,n\}}$$
$$< \frac{1}{k} + \frac{1}{\ell}.$$

Completeness of $S(\mathbb{R}^N)$ implies $f = \lim_n f_n$ for τ_S must be in $S(\mathbb{R}^N)$ $\square$.

§1.2. Tempered Distributions

Def: A tempered distribution is a linear and continuous map $\varphi : S(\mathbb{R}^N) \longrightarrow \mathbb{C}$:

- $\forall f, g \in S(\mathbb{R}^N)$, $\forall \lambda \in \mathbb{C}$, $\varphi(f + \lambda g) = \varphi(f) + \lambda \varphi(g)$
- $\forall \varepsilon > 0$, $\varphi^{-1}\{z \in \mathbb{C},\ |z| < \varepsilon\} \in \tau_S$.

Examples: 1) The Dirac "delta-function" is a tempered distribution. Indeed $f \mapsto f(0) =: \delta(f)$ is certainly linear and if $|||f|||_0 < \varepsilon$ then $|f(0)| < \varepsilon$.

2) The principle Cauchy value $\text{p.v.}\left(\frac{1}{x}\right)$ is a tempered distribution. Indeed, for $f \in S(\mathbb{R}^N)$

$$\left(\text{p.v.}(1/x)\right)(f) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{x}\, dx$$

$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{f(x) - f(-x)}{x}\, dx$$

$$= \lim_{\varepsilon \to 0^+} \left(\cancel{\ln(x)\left(f(x) - f(-x)\right)\Big|_\varepsilon^\infty} - \int_\varepsilon^\infty \ln(x)\left(f'(x) + f'(-x)\right) dx \right)$$

$$= \lim_{\varepsilon \to 0^+} \left(\cancel{x(\ln(x) - 1)\left(f'(x) + f'(-x)\right)\Big|_\varepsilon^\infty} + \int_\varepsilon^\infty x(\ln(x) - 1)\left(f''(x) - f''(-x)\right) dx \right)$$

$$= \int_{\mathbb{R}^+} x(\ln(x) - 1)\left(f''(x) - f''(-x)\right) dx$$

Thus $\left| \text{p.v.}(1/x)(f) \right| = \left| \int_{\mathbb{R}^+} x(\ln(x) - 1)(f''(x) - f''(-x))\,dx \right|$

$$\leq \int_{\mathbb{R}} \frac{|x(\ln|x| - 1)|}{(1+x^2)^2} (1+x^2)^2\, 2|f''(x)|\,dx$$

$$\leq \left(2 \int_{\mathbb{R}} \frac{|x(\ln|x| - 1)|}{(1+x^2)^2} \right) |||f|||_2$$

3) If $g(x) : \mathbb{R}^N \to \mathbb{C}$ is continuous and polynomially bounded we may consider

$$S(\mathbb{R}^N) \ni f \longmapsto \int_{\mathbb{R}^N} g(x)\, \partial^\alpha f(x)\, d\mu_L, \quad \alpha \in \mathbb{N}^N \text{ fixed.}$$

this defines an element in $S'(\mathbb{R}^N)$:

$$\left| \int_{\mathbb{R}^N} g(x)\, \partial^\alpha f(x)\, d\mu_L \right| \leq \int_{\mathbb{R}^N} |g(x)|\, |\partial^\alpha f(x)|\, d\mu_L$$

$$= \int_{\mathbb{R}^N} \frac{|g(x)|}{(1+x\cdot x)^n} (1+x\cdot x)^n |\partial^\alpha f(x)|\, d\mu_L$$

$$\leq \int_{\mathbb{R}^N} \frac{|g(x)|}{(1+x\cdot x)}\, d\mu_L\; |||f|||_n \quad \text{for } n \text{ large enough.}$$

Thm: let $H_{-n}(\mathbb{R}^N) := H_n'(\mathbb{R}^n)$. Then

$$S'(\mathbb{R}^N) = \bigcup_{n \geq 0} H_{-n}(\mathbb{R}^n)\Big|_{S(\mathbb{R}^N)}$$

proof: Let $\varphi \in S'(\mathbb{R}^N)$. Then

$0 \in \varphi^{-1}(B(0,1)) \in \tau_S$. Therefore, $\exists \varepsilon > 0$,

$\exists\, n \in \mathbb{N}$ s.t. $\{ f \in S(\mathbb{R}^N) \mid \|f\|_n < \varepsilon \}$
$\subset \varphi^{-1}(B(0,1))$

Hence, φ is in the topological dual of $H_n(\mathbb{R}^N)$. There is a unique and isometric extension Φ of φ to all $H_n(\mathbb{R}^N)$.

So, $\Phi \in H_n'(\mathbb{R}^N) = H_{-n}(\mathbb{R}^N)$

The reverse inclusion is clear $\square$.

So if $\varphi \in S'(\mathbb{R}^N)$, then there is $C > 0$, and $n \in \mathbb{N}$ s.t.

$$|\varphi(f)| \leq C \, |||f|||_n$$

the smallest possible $n \in \mathbb{N}$ is called the order of φ.

Schwartz representation thm: Let $\varphi \in S'(\mathbb{R}^N)$. Then there is a finite family $\{g_\alpha\}_{\alpha \in \mathbb{N}^N_{\leq n}}$ of continuous and polynomially bounded functions, s.t.

$$\varphi(f) = \int_{\mathbb{R}^N} \sum_{\alpha \in \mathbb{N}^N_{\leq n}} g_\alpha \, \partial^\alpha f(x) \, d\mu_L$$

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proof: By the previous thm we know that there is some $\Phi \in H_{-n}(\mathbb{R}^N)$ so that $\varphi(f) = \Phi(f)$.

By identifying $H_n(\mathbb{R}^N)$ with some closed subspace of $K = \bigoplus_{\alpha \in \mathbb{N}^N_{\leq n}} L^2(\mathbb{R}^N, (1+x\cdot x)^n d\mu_L)$ we know by Riesz & Fréchet that there is an element $(h_\alpha)_{\alpha \in \mathbb{N}^N_{\leq n}} \in K$ so that

$$\phi(f) = \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}^N} h_\alpha(x)(1+x\cdot x)^n \partial^\alpha f \, d\mu_L$$

For $x \in \mathbb{R}^N$, set $E_x := \{ y \in \mathbb{R}^N \text{ s.t. } y = \sigma x,\ \sigma \in [0,1]^N \}$

Define

$$\mathbb{R}^N \ni x \mapsto g_{\alpha+\bar{1}}(x) := (-1)^N \operatorname{sgn}(x) \int_{E_x} h_\alpha(y)(1+y\cdot y)^n d\mu_L$$

As an exercise, prove that

$$\phi(f) = \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}^N} g_{\alpha+\bar{1}} \, \partial^{\alpha+\bar{1}} f(x) \qquad \square.$$

<u>Remark:</u> a $\varphi \in S'(\mathbb{R}^N)$ has infinitely many such representations. e.g.:

$$\delta_0(f) = \int_{\mathbb{R}} (-\Theta(x)) f'(x) + a f'(x) + b f''(x) + \dots) \, d\mu_L$$

<u>Corollary:</u> Let $\varphi \in S'(\mathbb{R}^N)$. There is a sequence $(f_k)_{k\in\mathbb{N}} \subset S(\mathbb{R}^N)$ s.t. $\forall f \in S(\mathbb{R}^N)$

$$\varphi(f) = \lim_{k\to\infty} \int_{\mathbb{R}^N} f_k(x) f(x) \, d\mu_L$$

$$\left(= \lim_{k\to\infty} \varphi_{f_k}(f) \right)$$

proof: There is an $n \in \mathbb{N}$, and some $\Phi \in H_n{}'(\mathbb{R}^N)$ s.t. $\forall f \in S(\mathbb{R}^N)$, $\varphi(f) = \Phi(f)$. By Riesz and Fréchet, there is $g \in H_n(\mathbb{R}^N)$ s.t.

$\varphi(f) = \langle g, f \rangle_n$. Since $S(\mathbb{R}^N)$ is dense in $H_n(\mathbb{R}^N)$, there is a sequence $(h_k)_{k \in \mathbb{N}} \subset S(\mathbb{R}^N)$ s.t. $\varphi(f) = \langle g, f \rangle_n = \langle \lim_k h_k, f \rangle_n$

$$= \lim_k \langle h_k, f \rangle_n$$

$$= \lim_k \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}^N} \overline{\partial^\alpha h_k} \, (1+x\cdot x)^n \partial^\alpha f \, d\mu_L$$

We conclude by setting

$$f_k = \sum_{\alpha \in \mathbb{N}^N_{\leq n}} (-1)^{|\alpha|} \partial^\alpha \left(\overline{\partial^\alpha h_k} \, (1+x\cdot x)^n \right)$$

$\square$.

Remark: The sequence $(f_k)_{k \in \mathbb{N}}$ thusly obtained does not converge to some $f \in S(\mathbb{R}^N)$. It converges in the space $S'(\mathbb{R}^N)$.

This motivates the

Def: The weak*-topology $\tau(S', S)$ is defined as the collection of all sets $U \subset S'$ s.t. $\forall \varphi \in U, \exists f_1, \dots, f_n$ s.t.

$$\{ \psi \in S' \mid |\psi(f_k) - \varphi(f_k)| < 1 \ \forall k = 1, \dots, n \} \subset U$$

The reader may check that $\tau(S', S)$ is indeed a topology, and $S(\mathbb{R}N)$ is dense in $S'(\mathbb{R}N)$

for $\tau(S', S)$. $S'(\mathbb{R}^N)$ is also complete under $\tau(S', S)$

($\tau(S, S')$ is the weak topology)

§ 1.3. Fréchet Spaces

There is no norm which induces τ_S on $S(\mathbb{R}^N)$.

Def: A <u>metric space</u> is a pair (X, d), where X is a set and $d : X \times X \longrightarrow \mathbb{R}_+$, called <u>distance</u>, satisfies

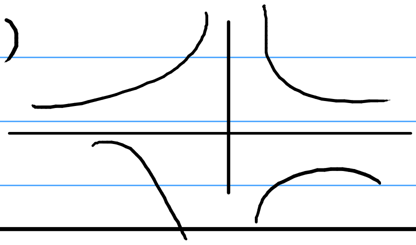
1) $\forall x, y \in X$, $d(x,y) = d(y,x)$ and $d(x,y) = 0$ $\Longleftrightarrow x = y$
2) $\forall x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$

In a metric space, open balls $B(x,r)$ are defined as $\{ y \in X \mid d(y,x) < r \}$ and $r \in \mathbb{R}_+^*$.
The topology τ_d on X is the collection of sets $U \subset X$ s.t. $\forall x \in U$, $\exists r > 0$ s.t. $B(x,r) \subset U$.
A sequence $(x_k)_{k \in \mathbb{N}} \subset X$ <u>converges</u> to $x \in X$ iff $\lim_{k \to \infty} d(x, x_k) = 0$.
If $\forall \varepsilon > 0$, $\exists n_k \in \mathbb{N}$ s.t. $k, \ell \geq n_k$ implies $d(x_k, x_\ell) < \varepsilon$, then $(x_k)_{k \in \mathbb{N}}$ is <u>Cauchy</u>.
(X, d) is said to be <u>complete</u> iff every Cauchy sequence converges in X.

