# Exercice sheet 3

Fourier transforms

**1.** Prove that if an associative multiplication $\times$ on $\mathcal{S}'(\mathbb{R})$ satisfies

- $\forall\varphi \in \mathcal{S}'(\mathbb{R}), \quad \varphi_0 \times \varphi = \varphi_0$ and $\varphi \times \varphi_1 = \varphi = \varphi_1 \times \varphi$,
- for any polynomially bounded and $\mu_L$-measurable functions $f, g$,
$$\varphi_f \times \varphi_g = \varphi_{fg} \quad \text{and} \quad \varphi_{f+g} = \varphi_f + \varphi_g,$$
- for any polynomially bounded and $\mu_L$-measurable functions $f, g$,
$$f(x) = \int_0^x g(t)\mu_L(dt) \quad \Rightarrow \quad D\varphi_f = \varphi_g,$$
- $\forall\varphi, \eta \in \mathcal{S}'(\mathbb{R}), \quad D(\varphi \times \eta) = (D\varphi) \times \eta + \varphi \times (D\eta)$,

then $\varphi_x\delta_0 = \varphi_0 = \delta_0 \times \varphi_x$ and $\varphi_x \times \text{P.v.}(\frac{1}{x}) = \varphi_1$, where $\delta_0 = D\varphi_\Theta$ and $\text{P.v.}(\frac{1}{x}) = D^2(x(\ln(|x|) - 1)$.

**2.** Prove the **Leibnitz integral rule**:

Let $U \subset \mathbb{R}$ be open, and let $(X, \Sigma, \mu)$ be a measure space. Suppose $f : U \times X \to \mathbb{K}$ satisfies:

- For each $t \in U$, $f \in \mathcal{L}^1(X, \mu)$,
- there is a $\mu$-null set $N \in \Sigma$, so that $\partial_t f(t, x)$ exists $\forall t \in U$ and $\forall x \in X \setminus N$,
- $\forall t \in U$, $\exists r > 0$, $\exists g_{t,r} \in \mathcal{L}^1(X, \mu)$, s.t. $|\partial_s f(s, x)| \leq g_{t,r}$ $\forall s \in ]t - r, t + r[$ and $\forall x \in X \setminus N$.

Then $\frac{d}{dt}\int_X f(t, x)d\mu = \int_{X\setminus N} \partial_t f(t, x)d\mu$.

Use the Leibnitz integral rule to show, that $\partial^\alpha \widehat{f(x)}(p) = \widehat{(-ix)^\alpha f}(x)(p)$ and that $(ip)^\alpha \widehat{f(x)}(p) = \widehat{\partial^\alpha f(x)}$.

**3.** Prove that the Fourier transform is a continuous endomorphism on $\mathcal{S}(\mathbb{R}^N)$.
(Hint: use continuity of the maps $\partial^\alpha$ and $(-ix)^\alpha$ previously proven.)

**4.** Use a contour integral to prove that
$$\mathcal{F}(\exp(-\frac{x \cdot x}{2}))(p) = \exp(-\frac{p \cdot p}{2}).$$

Consider then for $\epsilon > 0$ the integrals
$$I_\epsilon(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}} f(y)\exp(ik \cdot (x - y))\exp(-\epsilon^2\frac{k \cdot k}{2})\mu_L(dk \times dy).$$

Use Fubini and dominated convergence, to prove that
$$\mathcal{F} \circ \mathcal{F}^* = \mathbb{1}_{\mathcal{S}(\mathbb{R}^N)} = \mathcal{F}^* \circ \mathcal{F}.$$

**5.** For a given $\epsilon > 0$ define the following functions

$$\hat{\Delta}^C_{\epsilon,\pm} := \frac{-1}{(2\pi)^2} \frac{1}{(p_0 \mp i\epsilon + E_p)(p_0 \mp i\epsilon - E_p)},$$
$$\hat{\Delta}^F_{\epsilon,\pm} := \frac{-1}{(2\pi)^2} \frac{1}{(p_0 \mp i\epsilon + E_p)(p_0 \pm i\epsilon - E_p)},$$

where $E_p = \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}$. Show that $\hat{\Delta}^C_\pm := \lim_{\epsilon \to 0^+} \varphi_{\hat{\Delta}^C_{\epsilon,\pm}}$ and $\hat{\Delta}^F_\pm := \lim_{\epsilon \to 0^+} \varphi_{\hat{\Delta}^C_{\epsilon,\pm}}$ exist in $\mathcal{S}'(\mathbb{R}^4)$.
Show then that $(\Box + m^2)\Delta^C_\pm = \delta^{(4)}_0 = (\Box + m^2)\Delta^F_\pm$.