Remark: in general, $B(x,r)$ are not convex sets.

e.g. $X = \mathbb{R}^2$, $d(x,y) := |x_1-y_1|^{1/2} + |x_2-y_2|^{1/2}$

$B(0,1)$



Baire category theorem: Let (X,d) be a complete metric space. If $(U_k)_{k\in\mathbb{N}}$ is a sequence of dense and open sets in X. Then $\bigcap_{k\in\mathbb{N}} U_k$ is dense again.

proof: Let $W \subset X$ be open and non-empty. The proof is complete if we can show, that

$W \cap \left(\bigcap_{k\in\mathbb{N}} U_k\right) \neq \emptyset$.

The set $W \cap U_0$ is not empty and open. Therefore, there is an $x_0 \in X$ and an $r_0 < 1$ s.t.

$$B(x_0,r_0) \subset \overline{B(x_0,r_0)} \subset W \cap U_0$$

By induction, given the open $B(x_k,r_k) \subset W \cap \left(\bigcap_{l\leq k} U_l\right)$, then there is a point x_{k+1} and an $r_{k+1} < 2^{-k}$, so that

$$B(x_{k+1}, r_{k+1}) \subset \overline{B(x_{k+1},r_{k+1})} \subset B(x_k,r_k) \cap U_{k+1}$$

We end up with a nested sequence of open balls $\{B(x_k,r_k)\}_{k\in\mathbb{N}}$ s.t. $\forall k\in\mathbb{N}$,

$$x_{k+1} \in B(x_k,r_k) \subset \overline{B(x_k,r_k)} \subset W \cap \left(\bigcap_{l\leq k} U_l\right)$$

$(x_k)_{k\in\mathbb{N}}$ is Cauchy and by completeness converges to some $x\in X$. Clearly,
$x \in \overline{B(x_k, r_k)} \quad \forall k\in\mathbb{N}$

so $x\in \bigcap_k \overline{B(x_k,r_k)} \subset W\cap\left(\bigcap_{k\in\mathbb{N}} U_k\right)$

$\square$.

Consider the norms $\|\ \|_n$ on $\mathcal{S}(\mathbb{R}^N)$ and define

$$d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\frac{\|f-g\|_n}{1+\|f-g\|_n}$$

One can check that $d(f,g)$ is a distance on $\mathcal{S}(\mathbb{R}^N)$, is translationally invariant, $d(f+h, g+h) = d(f,g)$ and $\tau_d = \tau_S$

$\left(x \longmapsto \dfrac{x}{1+x} \right)$

$\mathcal{S}(\mathbb{R}^N)$ with this distance is an example of

Def: A **Fréchet space** is a pair (V,τ), where V is a vector space (over $\mathbb{K}$) and τ is a topology on V satisfying

1) τ is induced by a family $\{\|\ \|_j : j\in J\}$ of (semi-)norms on V,
2) τ is induced by a translationally invariant distance d on V.
3) (V,τ) is complete.

(if $U \ni 0$ is open and convex, define
$\|x\|^{-1} = \sup\{\lambda \in \mathbb{R} \mid \lambda x \in U\}$ which is a norm).

$\boxed{\begin{array}{l}\text{Uniform boundedness principle: Let } \mathcal{F} \text{ be a Fréchet space and let } \{T_j\}_{j\in J} \text{ be} \\ \text{a family of continuous and linear maps from } \mathcal{F} \text{ to some normed space } (V, \|\ \|). \text{ Suppose} \\ \text{that } \{T_j\}_{j\in J} \text{ is simply bounded:} \\ \qquad \forall x \in \mathcal{F}, \exists C>0 \text{ s.t. } \forall j \in J,\ \|T_j(x)\| < C. \\ \text{Then } \{T_j\}_{j\in J} \text{ is equicontinuous:} \\ \qquad \exists r>0,\ \forall j\in J,\ \forall x \in B(0,r),\ \|T_j(x)\| < 1\end{array}}$

<u>proof</u>: Set

$$E_n := \{x \in \mathcal{F} : \forall j \in J,\ \|T_j(x)\| \leq n\}$$
$$= \bigcap_{j\in J} T_j^{-1}\{v \in V \mid \|v\| \leq n\}$$

All E_n's are closed in $\mathcal{F}$ and $\bigcup_{n\in\mathbb{N}} E_n = \mathcal{F}$.
Hence, $\bigcap_{n\in\mathbb{N}} (\mathcal{F} \setminus E_n) = \emptyset$.
By Baire's category theorem, $\exists n \in \mathbb{N}$ s.t.
$\mathcal{F}\setminus E_n$ is not dense in $\mathcal{F}$. Therefore,
$\exists x \in \mathcal{F}$, $\exists r>0$ s.t. $B(x,r) \subset E_n$. Thus,
if $y \in B(0,r)$, then $\forall j \in J$, $\|T_j(x+y)\| \leq n$.
By translational invariance, $-y \in B(0,r)$ if
$y \in B(0,r)$ and $n \geq \|T_j(x-y)\|$
$\geq |\|T_j(x)\| - \|T_j(y)\||$ $\quad \forall j \in J$.

Therefore, $\forall y \in B(0,r)$, $\forall j \in J$
$\|T_j(y)\| \leq 2n$. If $\varepsilon = \frac{r}{3n}$, then
$\|T_j(y)\| < 1$ if $y \in B(0,\varepsilon)$ and $j \in J$.
$\square$.

<u>Corollary:</u> $S'(\mathbb{R}^N)$ is complete for the weak $*$- topology. $\square$.

§1.4. Continuous maps on $S(\mathbb{R}^N)$ and $S'(\mathbb{R}^N)$

<u>Def:</u> Let X and Y denote the spaces $S(\mathbb{R}^N)$ or $S'(\mathbb{R}^N)$ and let τ_X and τ_Y denote the topologies τ_S or $\tau(S,S')$.
Then $f: X \to Y$ is said to be continuous iff $\forall U \in \tau_Y$, $f^{-1}(U) \in \tau_X$.

<u>Thm:</u> Let $T: S(\mathbb{R}^N) \to S(\mathbb{R}^N)$ be linear. The following statements are equivalent:
1) T is continuous
2) For any norm $\| \ \|_n$ there is a positive constant $C > 0$ and a norm $\| \ \|_m$ s.t. $\forall f \in S(\mathbb{R}^N)$
$\|T(f)\|_n \leq C \|f\|_m$
3) For a sequence $(f_k)_{k\in\mathbb{N}} \subset S(\mathbb{R}^N)$ s.t. $\lim_{k\to\infty} f_k = 0$ for τ_S, $\lim_k T(f_k) = 0$.
4) For a sequence $(f_k)_{k\in\mathbb{N}} \subset S(\mathbb{R}^N)$ s.t. $\lim_k f_k = f$, $\lim_k T(f_k) = T(f)$.

proof: exercise $\triangleright$

As an exercise one may prove that

$\lim_k (f_k + g_k) = f + g$ and

$\lim_k (f_k g_k) = f \cdot g$ for τ_S.

if $(\varphi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}} \subset S'(\mathbb{R}^N)$ s.t.

$\lim_k \varphi_k = \varphi$ and $\lim_k \psi_k = \psi$ for $\tau(S', S)$

then $\lim_k (\varphi_k + \psi_k) = \varphi + \psi$

Other continuous maps are

$S(\mathbb{R}^N) \ni f \longmapsto \partial^\alpha f \in S(\mathbb{R}^N), \alpha \in \mathbb{N}^N$

$S(\mathbb{R}^N) \ni f \longmapsto x^\alpha f \in S(\mathbb{R}^N), \alpha \in \mathbb{N}^N$

$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_N^{\alpha_N} = x^\alpha$

$\boxed{\begin{array}{l}\text{Def: The space } \mathcal{O}_M(\mathbb{R}^N) \text{ of \underline{multipliers}} \\ \text{is the subset of } C^\infty(\mathbb{R}^N) \text{ consisting of} \\ \text{polynomially bounded and smooth functions}\end{array}}$

If $T: S(\mathbb{R}^N) \longrightarrow S(\mathbb{R}^N)$ is continuous and linear,

then $T^t: S'(\mathbb{R}^N) \longrightarrow S'(\mathbb{R}^N)$

$\varphi \longmapsto \varphi \circ T$

is continuous and linear for $\tau(S', S)$.

$\boxed{\begin{array}{l}\text{Def: For } \alpha \in \mathbb{N}^N \text{ and } \varphi \in S'(\mathbb{R}^N), \\ \quad D^\alpha \varphi := (-1)^{|\alpha|} (\partial^\alpha)^T (\varphi)\end{array}}$

Examples: 1) The Heavyside function $\Theta(x) = 1_{\mathbb{R}^+}$ defines a tempered distribution φ_Θ and $D^1\varphi_\Theta = \delta_0$. If $n \in \mathbb{N}^*$

$$D^n \varphi_\Theta = D^{n-1}\delta_0$$

$$(D^{n-1}\delta_0)(f) = (-1)^{n-1} f^{(n-1)}(0).$$

2) $x\delta_0(f) = \delta_0(x \cdot f) = 0. = \varphi_0$

3) $(x \cdot \mathrm{p.v.}(1/x))(f) = \mathrm{p.v.}(1/x)(xf)$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} x \frac{1}{x} f(x)\, d\mu_L = \int_{\mathbb{R}} f(x)\, d\mu_L$$

$$= \varphi_1(f).$$

Schwartz's impossibility result: There is no associative product $\times$ on $S'(\mathbb{R})$ s.t.

- $\forall \varphi \in S'(\mathbb{R}^N)$, $\varphi_0 \times \varphi = \varphi_0$, $\varphi_1 \times \varphi = \varphi \times \varphi_1 = \varphi$
- if f, g are polynomially bounded and μ_L-measurable, then $\varphi_f \times \varphi_g = \varphi_{fg}$ and $\varphi_{f+g} = \varphi_f + \varphi_g$
- if f, g are polynomially bounded and μ_L measurable, then if $f(x) = \int_0^x g(t)\,\mu_L(dt)$, then $D\varphi_f = \varphi_g$
- $\forall \varphi, \psi \in S'(\mathbb{R}^N)$, $D(\varphi \times \psi) = D(\varphi)\times\psi + \varphi \times D\psi$.

---

proof: For such a multiplication one may check that $\varphi_0 = \varphi_x \times \delta_0 = \delta_0 \times \varphi_x$ and

$$\varphi_x \times \mathrm{p.v.}(1/x) = \varphi_1.$$

But then, $\varphi_0 = (\delta_0 \times \varphi_x) \times \mathrm{p.v.}(1/x)$
$= \delta_0 \times (\varphi_x \times \mathrm{p.v.}(1/x)) = \delta_0$ $\square$

Remark. This "no-go" result can be generalised to $S'(\mathbb{R}^N)$. In QFT one would like to have objects like (ϕ^4) or $j^\mu A_\mu$, which no one knows how to define (at least in 3+1 dimensions). Glimm & Jaffe produced some interacting models in (2+1) dimensions. Colombeau defined a more general algebra of function, containing $S'(\mathbb{R}^N)$. In this algebra

$$\varphi_f \times \varphi_g = \varphi_{f\cdot g} \quad \text{only if} \quad f,g \in C^\infty(\mathbb{R}^N).$$

Convolution between two functions f, g is given by

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)\, g(y)\, \mu_L(dy)$$

this convolution is certainly well-defined for $f \in L^p(\mathbb{R}^N, \mu_L)$, $g \in L^q(\mathbb{R}^N, \mu_L)$ and $\frac{1}{p}+\frac{1}{q}=1$

We will be interested in the case $f \in S(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N, \mu_L)$.

In this case $(f*g)(x) \in C^\infty(\mathbb{R}^N)$
(use the Leibnitz integral rule)

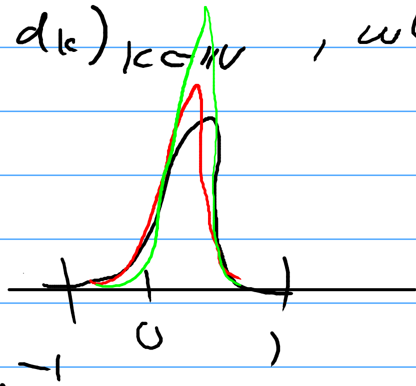
If $f, g \in S(\mathbb{R}^N)$ then
$(f * g)(x) \in S(\mathbb{R}^N)$ and if f is fixed,

then $\mathcal{S}(\mathbb{R}^N) \ni g \longmapsto f * g \in \mathcal{S}(\mathbb{R}^N)$ is linear and continuous (for τ_S)

Let $f \in C^\infty(\mathbb{R}^N)$ and $\operatorname{supp}(f) \subset B(0,1)$, and $f \geq 0$, and $\int_{\mathbb{R}^N} f(x)\,\mu_L(dx) = 1$

This allows the definition of a Dirac-sequence (mollifier) $(d_k)_{k\in\mathbb{N}}$, where

$$d_k(x) := k^N f(kx)$$



One can show, that

$$\lim_{k\to\infty} d_k * f = f \quad \text{for } \tau_S.$$

What happens to $\varphi \in \mathcal{S}'(\mathbb{R}^N)$?

Let's take φ_h, where h is polynomially bounded and continuous,

$$\int_{\mathbb{R}^N} h(x)\,(f*g)(x)\,\mu_L(dx) \quad \left(= \varphi_h(f*g)\right)$$

$$= \int_{\mathbb{R}^N} h(x) \int_{\mathbb{R}^N} f(x-y)\,g(y)\,\mu_L(dy)\,\mu_L(dx)$$

$$= \int_{\mathbb{R}^N} g(y) \int_{\mathbb{R}^N} h(x)\,f(x-y)\,\mu_L(dx)\,\mu_L(dy)$$

$$= \int_{\mathbb{R}^N} g(y)\,(P^+ f * h)\,\mu_L(dy)$$

where $P(x) = -x$ and $P^+ f(x) = f(-x)$

So, $\forall g \in S(\mathbb{R}^N)$, $(f * \varphi)(g) := \varphi(P^+(f) * g)$

Convolution with a fixed $f \in S(\mathbb{R}^N)$ becomes a continuous map on $S'(\mathbb{R}^N)$ for the weak $*$-topology.

If $(d_k)_{k \in \mathbb{N}}$ is a Dirac-sequence then one can show that the sequence

$$(d_k * \varphi)_{k \in \mathbb{N}} \subset S'(\mathbb{R}^N)$$

converges to φ for the weak $*$-topology.
Moreover for $k \in \mathbb{N}$ fixed, $d_k * \varphi \in C^\infty(\mathbb{R}^N)$
This is called regularisation of φ.

§1.5. The Fourier transform:

Def: We define linear maps $\mathcal{F}, \mathcal{F}^*$ by

$$(\mathcal{F} f)(p) = \hat{f}(p) := \frac{1}{(2\pi)^{N/2}} \int e^{-ip\cdot x} f(x)\, \mu_L(dx)$$

$$(\mathcal{F}^* f)(p) = \check{f}(p) := \frac{1}{(2\pi)^{N/2}} \int e^{ip\cdot x} f(x)\, \mu_L(dx)$$

$\mathcal{F}, \mathcal{F}^*$ are well-defined on $S(\mathbb{R}^N)$. Moreover,

$$\widehat{\partial^\alpha f}(p) = (ip)^\alpha \hat{f}(p), \qquad \check{\partial^\alpha f}(p) = (-ip)^\alpha \check{f}(p),$$

$$\widehat{(-ix)^\alpha f}(p) = \partial^\alpha \hat{f}(p), \qquad \check{(ix)^\alpha f}(p) = \partial^\alpha \check{f}(p)$$

$\mathcal{F}$ and $\mathcal{F}^*$ are also continuous endomorphisms on $S(\mathbb{R}^N)$.

Moreover:

Thm: If $f, g \in S(\mathbb{R}^N)$ then

$$\mathcal{F}(f * g)(p) = (2\pi)^{N/2} \hat{f}(p)\, \hat{g}(p)$$

proof:
$$\mathcal{F}(f * g)(p) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(-ip\cdot x)\,(f*g)(x)$$

...... $\square$.

Lemma $\mathcal{F}$ is inverse to $\mathcal{F}^*$:

$$\mathcal{F} \circ \mathcal{F}^* = \mathbb{1}_{S(\mathbb{R}^N)} = \mathcal{F}^* \circ \mathcal{F}$$

proof: For $\varepsilon > 0$, consider

$I_\varepsilon(x) :=$

$$\frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} f(y) \exp(ik\cdot(x-y)) \exp\left(-\frac{k^2}{2}\varepsilon^2\right) \mu_L(dk\,dy)$$

Apply Fubini's theorem and dominated convergence.
Details are left to the reader $\square$.

Corollary: $\forall f, g \in S(\mathbb{R}^N)$, $\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2}$

Hence, $\|g\|_{L^2} = \|\hat{g}\|_{L^2} = \|\check{g}\|_{L^2}$

One notices that $\hat{\check{f}} = \overline{\check{f}}$ and details like Fubini $\square$.

In the case where $\varphi_g \in S'(\mathbb{R}^N)$ with $g \in S(\mathbb{R}^N)$, then we have

$$\varphi_{\hat{g}}(f) = \langle \overline{\hat{g}}, f \rangle_{L^2} = \langle \check{\bar{g}}, f \rangle_{L^2}$$
$$= \langle \bar{g}, \hat{f} \rangle_{L^2} = \varphi_g(\hat{f})$$

<u>Def</u>: On $S'(\mathbb{R}^N)$ we define for $\varphi \in S'(\mathbb{R}^N)$

$$\mathcal{F}(\varphi)(f) := \varphi(\mathcal{F}(f))$$

or $\hat{\varphi}(f) = \varphi(\hat{f})$

Clearly, $\mathcal{F}$ is weak*-continuous. One also has

$$\mathcal{F}(D^\alpha \varphi) = (ip)^\alpha \mathcal{F}(\varphi) \quad \text{and}$$

$$\mathcal{F}((-ix)^\alpha \varphi) = D^\alpha \mathcal{F}(\varphi)$$

<u>Examples</u>:

1) Let's consider φ_Θ. Manifestly,

$$\lim_{\varepsilon \to 0^+} \varphi_{e^{-\varepsilon x}\Theta(x)} = \varphi_\Theta$$

By weak*-continuity,

$$\mathcal{F}(\varphi_\Theta) = \lim_{\varepsilon \to 0^+} \mathcal{F}(\varphi_{e^{-\varepsilon x}\Theta(x)}).$$

For $f \in S(\mathbb{R}^N)$, we have

$$\mathcal{F}(\varphi_{e^{-\varepsilon x}\Theta(x)})(f) = \varphi_{e^{-\varepsilon x}\Theta(x)}(\mathcal{F}(f))$$
$$= \langle \overline{e^{-\varepsilon x}\Theta(x)}, \mathcal{F}(f) \rangle_{L^2}$$
$$= \langle \mathcal{F}^*(\overline{e^{-\varepsilon x}\Theta(x)}), f \rangle_{L^2}$$
$$= \langle \overline{\mathcal{F}(e^{-\varepsilon x}\Theta(x))}, f \rangle_{L^2} .$$

But :

$$\mathcal{F}(e^{-\varepsilon x}\Theta(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipx} e^{-\varepsilon x}\Theta(x)\, \mu_L(dx)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-ipx - \varepsilon x)\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{ip+\varepsilon} = \frac{1}{\sqrt{2\pi}} \frac{\varepsilon - ip}{p^2+\varepsilon^2}.$$

We have seen that $\lim_{\varepsilon \to 0} \frac{p}{p^2+\varepsilon^2} = \text{p.v.}\left(\frac{1}{p}\right)$

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{\varepsilon}{p^2+\varepsilon^2} f(p)\, \mu_L(dp) = -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \operatorname{Arctan}\left(\frac{p}{\varepsilon}\right) f'(p)\, dp$$
$$= -\int_{\mathbb{R}} \lim_{\varepsilon \to 0^+} \operatorname{Arctan}\left(\frac{p}{\varepsilon}\right) f'(p)\, dp = -\frac{\pi}{2} \int_{\mathbb{R}} \operatorname{sgn}(p)\, f'(p)\, dp$$
$$= \pi \delta_0(f)$$

In conclusion:

$$\mathcal{F}(\varphi_\Theta) = \frac{1}{\sqrt{2\pi}} \left(\pi \delta_0 - i\, \text{p.v.}\left(\frac{1}{p}\right) \right)$$

2) If $f \in \mathcal{L}^1(\mathbb{R}, \mu_L)$ and set

$$F(x) := \int_{-\infty}^x f(t)\, d\mu_L .$$

Observe that $F(x) = (\Theta * f)(x)$, at least in the distributional sense. Hence $g \in S(\mathbb{R})$

$$\mathcal{F}(\varphi_F)(g) = \varphi_F(\check{g}) = \varphi_{f*\Theta}(\check{g})$$

$$= \varphi_\Theta(p^*(f) * \check{g}) = \mathcal{F}(\varphi_\Theta)(\mathcal{F}^*(p^*(f) * \check{g}))$$

$$= \sqrt{2\pi}\, \mathcal{F}(\varphi_\Theta)(\check{f} g) \text{. Therefore}$$

$$\mathcal{F}(\varphi_F) = \left(\pi\, \check{f}(p)\, \delta_0 - i\, \text{p.v.}\left(\frac{\check{f}(p)}{p}\right)\right)$$

3) Fourier transforms may be used to find solution (ie the distributional sense) of partial linear diff. eq.

Klein-Gordon: $(\Box + m^2)\, g = (\partial_\nu \partial^\nu - m^2) g = f$

Let's solve this for $f \in S(\mathbb{R}^N)$ fixed.

Write $f = \delta_0^{(4)} * f$, where $\delta_0^{(4)} := \mathcal{F}^*\left(\frac{1}{(2\pi)^2}\right)$

We will find solutions if

$$(\Box + m^2)\, G = \delta_0^{(4)}$$

$$\Rightarrow \quad g = G * f$$

G is then called a Green's "function" for $(\Box + m^2)$

After a Fourier transform,

$$(p_\mu \eta^{\mu\nu} p_\nu - m^2)\, \tilde{G} = \frac{-1}{(2\pi)^2}$$

$$\text{where } \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -\mathbb{1}_3 & \\ 0 & & & \end{pmatrix}$$

We start by setting $E_p := \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}$

so that, $p_\mu \eta^{\mu\nu} p_\nu - m^2 = (p_0 + E_p)(p_0 - E_p)$

For $\varepsilon > 0$ we define

$$g^{c}_{\varepsilon,\pm} := \frac{-1}{(2\pi)^2} \frac{1}{(p_0 \mp i\varepsilon + E_p)(p_0 \mp i\varepsilon - E_p)}$$

$$g^{F}_{\varepsilon,\pm} = \frac{-1}{(2\pi)^2} \frac{1}{(p_0 \mp i\varepsilon + E_p)(p_0 \pm i\varepsilon - E_p)}$$

First step: check that $\lim_{\varepsilon \to 0^+} \varphi_{g^{c,F}_{\pm,\varepsilon}}$ exist in $S'(\mathbb{R}^N)$

Second step: $(p_\mu \eta^{\mu\nu} p_\nu - m^2) \lim_{\varepsilon \to 0^+} \varphi_{g^{c,F}_{\pm,\varepsilon}} = \frac{-1}{(2\pi)^2}$

§1.6. The Fourier–Laplace transform

The idea is to compute

$$\frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(-ix\cdot z)\, f(x)\, d\mu_L$$

In order to give sense to this we will apply it to $f \in S(\mathbb{R}^N)$ s.t. $\mathrm{supp}(f)$ is compact.