**6.** Consider the Schrödinger equation $i\partial_t \psi(t) = -\frac{1}{2m}\Delta\Psi(t)$ together with an $L^2(\mathbb{R}^3, \mu_L)$ solution $\Psi(t)$ to it, i.e.:

- $(1 + p_1^2 + p_2^2 + p_3^2)\widehat{\Psi(0)}(p) \in L^2(\mathbb{R}^3, \mu_L)$,
- $\forall t \in \mathbb{R}, \quad \widehat{\Psi(t)}(p) = \exp(-i\frac{t(p_1^2+p_2^2+p_3^2)}{2m})\widehat{\Psi(0)}(p)$.

Suppose $\mathrm{supp}(\Psi(0)) \subset B(0, r)$. What may one conclude on $\mathrm{supp}(\Psi(t))$?
(Hint: use Schwarz's Paley & Wiener theorem).

**7.** Consider the relativistic Fourier transform $\hat{f}(p) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \exp(-ip^t\eta x) f(x)\mu_L(dx)$.
For a fixed $f \in \mathcal{S}(\mathbb{R}^4)$, let $C_f := \{y \in \mathbb{R}^4 \,:\, \forall x \in \mathrm{supp}(f),\ y^t\eta x \leq 0\}$.
Show that $C_f$ is convex and that on $p + iy \in \mathbb{R} + i\mathring{C}_f$, $\mathcal{L}(f, y)(p)$ is well-defined, holomorphic and obeys the same estimates as in Paley & Wiener's theorem, but for the exponential term.

**8.** Use the previous exercice to show, that for $f \in \mathcal{S}(\mathbb{R}^4)$ with $\sup(f) \subset \mathbb{R}_\pm \times \mathbb{R}^3$, $\mathcal{L}(f, y)(p)$ is holomorphic in $p_0 + iy$ if $y \in \mathbb{R}^*_\mp \times \{(0, 0, 0)\}$.
Show that if $f \in \mathcal{S}(\mathbb{R}^4)$ with $\sup(f) \subset \mathbb{R}_\mp \times \mathbb{R}^3$, $\mathcal{L}^*(f, y)(p) := \mathcal{L}(f, -y)(-p)$ is holomorphic in $p_0 + iy$ if $y \in \mathbb{R}^*_\mp \times \{(0, 0, 0)\}$.
Use this to show, that $\Delta^C_\pm$ have causal supports.

**9.** A solution to the Cauchy problem

$$(\Box + m^2)\varphi = 0, \quad \varphi(0) = f \text{ and } (\partial_t\varphi)(0) = f_t, \quad f, f_t \in \mathcal{S}'(\mathbb{R}^3)$$

is a function $\varphi : \mathbb{R} \to \mathcal{S}'(\mathbb{R}^3)$, so that $\lim_{\tau \to 0} \frac{\varphi(t+\tau)-\varphi(t)}{\tau} = \varphi_t(t)$ and $\lim_{\tau \to 0} \frac{\varphi_t(t+\tau)-\varphi_t(t)}{\tau} = \varphi_{tt}(t)$ exist in the weak*-topology and so that $\varphi_{tt}(t) - \Delta\varphi(t) + m^2\varphi(t) = 0$.
Use Schwartz's Paley & Wiener theorem to show, that if both $f$ and $f_t$ have compact support, then $\varphi(t)$ has causal support.
What happens if one considers positive and negative energy frequencies separately?
(Hint: $\sqrt{a + ib} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)$)

**10.** Let $f, f_t \in \mathcal{S}(\mathbb{R}^3)$ and show, that the solution obtained in this case by the previous exercice reads

$$\lim_{k \to \infty} \left((D_t\Delta^C_+) * (d_k(t)f) + \Delta^C_+ * (d_k(t)f_t)\right),$$

where $(d_k)_{k \in \mathbb{N}^*} \subset \mathcal{S}(\mathbb{R})$ is a Dirac sequence.
What happens when one substitutes $\Delta^C_+$ by $\Delta^C_-$ or $\Delta^F_\pm$?

(Hints: compute the Fourier transform of the convolution of a tempered distribution with a test function. Apply Paley & Wiener's theorem to the Dirac sequence and use a contour integral on $p_0$.)