We have the

Paley & Wiener's theorem: Let $f \in S(\mathbb{R}^N)$ with $\mathrm{supp}(f) \subset B(0,r)$. Then the Fourier-Laplace transform $\mathcal{L}(f, q)(p)$ is an entire function $F(p+iq)$ that satisfies

$$\forall n \in \mathbb{N}, \quad |F(z)| \leq C_{n,f} \, (1+|z|)^{-n} \exp(r\,|q|).$$

Conversely, any entire function $F(z)$ with the above estimations then it is the Fourier-Laplace transform of some $f \in S(\mathbb{R}^N)$ with $\mathrm{supp}(f) \subset B(0,r)$

<u>proof:</u> Suppose $\mathrm{supp}(f) \subset B(0,r)$, so that $\exp(q\cdot x)$ remains bounded on $B(0,r)$ and $f(x)\exp(-ix\cdot(p+iq))$ is still a Schwartz function.

By definition

$$\mathcal{L}(f,q)(p) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(-ix\cdot(p+iq))\, f(x)\, \mu_L(dx).$$

Note that on $B(0,r)$, $\exp(-ix\cdot(p+iq)) = \sum_{n\geq 0} \frac{(-ix\cdot(p+iq))^n}{n!}$

$= \sum_{\alpha\in\mathbb{N}^N} \frac{1}{\alpha!} z^\alpha (-ix)^\alpha$ and the limit is uniform.

Therefore,

$$\mathcal{L}(f,q)(p) = \frac{1}{(2\pi)^{N/2}} \sum_{\alpha\in\mathbb{N}^N} \frac{1}{\alpha!} z^\alpha \int_{\mathbb{R}^N} (-ix)^\alpha f(x)\, \mu_L(dx)$$

$$= \frac{1}{(2\pi)^{N/2}} \sum_{\alpha\in\mathbb{N}^N} \frac{r^{|\alpha|}}{\alpha!} z^\alpha \int_{\mathbb{R}^N} \left(-\frac{ix}{r}\right)^\alpha f(x)\, \mu_L(dx),$$

which defines an entire function $F(z)$.

For $\alpha \in \mathbb{N}^N_{\leq n}$, we have by integration by parts:

$$(-iz)^\alpha F(z) = \frac{1}{(2\pi)^{N/2}} \int_{B(0,r)} \left(\partial^\alpha \exp(-ix\cdot z)\right) f(x)\, \nu_L(dx)$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{N/2}} \int_{B(0,r)} \exp(-ix\cdot z)\, \partial^\alpha f(x)\, \nu_L(dx).$$

Hence,

$$|z^\alpha|\,|F(z)| \leq \frac{1}{(2\pi)^{N/2}} \int_{B(0,r)} \exp(q\cdot x)\, |\partial^\alpha f(x)|\, \nu_L(dx)$$

$$\leq \underbrace{\frac{1}{(2\pi)^{N/2}} |||f|||_n \left(\int_{B(0,r)} \frac{1}{(1+x\cdot x)^n} \nu_L(dx)\right)}_{C'_{n,f}} e^{r|q|}$$

So,

$$(1+|z|)^n\, |F(z)| \leq \left(1 + \sum_{k=1}^{N} |z_k|\right)^n |F(z)|$$

$$= \sum_{\beta \in \mathbb{N}^N_{\leq n}} \frac{n!}{(n-|\beta|)!\,\beta!} |z^\beta|\, |F(z)|$$

$$\leq \underbrace{\sum_{\beta \in \mathbb{N}^N_{\leq n}} \frac{n!}{(n-|\beta|)!\,\beta!} C'_{n,f}}_{C_{n,f}} e^{r|q|}$$

Conversely, let $F(z)$ is entire with the indicated inequalities. These imply

$$\check{F}(x) := \int_{\mathbb{R}^N} \exp(ix\cdot p)\, F(p)\, \nu_L(dp)$$

is well-defined.

Leibnitz integral rule proves the smoothness of $\check{F}(x)$.

Let then $|x|>r$ and choose $q\in\mathbb{R}^N$ s.t. $q=\lambda x$, $\lambda\in\mathbb{R}$. We then have

$$|\exp(ix\cdot(p+iq))F(p+iq)| \leq C_{n}(1+|p+iq|)^{-n} \times \exp((r-|x|)|x|\lambda)$$

Note that

$$\check{F}(x) = \int_{\mathbb{R}^N} \exp(ix(p+iq))F(p+iq)\,\mathcal{N}_L(dp)$$

by multiple contour integration and entirety of $F(z)$.

Therefore,

$$|\check{F}(x)| \leq C_N\left(\int_{\mathbb{R}^N} \frac{1}{(1+|p+iq|)^N}\,\mathcal{N}_L(dp)\right)e^{(r-|x|)|x|\lambda}$$

Taking the limit when $\lambda\to\infty$ yields

$$\check{F}(x)=0. \qquad \square.$$

Def: Let $\varphi\in\mathcal{S}'(\mathbb{R}^N)$. For an open set $U\subset\mathbb{R}^N$ define $\varphi_U:=\{\varphi(f)\mid f\in\mathcal{S}(\mathbb{R}^N),\ \mathrm{supp}(f)\subset U\}$

Define

$$\mathrm{supp}(\varphi):=\mathbb{R}^N\setminus\Big(\bigcup_{\substack{U\subset\mathbb{R}^N,\ U\text{ open},\\ \varphi_U=\{0\}}} U\Big)$$

$$\int_U g\cdot f\,d\mu = 0 \quad \forall f\ \ \mathrm{supp}(f)\subset U$$

$$\Rightarrow g|_U=0 \text{ a.e.}$$

Suppose $\varphi\in\mathcal{S}'(\mathbb{R}^N)$ with compact support.

This means, $\exists r > 0$ s.t. $\varphi(f) = 0$ as soon as $\mathrm{supp}(f) \cap B(0,r) = \emptyset$.

For $0 < r < r'$ define the function

$$\rho_{r,r'}(x) := \begin{cases} 1 & \text{if } |x| \leq r \\ 0 & \text{if } |x| \geq r' \\ \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{|x| - \frac{r+r'}{2}}{(|x|-r)(|x|-r')}\right) & \text{if } r < |x| < r' \end{cases}$$

Such a function is called a smooth step-function.

Obviously, $1 = \rho_{r,r'} + (1 - \rho_{r,r'})$ and

$$\varphi(f) = \varphi(\rho_{r,r'} f). \text{ As a consequence,}$$

$$\exp(q \cdot x)\rho_{r,r'}\varphi = \exp(q\cdot x)\varphi$$

<u>Def:</u> Let $\varphi \in S'(\mathbb{R}^N)$ of compact support in $B(0,r)$. The Fourier-Laplace transform is defined as

$$\mathcal{L}(\varphi, q) := \mathcal{F}(\exp(q\cdot x)\varphi)$$

<u>Remarks:</u>

- Schwartz's representation theorem states that

$$\varphi(f) = \int_{\mathbb{R}^N} g(x)\, \partial^\alpha f(x)\, \mu_L(dx)$$

with g polynomially bounded and continuous.

If $\mathrm{supp}(\varphi) \subset B(0,r)$, then $\varphi(f) = \varphi(\rho_{r,r'} f)$ and $g(x)$ may just as well be chosen to be supported in $B(0,r')$

• if $\mathrm{supp}(\varphi) \subset B(0,r)$ is of order n $(|\varphi(f)| \leq C \,|||f|||_n)$. But,

$$|\varphi(f)| \leq C\,|||\rho_{r,r'} f|||_n \leq C'\,|||\rho_{r,r'}|||_{n,r'} \times |||f|||_{n,r'}$$

where $|||f|||_{n,r'} = \sup\{\, |\partial^\alpha f(x)| \mid \alpha \in \mathbb{N}^N_{\leq n}, x \in B(0,r')\}$

• if $\mathrm{supp}(\varphi) \subset B(0,r)$ there is $r' < r$ s.t. $\mathrm{supp}(\varphi) \subset \overline{B(0,r')}$. We can therefore chose $r' = r$ in the above remark.

Schwartz's Paley & Wiener theorem: Let $\varphi \in S'(\mathbb{R}^N)$ be of $\mathrm{supp}(\varphi) \subset B(0,r)$. Then $\mathcal{L}(\varphi, q) = \varphi_{F(z)}$, where $F(z)$ is entire and satisfies

$$|F(z)| \leq C_n (1 + |z|)^n \exp(r|q|)$$

for some C_n and some $n \in \mathbb{N}$ and where n is the order of φ.

Conversely, if $F(z)$ is entire and as above, it is the Fourier-Laplace transform of some $\varphi \in S'(\mathbb{R}^N)$ with $\mathrm{supp}(\varphi) \subset B(0,r)$

proof: Suppose $\varphi \in S'(\mathbb{R}^N)$ of order n with $\mathrm{supp}(\varphi) \subset B(0,r)$. By the previous remarks, there is an $\alpha \in \mathbb{N}^N$ and a family $\{g_\beta\}_{\beta \leq \alpha}$ of continuous functions with $\mathrm{supp}(g_\beta) \subset B(0,r)$

s.t. $\forall f \in S(\mathbb{R}^N)$

$$\varphi(f) = \sum_{\beta \leq \alpha} \int_{\mathbb{R}^N} g_\beta \left(\partial^\beta f\right) \mu_L(dx).$$

By definition, $\alpha(\varphi, q) = \mathcal{F}\left(\rho \exp(q\cdot x)\, \varphi\right)$, where $\rho = \rho_{r', r}$, $r' < r$. Therefore,

$$\alpha(\varphi, q)(f) = \mathcal{F}(\rho \exp(q\cdot x)\, \varphi)(f)$$

$$= \sum_{\beta \leq \alpha} \int_{B(0,r)} g_\beta\, \partial_x^\beta \left(\frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(q\cdot x - i x\cdot y) f(y) \mu_L(dy) \right) \mu_L(dx)$$

By Fubini's thm:

$$\alpha(\varphi, q)(f) = \sum_{\beta \leq \alpha} \int_{\mathbb{R}^N} f(y) \left(\frac{1}{(2\pi)^{N/2}} \int_{B(0,r)} g_\beta(x)\, \partial_x^\beta \exp(q\cdot x - i x\cdot y)\, d\mu_x \right) d\mu_y$$

$$= \int_{\mathbb{R}^N} f(y) \frac{1}{(2\pi)^{N/2}} \varphi\left(\rho \exp(q\cdot x - i x\cdot y)\right) d\mu_y.$$

write $z = y + iq$.

$$\alpha(\varphi, q)(f) = \int_{\mathbb{R}^N} f(y)\, F(y, q)\, d\mu_y$$

where

$$F(z) = \frac{1}{(2\pi)^{N/2}} \varphi\left(\rho \exp(-i x\cdot z)\right)$$

On $B(0,r)$, the Schwartz function $\rho \exp(-i x\cdot z)$ is the uniform limit of $\sum_{\alpha \in \mathbb{N}^N} \rho \frac{1}{\alpha!} (-ix)^\alpha \cdot z^\alpha$ this holds also for all derivatives with $\beta \leq \alpha$

Since φ is of order n:

$$F(z) = \frac{1}{(2\pi)^{N/2}} \varphi\left(\sum_{\alpha \in \mathbb{N}^N} \rho \frac{1}{\alpha!} (-ix)^\alpha z^\alpha \right)$$

$$= \sum_{\alpha \in \mathbb{N}^N} \frac{1}{(2\pi)^{N/2}} \varphi(\rho\, (-ix)^\alpha z^\alpha)$$

$$= \sum_{\alpha \in \mathbb{N}^N} \frac{r^{|\alpha|}}{(2\pi)^{N/2}\alpha!} \varphi\left(\rho\left(-i\frac{x}{r}\right)^\alpha\right) \cdot z^\alpha .$$

Since $\left|\varphi\left(\rho\left(-i\frac{x}{r}\right)^\alpha\right)\right| \leq C' |||\rho|||_{u,r}\ |||\left(-i\frac{x}{r}\right)^\alpha|||_{u,r}$

this series defines an entire function.

Furthermore,

$$|F(z)| = \left|\frac{1}{(2\pi)^{N/2}} \varphi(\rho \exp(-ix\cdot z)\right|$$

$$\leq \underbrace{\frac{1}{(2\pi)^{N/2}} C' |||\rho|||_{u,r}}_{C_u}\ |||\exp(-ix\cdot z)|||_{u,r}$$

$$\leq C_u\, (1+|z|)^u \exp(r\,|q|)$$

Conversely, let $F(z)$ be entire which satisfies the given inequalities. Then $F(x)$ is continuous and polynomially bounded. There is a $\varphi \in S'(\mathbb{R}^N)$ so that $\mathcal{F}(\varphi) = \varphi_{F(x)}$. Let us show that $\mathrm{supp}(\varphi) \subset B(0,r)$.

Consider a Dirac-sequence $(d_k)_{k\in\mathbb{N}^*}$ and $f \in S(\mathbb{R}^N)$ supported outside $\overline{B(0,r)}$.

By Paley & Wiener, $\alpha(d_k, q)(y)$ are entire functions satisfying

$$\forall m \in \mathbb{N},\ |\alpha(d_k,q)(y)| \leq C_{k,m} \left(1+|y+iq|\right)^{-m} e^{\frac{1}{k}|q|}$$

Note also that

- $\lim_k \varphi_{\alpha(d_k,0)(y)} = \varphi_{\frac{1}{(2\pi)^{N/2}}}$ in the weak*-topology. Hence,

$$\lim_k \varphi_{F(x)\alpha(d_k,0)(y)} = \varphi_{\frac{F(x)}{(2\pi)^{N/2}}}$$

in the weak*-topology.

- $\forall k \in \mathbb{N}^*$ $F(y+iq)\,\alpha(d_k,q)(y)$ are entire functions so the are themselves Fourier-Laplace transforms of some smooth functions f_k with $\mathrm{supp}(f_k) \subset B(0, r+\frac{1}{k})$.

Hence, for k sufficiently large, $\mathrm{supp}(f_k) \cap \mathrm{supp}(f) = \emptyset$. Therefore

$$\varphi(f) = \mathcal{F}(\varphi)(f^\vee) = \varphi_{F(x)}(f^\vee)$$

$$= \lim_k \varphi_{F(x)\alpha(d_k,0)(x)}\left((2\pi)^{N/2} f^\vee\right)$$

$$= \lim_k \varphi_{\hat{f_k}}\left((2\pi)^{N/2} f^\vee\right) = \lim_k \varphi_{f_k}\left((2\pi)^{N/2} f\right) = 0$$

$\square$.

Chapter 2. Functional Integration

- Brown discovered erratic movements when observing pollens in water solution. This lead to the hypothesis of atomistic structure of matter.

"Brownian motion" $\longleftrightarrow$ "erratic movement"

§2-1. Einstein's derivation of the dispersion equation

Let $u(t,x)$ be the density of the pollen grains per unit of length at some given time t. at the point x (for simplicity we consider a 1-dimensional space).

Denote by $p(\tau,y)$ the proportion of pollen grains moving from x to $x+y$ in a time intervall τ.

Suppose $\int_{\mathbb{R}} p(\tau,y)\, y^2\, dy = D\tau \quad , D>0$

$$u(t+\tau,x) = \int_{\mathbb{R}} u(t,x-y)\, p(\tau,y)\, dy$$

By a Taylor approximation we get

$$\cancel{u(t,x)} + \tau u_t(t,x) + O(\tau)$$

$$= \int_{\mathbb{R}} \Big(\cancel{u(t,x)} \cancel{- y\, u_x(t,x)} + \frac{1}{2} y^2 u_{xx}(t,x) + O(y^2) \Big) p(\tau,y)\, dy$$

by symmetry

we finally get $\partial_t u(t,x) = \frac{D}{2} \Delta u(t,x)$
which is the well-known heat-equation.
if $u(0,x) = \delta_0$ then we have

$$u(t,x) = \frac{1}{\sqrt{2\pi D t}} \exp\left(-\frac{1}{2Dt}(x\cdot x)\right)$$

heat kernel.

$$\partial_t u = \frac{D}{2}\Delta u \longleftrightarrow i\hbar\partial_t \Psi = -\frac{\Delta}{2m}\Psi$$

It looks like the Schrödinger equation is the heat equation for "imaginary times" it.

§2.2. Constructions of Brownian motions

Def: A probability space is a tripple (Ω, Σ, μ), where Ω is some set, Σ is some σ-algebra over Ω and $\mu : \Sigma \to [0,1]$ is some measure on Σ with $\mu(\Omega) = 1$.
A random variable is a Σ-measurable function $X : \Omega \to \mathbb{R}$.
$(X^{-1}[a,b] \in \Sigma)$
For $X \in \mathcal{L}^1(\Omega,\mu)$ one defines the expected value of X by $\mathbb{E}(X) := \int_\Omega X \, d\mu$
The variance $\mathbb{V}(X) := \mathbb{E}(X - \mathbb{E}(X))^2$ if $X \in L^2(\Omega,\mu)$

Each random variable X on (Ω, Σ, μ) has a cumulative probability distribution (c.p.d.)

given by $F_X(x) := \mu(X^{-1}]-\infty, x])$

A c.p.d is a càdlàg function (from the french "continue à droite, limite à gauche")

i.e.

- $\forall x,y \in \mathbb{R} \quad 0 \leq F_X(x) \leq F_X(y) \leq 1$ if $x \leq y$
- $\forall x \in \mathbb{R} \quad \lim_{y \to x^+} F_X(y) = F_X(x)$
- $\forall x \in \mathbb{R} \quad \lim_{y \to x^-} F_X(y) \leq F_X(x)$

A probability density function $f_X(x)$ is a distribution defined $f_X(x) = D_x F_X(x)$

Def: Two random variables X, Y on some p.s (Ω, Σ, μ) are said to be independent iff $\mu(X^{-1}]-\infty, x] \cap Y^{-1}]-\infty, y]) = F_{X,Y}(x,y)$
$= F_X(x) F_Y(y)$.

Def: A (one-dimensional) Brownian motion is a family $\{B_t\}_{t \in \mathbb{R}_+}$ of random variables defined on some p.s. (Ω, Σ, μ), so that

- $\forall \omega \in \Omega \quad t \mapsto B_t(\omega)$ is continuous, starting at 0 μ-a.c.
- $\forall t, \tau \in \mathbb{R}_+, \ \mathbb{E}(B_{t+\tau} - B_t) = 0$ and $\mathbb{V}(B_{t+\tau} - B_t) = \tau$
- $\forall t, \tau \in \mathbb{R}_+$, $B_{t+\tau} - B_t$ and B_s with $s \leq t$ are independent

§2.2.1. Random walk

Let us define (Ω, Σ, ν) as

$\Omega = (\pm 1)^{\mathbb{N}^*} = \{ \omega : \mathbb{N}^* \to \{\pm 1\}\}$

Observe that the map

$$f : \Omega \to [0,1], \quad \omega \mapsto \sum_{n \geq 1} \frac{1+\omega(n)}{2^{n+1}}.$$

This map is surjective (injective?).
Consider then the Lebesgue σ-algebra Σ_L on $[0,1]$ and the Lebesgue measure ν_L.
These objects are then pulled-back:

$\Sigma := \{ f^{-1}(E) \mid E \in \Sigma_L \}$

$\forall F \in \Sigma, \quad \nu(F) := \nu_L(f(F))$

if $t \in \{\pm 1\}^n$ and if

$$A_t := \{ \omega \in \Omega \mid \omega(k) = t(k) \text{ for } k = 1, \dots, n\}$$

then $\nu(A_t) = \left(\frac{1}{2}\right)^n$

On this p.s. the random walk is the family of random variables $(S_n)_{n \geq 1}$ given by $S_n(\omega) := \sum_{k=1}^{n} \omega(k)$

One may verify, that $\mathbb{E}(S_n) = 0 \ \forall n \geq 1$

$\mathbb{V}(S_n) = n$ and

$$\nu(S_n^{-1}\{n-2k\}) = 2^{-n}\binom{n}{k}, \ k = 0, \dots, n$$

§2.2.2 Wiener process

Def: Let X be a random variable defined on a p.s. (Ω, Σ, μ). The characteristic function of X is given by

$$\Phi_X(t) := \mathbb{E}(\exp(itX))$$

Let us call Σ_B the σ-algebra generated by all open sets of $\mathbb{R}$. For some open intervall $I \subset \mathbb{R}$ define $\mu_X(I) := \mu(X^{-1}(I))$. As an exercise, show that μ_X is uniquely extensible to Σ_B. It is then easy to show, that

$$\Phi_X(t) = \int_{\mathbb{R}} e^{itx} \mu_X(dx)$$

$$F_X(t) = \mu(X^{-1}]-\infty, t]) = \mu_X(-\infty, t]$$

$$\mathcal{F}(\varphi_{\Phi_X(t)}) = \sqrt{2\pi}\, D\varphi_{F_X(t)}$$

If $f \in \mathcal{L}^1(\mathbb{R}, \mu_X)$ then $\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) \mu_X(dx)$

If, X, Y are independent random variables, then

$$\mu(X \in I, Y \in J) = \mu_X(I)\mu_Y(J)$$

We see that $\Phi_X(t) = \Phi_Y(t)$ iff $\mu_X = \mu_Y$ and if X and Y are ind. then $\Phi_{X+Y}(t) = \Phi_X(t)\,\Phi_Y(t)$.

Thm: The only possible Brownian motion is a Wiener process, i.e. it is a family $\{W_t\}_{t\in\mathbb{R}_+}$ of random variables on $(\Omega, \mathcal{E}, \mu)$ s.t.

- $\forall \omega\in\Omega$, $t\mapsto W_t(\omega)$ is a continuous map starting at 0 a.e.
- $\forall t,\tau\in\mathbb{R}_+$, $W_{t+\tau}-W_t$ is a gaussian random variable with 0 mean and $\mathbb{V}(W_{t+\tau}-W_t)=\tau$
- $\forall t,\tau\in\mathbb{R}_+$, $W_{t+\tau}-W_t$ are indep. of W_s, $s<t$.

Sketch of proof: Let $\{B_t\}_{t\in\mathbb{R}_+}$ be a Brownian motion. Let us show that $\Phi_{B_1}(t) = e^{-t^2/2}$.

Write

$$B_1 = B_{1/N} + (B_{2/N}-B_{1/N}) + \dots + (B_1 - B_{\frac{N-1}{N}})$$

Therefore, $\Phi_{B_1}(t) = \left(\Phi_{B_{1/N}}(t)\right)^N$. But,

$$\Phi_{B_{1/N}}(t) = \mathbb{E}(\exp(itB_{1/N}))$$ and

$$\exp(iy) = 1 + iy - \frac{y^2}{2} + y^2 r(y)$$ with

$r(y)$ is continuous, bounded an $\lim_{y\to 0^+} r(y) = 0$.

(exercise)

$$\Phi_{B_1}(t) = \left(1 - \frac{t^2}{2N} + t^2 h_N(t)\right)^N,$$ where

$$h_N(t) = \mathbb{E}(B^2_{1/N}\, r(tB_{1/N}))$$

For given $\varepsilon, K>0$ and $|t|\le K$, there is $\delta>0$, s.t. that $|x|\le\delta \Rightarrow |r(tx)|<\varepsilon$. Thus,

$$\mathbb{E}(\exp(itB_{1/N})) - \left(1 - \frac{t^2}{2N}\right) = t^2 \underbrace{\mathbb{E}\left(B_{1/N}^2\, r(tB_{1/N})\right)}_{h_N(t)},$$

$$= t^2\left(\frac{\varepsilon}{N} + \mathbb{E}\left(B_{1/N}^2\, r(tB_{1/N})\, \mathbb{1}_{\{|B_{1/N}| \geq \delta\}}\right)\right)$$

$$< t^2\left(\frac{\varepsilon}{N} + M\, \mathbb{E}\left(B_{1/N}^2\, \mathbb{1}_{\{|B_{1/N}| \geq \delta\}}\right)\right)$$

$$\leq t^2\left(\frac{\varepsilon}{N}\right)$$ by dominated convergence and the fact that $\lim_{N\to\infty} B_{1/N}(\omega) = 0$ a.e.

Therefore, $\forall \varepsilon > 0$, $\forall K > 0$, $|t| \leq K$

$$\left|\Phi_{B_{1/N}}(t) - \left(1 - t^2/2N\right)\right| \leq |h_N(t)| \leq \frac{\varepsilon}{N} t^2$$

Hence,

$$\left|\left(1 - \frac{t^2}{2N} + \frac{t^2}{N} h(t)\right)^N - \left(1 - \frac{t^2}{2N}\right)^N\right|$$

$$\leq \sum_{k=1}^{N} \binom{N}{k} \left(1 - \frac{t^2}{2N}\right)^{N-k} \left|\frac{t^2}{N} h(t)\right|^k$$

$$\leq \left(1 + \frac{t^2 |h(t)|}{N}\right)^N - 1 \leq \left(1 + \frac{t^2\varepsilon}{N}\right)^N - 1$$

$$\leq e^{t^2\varepsilon} - 1.$$ This being true for any $\varepsilon > 0$, with $N \to \infty$, $\Phi_{B_1}(t) = e^{-t^2/2}$ $\square$.

§2.2.3. Wiener's construction:

A topological space (Ω, τ), where $\tau \subset \mathcal{P}(\Omega)$ s.t. 1. $\emptyset \in \tau$ 2. $\Omega \in \tau$

3. if $\mathcal{F} \subset \tau$ then $\bigcup_{V \in \mathcal{F}} V \in \tau$

4. if $\mathcal{F} \subset \tau$, $|\mathcal{F}| \in \mathbb{N}$, $\bigcap_{V \in \mathcal{F}} V \in \tau$.

A **basis** for a topology τ is a set $B \subset \tau$ s.t. $\forall V \in \tau \; \exists \mathcal{F} \subset B$ s.t. $V = \bigcup_{U \in \mathcal{F}} U$

A **sub-basis** for a topology τ is a set $S \subset \tau$ so that finite intersections of elements in S form a basis for τ.

A topological space is **quasi-compact** iff. For every cover $C \subset \tau$ of Ω, there is a finite cover $F \subset C$, $|F| \in \mathbb{N}$, of Ω.

A topological space is called **Hausdorff** iff, $\forall x, y \in \Omega$ there is $U, V \in \tau$ s.t. $x \in U$, $y \in V$, $U \cap V = \emptyset$.

A topological space is called **compact** if it is both quasi-compact and Hausdorff.

Alexander's compactness thm.: For a topological space (Ω, τ) the following statements are equivalent:

1) (Ω, τ) is quasi-compact
2) There is a sub-basis $\mathcal{B} \subset \tau$ so that any cover of Ω by elements in $\mathcal{B}$ has a finite sub-cover

proof: 1) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 1) Suppose we are given a sub-base $\mathcal{B} \subset \tau$ so that any cover of Ω by elements in $\mathcal{B}$ has a finite sub-cover.

Suppose (Ω, τ) is not quasi-compact and we shall find a contradiction.
There are therefore covers $C \subset \tau$ of Ω with no finite sub-covers.
Let's consider $\mathcal{G}$ the collection of all covers $C \subset \tau$ with no finite sub-covers. $\mathcal{G}$ is not empty and partially ordered by inclusion.
A totally ordered (by inclusion) subset of covers in $\mathcal{G}$ has its union as an upper bound.
By Zorn's lemma, $\mathcal{G}$ has a maximal element, say $M \in \mathcal{G}$.
Consider $M \cap \beta$. This cannot cover Ω. So there is $x \in \Omega$ not covered by $M \cap \beta$. But since M covers Ω, there is some $U \in M$ s.t. $x \in U$. Since β is a sub-basis for τ, there are $B_1, \dots, B_n \in \tau$ s.t.

$$x \in B_1 \cap B_2 \cap \dots \cap B_n \subset U.$$

Since $M \cap \beta$ does not cover x, $\forall k = 1, \dots, n$, $B_k \notin M$. By maximality of M, $M \cup \{B_k\}$ has a finite subcover of Ω of the form

$$M_k \cup \{B_k\}, \quad M_k \subset M, \quad |M_k| \in \mathbb{N}.$$

The finit set $\left(\bigcup_{k=1}^{n} M_k\right) \cup \left\{\bigcap_{k=1}^{n} B_k\right\}$ covers Ω (exercise)
But then $\left(\bigcup_{k=1}^{n} M_k\right) \cup \{U\}$ covers also Ω and is a subset of M. But then

$$M \notin \mathcal{G} \quad \lightning \quad \square.$$

1 $\{(\Omega_i, \tau_i)\}_{i\in I}$ of topological spaces then the cartesian product $\prod_{i\in I} \Omega_i$ is naturally equipped with the product topology, which is the collection of unions of sets of the form $\prod_{i\in I} V_i$, $V_i \in \tau_i$ and $V_i \neq \Omega_i$ for a finite number of i's.

For some fixed $i\in I$, let $\pi_i : \prod_{j\in I} \Omega_j \to \Omega_i$, $(x_j)_{j\in I} \longmapsto x_i$, called the canonical projection of $\prod_{i\in I} \Omega_i$ to Ω_i

then sets of the form $\{\pi_i^{-1}\{U\} \mid U \in \tau_i\}$ form a sub-basis for the product topology.

One may check, that the product topology on $\prod_{i\in I} \Omega_i$ is the weakest topolgy for which all π_i's are continuous.

Tychonoff's theorem: let $\{(\Omega_i, \tau_i)\}_{i\in I}$ be a family of quasi-compact spaces, then $\prod_{i\in I} \Omega_i$ is also quasi-compact for the product topology

proof: Suppose C is a collection of open sets in the product topology with no finite sub-cover, and we shall prove, that C is not a cover of $\prod_{i\in I} \Omega_i$. For each $i\in I$, define

$$C_i := \{U \in C \mid \exists V \in \tau_i \text{ s.t. } U = \pi_i^{-1}\{V\}\}$$

For each $i \in I$, C_i has no finite sub-cover for $\prod_{i\in I} \Omega_i$.
Therefore, $\pi_i(C_i)$ has no finite subcover for Ω_i. By compactness of Ω_i, $\pi_i(C_i)$ does not cover Ω_i. Therefore $\exists\, x_i \in \Omega_i$ not covered by $\pi_i(C_i)$.
Consider $(x_i)_{i\in I} \in \prod_{i\in I} \Omega_i$. This element is not covered by $\bigcup_{i\in I} C_i$.
Therefore, any collection of sets in a sub-base of the cartesian product has a finite sub-cover as soon as it covers $\prod_{i\in I} \Omega_i$.
By Alexander's compactness theorem, we are done $\square$.

Wiener's construction of the Brownian motion starts with $\Omega_c := \prod_{t\geq 0} (\mathbb{R}^N)$. This Ω_c is compact for the product topology by Tychonoff's theorem.
Recall that a measure μ on a topological space (Ω, τ) is called <u>regular</u> iff it is defined on a σ-algebra $\Sigma \supset \tau$ s.t. $\forall E \in \Sigma$

- $\mu(E) = \sup\{\mu(K) \mid K \subset E,\ K \text{ compact}\}$
- $\mu(E) = \inf\{\mu(V) \mid V \in \tau,\ E \subset V\}$

If (Ω, τ) is compact, one has the

Riesz & Markov: Let (Ω, τ) be compact and let $\Phi : C(\Omega, \mathbb{R}) \to \mathbb{R}$ be a positive and linear functional. Then there is a unique complete measure space (Ω, Σ, μ), with $\tau \subset \Sigma$ so that μ is regular and

$$\phi(f) = \int_\Omega f \, d\mu$$

Here, μ is called <u>complete</u> iff $\forall F \subset E$ with $E \in \Sigma$ s.t. $\mu(E) = 0 \implies F \in \Sigma$ and $\mu(F) = 0$.

We will define a $\Phi : C(\Omega_L, \mathbb{R}) \to \mathbb{R}$ which is positive. It will be enough to define it for a dense subset of $C(\Omega_L, \mathbb{R})$ (in the sense of uniform convergence)

Stone-Weierstrass's theorem: Let (Ω, τ) be a compact space and let $\mathcal{F} \subset C(\Omega, \mathbb{R})$ be s.t.

- $\forall x, y \in \Omega$, $\exists f \in \mathcal{F}$ s.t. $f(x) \neq f(y)$ if $x \neq y$
- $\forall x \in \Omega$, $\exists f \in \mathcal{F}$ s.t. $f(x) \neq 0$

Then $\mathcal{F}$ is uniformly dense in $C(\Omega, \mathbb{R})$

Suppose $f \in C(\prod_{i=1}^{m} (\mathbb{R}^N))$ and suppose $0 < t_1 \le t_2 \le \dots \le t_m$ be fixed "time-points". Then we define

$$F_{f, t_1, \dots, t_m} : \Omega_L \to \mathbb{R},$$

$$\omega = (\omega_t)_{t \ge 0} \longmapsto F(\omega) := f(\omega_{t_1}, \dots, \omega_{t_m})$$

Such functions are called finite functions and the collection of those will be denoted by $C_{fi}(\Omega_c)$

Therefore, by Stone-Weierstrass, $C_{fi}(\Omega_c)$ is uniformly dense in $C(\Omega_c)$

The program is the following:

- Define an integral $I: C_{fi}(\Omega_c) \to \mathbb{R}$
- Extend I to a positive functional $J: C(\Omega_c) \to \mathbb{R}$
- Apply Riesz & Markov to find a regular measure μ_W s.t. $J(f) = \int_{\Omega_c} f \, d\mu_W$
- Use regularity of μ_W to prove that it is carried by $\{f \in C(\mathbb{R}_+) : f(0) = 0\}$.

Fix a $x_0 \in \mathbb{R}^N$ and $t_0 = 0$. Define for $F_{f, t_1, \dots, t_m} \in C_{fi}(\Omega_c)$

$$I_{x_0}(F_{f, t_1, \dots, t_m}) := \int_{\mathbb{R}^{mN}} P^{x_0}_{t_1, \dots, t_m}(x_1, \dots, x_m) f(x_1, \dots, x_m) \mu_L(dx^{mN})$$

with

$$P^{x_0}_{t_1, \dots, t_m}(x_1 \dots, x_m) = \prod_{k=1}^{m} \frac{1}{(2\pi(t_k - t_{k-1}))^{N/2}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}}$$

One may verify that $I_{x_0}(1) = 1$.

For $F_{f, t_1, \dots, t_m}$ and $0 < t$ with $\{t_1 \dots, t_m\} \cap \{t\} = \emptyset$

Define $G_{f, t_1, \dots, t_k, t, t_{k+1}, \dots, t_m}(\omega) := F_{f, t_1, \dots, t_m}(\omega)$

Then we have that

$$I_{x_0}(G_{f, t_1, \dots, t_k, t, t_{k+1}, \dots, t_m}) = I_{x_0}(F_{f, t_1, \dots, t_m})$$

this implies, that I_{x_0} is linear on $\mathcal{C}_{fi}(\Omega_c)$ and that $I_{x_0}(F_{f,t_1,\dots,t_m}) \in [0,1]$ if $\mathrm{Im}(f) \subset [0,1]$

If $\varepsilon > 0$ and $F_{f,t_1,\dots,t_m}$, $G_{g,t_1,\dots,t_m} \in \mathcal{C}_{fi}(\Omega_c)$ are such that

$$\| F_{f,t_1,\dots,t_m} - G_{g,t_1,\dots,t_m} \|_\infty < \varepsilon$$

then

$$| I_{x_0}(F_{f,\dots t_m}) - I_{x_0}(G_{g,t_1,\dots,t_m}) | < \varepsilon .$$

Thus, I_{x_0} defines a continuous and positive functional on $\mathcal{C}_{fi}(\Omega_c)$ with norm equal to 1.
By Stone & Weierstrass, one may uniquely extend I_{x_0} to a functional $J_{x_0} : C(\Omega_c) \to \mathbb{R}$ with $\| J_{x_0} \| = 1$.

We now invoke the theorem by Riesz & Markov and conclude, that there is a regular measure $\mu_{x_0,W}$ on some σ-algebra $\Sigma_{x_0,W} \supset \mathcal{T}$ s.t. $\forall f \in C(\Omega_L)$, $J_{x_0}(f) = \int_{\Omega_L} f \, d\mu_{x_0,W}$.
Obviously

$$\mu_{x_0,W}(\Omega_c) = 1$$

and if $E \subset \mathbb{R}^N$, Lebesgue-measurable, then

$$\mu_{x_0,W}(\{\omega \in \Omega_L \mid \omega_1 \in E\}) = \frac{1}{(2\pi)^{N/2}} \int_E \exp\left(-\frac{(x-x_0)\cdot(x-x_0)}{2} \right) \mu_L(dx)$$

$\mathcal{N}_{x_0,\omega}$ is therefore a promising candidate to define a Brownian motion. This will be the family $\{\omega_t \mid t \geq 0\}$ with $\omega_t(\omega) = \omega_t$

It remains to be shown, $\mathcal{N}_{x_0,\omega}(C_{x_0}) = 1$ where $C_{x_0} = \{ f \in C(\mathbb{R}_+) \mid f(0) = x_0 \} \subset \mathcal{R}_2$

Lemma 1: Let Γ be a family of open sets for the product topology which is closed under finite unions. Then

$$\mathcal{N}_{x_0,\omega}\left(\bigcup_{V \in \Gamma} V \right) = \sup \{ \mathcal{N}_{x_0,\omega}(V) \mid V \in \Gamma \}$$

proof: By inner regularity of $\mathcal{N}_{x_0,\omega}$,

$$\mathcal{N}_{x_0,\omega}\left(\bigcup_{V \in \Gamma} V \right) = \sup \{ \mathcal{N}_{0,\omega}(K) \mid K \subset \bigcup_{V \in \Gamma} V, \text{ and } K \text{ compact} \}$$

$\leq \sup \{ \mathcal{N}_{0,\omega}(V) \mid V \in \Gamma \}$, since K is covered by Γ, so by by compactness is covered by a finite union of sets in Γ, which is again in Γ.

Since $\mathcal{N}_{x_0,\omega}(V) \leq \mathcal{N}_{x_0,\omega}\left(\bigcup_{V \in \Gamma} V \right)$, we conclude $\square$.

By a direct computation (exercise) one can check that

$$\mathcal{N}_{x_0,\omega}\left(\{ \omega \in \mathcal{R}_2 \mid |\omega_t - x_0| > \varepsilon \} \right) \leq \frac{C}{\varepsilon} e^{-\frac{\varepsilon^2}{2Nt}}$$

if we set

$$\rho(\varepsilon,\delta) := \sup\{\mu_{x_0,\omega}(\{\omega\in\Omega_L \mid |\omega_t - x_0| > \varepsilon\}) : 0 < t \le \delta\}$$

then $\lim_{\delta\to 0^+} \frac{1}{\delta}\rho(\varepsilon,\delta) = 0.$

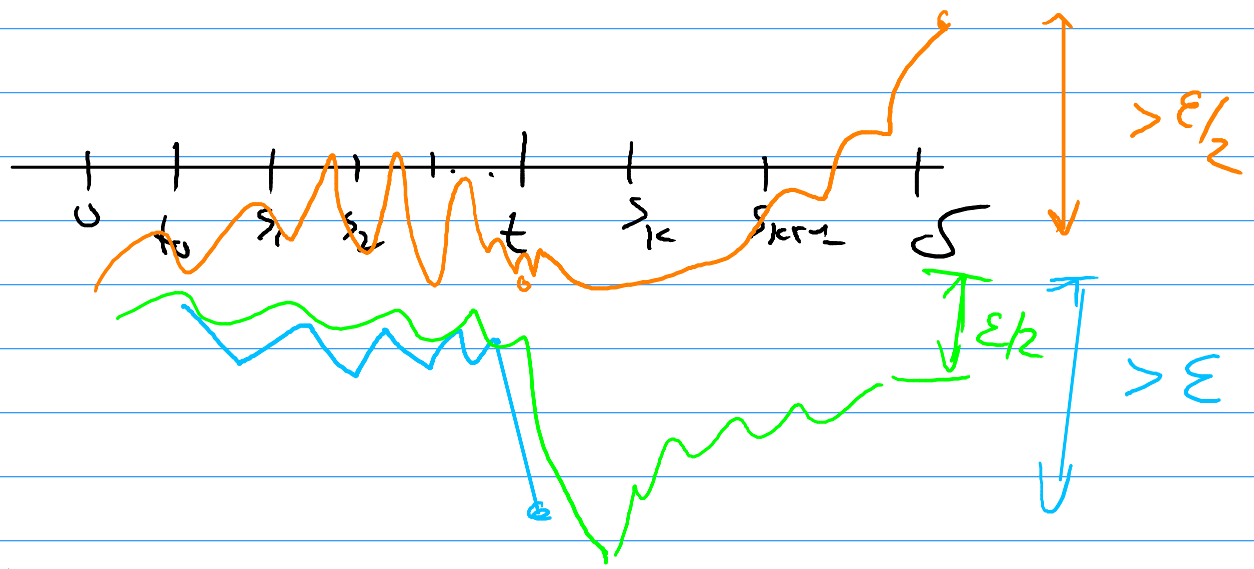
Lemma 2: let $0 \le t_0 < t \le \delta$ and let S a finite subset of $]t_0,\delta[$. For $\varepsilon > 0$, define

$$C_{t,\delta} := \{\omega\in\Omega_L \mid |\omega_t - \omega_\delta| > \tfrac{\varepsilon}{2}\},$$

$$D_{t_0,t,S,\delta} := \{\omega\in\Omega_L \mid \forall s\in S\cap]t_0,t[,\ |\omega_{t_0}-\omega_s| \le \varepsilon < |\omega_{t_0}-\omega_t|\}$$

then $\forall t\in]0,\delta]$

$$\mu_{x_0,\omega}(C_{t,\delta}\cap D_{t_0,t,S,\delta}) \le \rho(\tfrac{\varepsilon}{2},\delta)\,\mu_{x_0,\omega}(D_{t_0,t,S,\delta})$$



proof: $\mathbb{1}_{C_{t,\delta}} = F_{f,t,\delta}$, $f(x,v) := \mathbb{1}_{|x-v|>\frac{\varepsilon}{2}}$

Similarly, if $S\cap]t_0,t[= \{t_1,\dots,t_m\}$ then

$\mathbb{1}_{D_{t_0,t,S,\delta}} = G_{g,t_0,t_1,\dots,t_m,t}$ with

$$g(y, x_1, \dots, x_m, x) := \mathbb{1}_{|x-y|>\varepsilon} \prod_{k=1}^{m} \mathbb{1}_{|x_k - y| \le \varepsilon}$$

$$N_{u,w}(C_{t,\sigma} \cap D_{t_0,t,S,\sigma})$$

$$= \int_{\mathbb{R}^{N(m+3)}} p^{x_0}_{t_0,t_1,\dots,t_m,t,\sigma}(y, x_1, \dots, x_m, x, u)\, g(y, x_1, \dots, x_m, x)\, \mathcal{J}(x,u)\, N_L(dy \times dx_1 \times \dots \times du)$$

Observe that

$$p^{x_0}_{t_0,t_1,\dots,t_m,t,\sigma}(y, x_1, \dots, x_m, x, u) = p^{x_0}_{t_0,t_1,\dots,t_m,t}(y, x_1, \dots, x_m, x) \times p^{x}_{t,\sigma}(x, u)$$

Integrate first over u and notice that the result is bounded by $\rho(\frac{\varepsilon}{2}, \delta)$

Integrating over the remaining variables yields

$$N_{x_0,w}(D_{t_0,t,S,\sigma}) \qquad \square.$$

Lemma 3: Let $\sigma, \varepsilon > 0$, $t_0 \geq 0$ and consider a finite set $S \subset]t_0, \sigma[$. Let

$$A_{t_0,\sigma,\varepsilon,S} := \{ \omega \in \Omega_L \mid \exists s \in S \cup \{\sigma\} \text{ s.t. } |\omega_s - \omega_{t_0}| > \varepsilon \}$$

Then $$N_{x_0,w}(A_{t_0,\sigma,\varepsilon,S}) \le 2\rho(\frac{\varepsilon}{2}, \sigma)$$

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proof: Define

$$B_{t_0,\sigma,\varepsilon} := \{ \omega \in \Omega_L \mid |\omega_{t_0} - \omega_\sigma| > \varepsilon/2 \}$$

and let $C_{t,\sigma}$, $D_{t_0,t,S,\sigma}$ be as before.

If $\omega \in D_{t_0,t,S,\sigma}$ for some $t \in S \cup \{\sigma\}$

then if $\omega \notin B_{t_0,\delta,\varepsilon}$ then $\omega \in C_{t,\delta}$ for some t, since ω has to move a distance at least $\varepsilon/2$ to go back from outside the ball of radius ε to inside the ball of radius $\varepsilon/2$.

Therefore

$$A_{t_0,\delta,\varepsilon,S} \subset B_{t_0,\delta,\varepsilon} \cup \left(\bigcup_{t\in S} C_{t,\delta} \cap D_{t_0,t,S,\delta} \right)$$

thus

$$\mu_{x_0,\omega}(A_{t_0,\delta,\varepsilon,S}) \leq \mu_{x,\omega}(B_{t_0,\delta,\varepsilon}) + \sum_{t\in S} \mu_{x_0,\omega}(C\cap D)$$

$$\leq \mu_{x_0,\omega}(B_{t_0,\delta,\varepsilon}) + \rho\left(\frac{\varepsilon}{2},\delta\right) \sum_{t\in S} \mu_{x_0,\omega}(D_{t_0,t,S,\delta})$$

by Lemma 2. Since $D_{t_0,t,S,\delta} \cap D_{t_0,t',S,\delta} = \emptyset$ we get

$$\mu_{x_0,\omega}(A_{t_0,\delta,\varepsilon,S}) \leq 2\rho\left(\frac{\varepsilon}{2},\delta\right)$$ ☐

Notice that for some finite $S \subset]t_0, \delta[$ with $0 \leq t_0 < \delta$ if one defines

$$E_{t_0,\delta,\varepsilon,S} := \{\omega \in \Omega_2 \mid \exists t,s \in S \cup \{t_0,\delta\} \text{ s.t. } |\omega_s - \omega_t| > 2\varepsilon \}$$

then $E_{t_0,\delta,\varepsilon,S} \subset A_{t_0,\delta,\varepsilon,S}$

since if $|\omega_t - \omega_s| > 2\varepsilon$ then $|\omega_s - \omega_{t_0}|, |\omega_t - \omega_{t_0}| \leq \varepsilon$ cannot be both true.

So,

$$\mu_{x_0,\omega}(E_{t_0,\delta,\varepsilon,S}) \leq 2\rho\left(\frac{\varepsilon}{2},\delta\right)$$

Lemma 4: Let $\delta, \varepsilon > 0$ and define

$$E_{\delta,\varepsilon} := \{ \omega \in \Omega_L \mid \exists\, s,t \in [0,\delta] \text{ s.t. } |\omega_t - \omega_s| > 2\varepsilon \}$$

Then

$$\mu_{x_0,w}(E_{\delta,\varepsilon}) \leq 2\rho\left(\frac{\varepsilon}{2}, \delta\right)$$

proof: Note the set $E_{t_0,\delta,\varepsilon,S}$ are open in the product topology for any finite $S \subset]t_0,\delta[$.

Also,

$$\mu_{x_0,w}\left(E_{t_0,\delta,\varepsilon,S}\right) \leq 2\rho\left(\frac{\varepsilon}{2}, \delta\right) \text{ and}$$

$\bigcup_{t_0 \in [0,\delta],\, S \subset]t_0,\delta[} E_{t_0,\delta,\varepsilon,S} = E_{\delta,\varepsilon}$. By Lemma 1,

$$\mu_{x_0,w}(E_{\delta,\varepsilon}) \leq 2\rho\left(\frac{\varepsilon}{2}, \delta\right) \quad \square.$$

In the exercises, one may prove that

$$\mu_{x_0,w}\left(E_{t_0,t_1,\varepsilon}\right) \leq 2\rho\left(\frac{\varepsilon}{2}, \delta\right)$$

where

$$E_{t_0,t_1,\varepsilon} := \{ \omega \in \Omega_L \mid \exists\, s,t \in [t_0,t_1) \text{ st. } |\omega_s - \omega_t| > 2\varepsilon \text{ and } |t_0 - t_1| \leq \delta \}$$

Wiener's theorem: The Wiener measure $\mu_{x_0,w}$ is carried by the set

$$C_{x_0} := \{ f \in C(\mathbb{R}_+, \mathbb{R}) : f(0) = x_0 \} \text{ i.e.}$$

$$\mu_{x_0,w}(C_{x_0}) = 1$$

proof: For $\varepsilon > 0$ and $n, k \in \mathbb{N}^*$, set $\delta := \frac{1}{n}$ and consider

$$F_{k,\varepsilon,\delta} := \{ \omega \in \Omega_2 \mid \exists s,t \in [0,k] \text{ s.t. } |\omega_t - \omega_s| > 4\varepsilon \text{ and } |t-s| \leq \delta \}$$

We claim that

$$\mu_{x_0,\omega}(F_{k,\varepsilon,\delta}) \leq 2k \frac{1}{\delta} \rho(\tfrac{1}{2}\varepsilon, \delta).$$

Indeed $[0,k)$ is the union of the intervals $[0,\delta], [\delta, 2\delta], \dots, [(nk-1)\delta, k]$. If $t,s \in [0,k]$ s.t. $|t-s| < \delta = \frac{1}{n}$, then t,s must be element of some interval $[i\delta, (i+1)\delta]$ or they are in two adjacent ones.

But then there must be some intervals $[i\delta, (i+1)\delta]$ and $[(i+1)\delta, (i+2)\delta]$, s.t.

$|\omega_t - \omega_{(i+1)\delta}| \leq 2\varepsilon$ or $|\omega_s - \omega_{(i+1)\delta}| \leq 2\varepsilon$ cannot hold simultanously if $\omega \in F_{k,\varepsilon,\delta}$.

Therefore, there is some $i = 0, \dots, nk-1$ s.t.

$$\omega \in E_{i\delta,(i+1)\delta,\varepsilon}$$
$$:= \{ \omega \in \Omega_2 \mid \exists s,t \in [i\delta, (i+1)\delta) \text{ s.t. } |\omega_t - \omega_s| > 2\varepsilon \}$$

But,

$$\mu_{x_0,\omega}(E_{i\delta,(i+1)\delta,\varepsilon}) \leq 2\rho(\tfrac{\varepsilon}{2}, \delta)$$

and so

$$\mu_{x_0,\omega}(F_{k,\varepsilon,\delta}) \leq nk\, \mu_{x_0,\omega}(E_{i\delta,(i+1)\delta,\varepsilon})$$
$$\leq 2k \frac{1}{\delta} \rho(\tfrac{\varepsilon}{2}, \delta).$$

If $\omega \in \Omega_2$ is a continuous path on $\mathbb{R}_+$ then it must be uniformly continuous when restricted to $[0, k]$. Hence, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\omega \in \Omega_2 \setminus F_{k,\varepsilon,\delta}$

As a consequence

$$C_{x_0} = \bigcap_{k \geq 1} \bigcap_{p \geq 1} \bigcup_{n \geq 1} \Omega_2 \setminus F_{k, 1/p, 1/n}$$

$$\mu_{x_0,w}(\Omega_2 \setminus C_{x_0}) = \mu_{x_0,w}\Big(\bigcup_{k \geq 1} \bigcup_{p \geq 1} \bigcap_{n \geq 1} F_{k, 1/p, 1/n}\Big)$$

which is a countable union of $\bigcap_{n \geq 1} F_{k, 1/p, 1/n}$.

For fixed $k, p \in \mathbb{N}^*$,

$$\mu_{x_0,w}\Big(\bigcap_{n \geq 1} F_{k, 1/p, 1/n}\Big) = \lim_{n \to \infty} \mu_{x_0,w}(F_{k, 1/p, 1/n})$$

$$\leq \lim_{\delta \to 0} 2k \frac{1}{\delta} \varphi\left(\frac{\varepsilon}{2}, \delta\right) = 0.$$

So, $\mu_{x_0,w}(\Omega_2 \setminus C_{x_0}) = 0$ and

$$\mu_{x_0,w}(C_{x_0}) = 1 \qquad \square.$$

Let us consider $N = 1$. We have then

<u>Stroock's theorem</u>: The Wiener measure is carried by the set

$$\mathcal{W}_{x_0} := \Big\{ f \in C_{x_0} : \int_{\mathbb{R}_+} \frac{1}{(1+t^2)} |f(t)| \, dt < \infty \Big\}$$

proof: the expectation value of $\omega \in C_{x_0}$ under the Wiener measure is

$$\mathbb{E}_{x_0,\omega}(|\omega_t|) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2t}\right) |x| \, \mu_L(dx)$$

$$\underset{y=\frac{x}{\sqrt{t}}}{=} \frac{2\sqrt{t}}{(2\pi)^{1/2}} \int_{\mathbb{R}_+} \exp\left(-\frac{y^2}{2}\right) y \, dy = C \cdot \sqrt{t}$$

By Fubini's theorem:

$$\infty > \int_{\mathbb{R}_+} \mathbb{E}_{x_0,\omega}(|\omega_t|) \frac{1}{1+t^2} \, dt$$

$$= \mathbb{E}_{x_0,\omega}\left(\int_{\mathbb{R}_+} |\omega_t| \frac{1}{1+t^2} \, dt\right)$$

So,

$$\mu_{x_0,\omega}\left(\left\{ f \in C_{x_0} \;\middle|\; \int |f| \frac{1}{1+t^2} \, dt = \infty \right\}\right) = 0$$

$\triangleright$

As a consequence, $\mu_{x_0,\omega}$ is a measure on $S'(\mathbb{R})$ with characteristic function

$$\Phi(g) = \int_{S'(\mathbb{R})} \exp(i \varphi_f(g)) \, \mu_{x_0,\omega}(df)$$

with $g \in S(\mathbb{R})$

with $\Phi(0) = 1$, $\Phi : S(\mathbb{R}) \longrightarrow \mathbb{C}$

is continuous for τ_S

More generally, we have

Bochner & Minlos theorem: A function

$\Phi : S(\mathbb{R}^N) \longrightarrow \mathbb{C}$ is the characteristic function of some probability measure on $S'(\mathbb{R}^N)$

iff
- $\phi(0) = 1$
- ϕ is continuous for τ_S
- $\forall n \in \mathbb{N}^*$, $\forall (z_k) \in \mathbb{C}^n$ and $\forall (f_k) \in S(\mathbb{R}^N)^n$

$$\sum_{k,\ell=1}^{n} \overline{z_k}\, z_\ell\, \Phi(f_k - f_\ell) \geq 0.$$

For the Wiener measure

$$\phi(g) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |g|^2 \, dt\right)$$

$$\left| (\mathbb{R}^N)^{\mathbb{R}_+} \right| < \left| \mathbb{C}^{\mathbb{R}^N} \right|$$

1. Construction of $S(\mathbb{R}^N)$ via $H_n(\mathbb{R}^N)$ and τ_S.

$H_n \hookrightarrow H_m$ Nathan

(+ nuclear maps)

2. Schwartz's representation theorem

$\Phi(g) = \int f \, \partial g$, $\lim_{n \to \infty} \int g_n f$ Brian

(+ conditions on (g_n))

3. Uniform boundedness principle and applications

(+ example of non-application)

4. Fourier transform (on S and S')

def., properties, convolutions, $\frac{d}{dx}$

(+ eigenfunctions and spectral decomposition)

Armand & Thibaud

5. Green's function especially, for $\Box + m^2$, Luca Dugaiasu

causal, Feynman

(+ why Feynman?)

6. Paley & Wiener thms (esp. S')

(+ polynomial growth $\Rightarrow$ order of Φ)

7. Levy's uniqueness theorem Adam

(+ Brownian motion $\Rightarrow$ Wiener process)

8. <u>Tychonoff's theorem</u>

(+ Banach paradox)

9. <u>Construction of $\mu_{x_0,w}$</u>

(principle steps, independence and stationarity of increments) Ignacio