Solutions to exercice sheet 2

Fréchet spaces and continuous maps

1. Let (X,d) be a metric space. Show that

$$\tau_d := \{U \subset X \,:\, \forall x \in U,\, \exists r > 0 \text{ s.t. } B(x,r) \subset U\}$$

defines a topology on X. Show that if $(x_k)_{k\in\mathbb{N}}$ is a sequence in X so that $\lim_k d(x,x_k) = 0$, then $\lim_l d(x_l,x_k) = d(x,x_k)$ for all $k \in \mathbb{N}$.
For (X,d) a Fréchet space, show that for $n \in \mathbb{N}$ and $r > 0$, $nB(0,r) \subset B(0,nr)$ ($nB(0,r) := \{ny \,:\, y \in B(0,r)\}$).

Let $\{U_1,\ldots,U_n\} \subset \tau_d$ and let $x \in \cap_{k=1,\ldots,n} U_k$. Then $\forall k = 1,\ldots,n$, $\exists r_k > 0$ s.t. $B(x,r_k) \subset U_k$. Set $r := \min\{r_1,\ldots,r_n\}$. Obviously, $\forall k = 1,\ldots,n$ one has $B(x,r) \subset B(x,r_k) \subset U_k$ and $B(x,r) \subset \cap_{k=1,\ldots,n} U_k$.
Let $\mathcal{F} \subset \tau_d$ and $x \in \cup_{U\in\mathcal{F}} U$. By definition, there is then some $U \in \mathcal{F}$ s.t. $x \in U$. Again by definition, there is then some $r > 0$ s.t. $B(x,r) \subset U$ and consequently, $B(x,r) \subset \cup_{U\in\mathcal{F}} U$.
Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X so that $\lim_k d(x,x_k) = 0$, and let $\epsilon > 0$. Then there is an $N_\epsilon \in \mathbb{N}$ s.t. $l \geq N_\epsilon$ implies $d(x,x_l) < \epsilon$. Since $d(x,x_k) \leq d(x,x_l) + d(x_l,x_k)$ and $d(x_l,x_k) \leq d(x_l,x) + d(x,x_k)$, one has

$$-\epsilon < -d(x,x_l) \leq d(x_k,x_l) - d(x,x_k) \leq d(x_l,x) < \epsilon$$

and hence, for $l \geq N_\epsilon$, $|d(x,x_k) - d(x_l,x_k)| < \epsilon$.
If (X,d) is Fréchet and if $y \in B(0,r)$, then $d(0,ny) \leq \sum_{k=0}^{n-1} d(ky,(k+1)y) = \sum_{k=0}^{n-1} d(0,y) < nr$.

2. Let $\{\|\|_n\}_{n\in\mathbb{N}}$ be a family of norms on a vector space V. Show that

$$d(v,w) := \sum_{n\geq 0} 2^{-n} \frac{\|v-w\|_n}{1+\|v-w\|_n}$$

defines a translation invariant distance on V (what can you say about the real function $\mathbb{R}_+ \ni x \mapsto \frac{x}{1+x}$?).
Show that in the case of $V = \mathcal{S}(\mathbb{R}^N)$ one has $\tau_d = \tau_{\mathcal{S}}$.

d is obviously positive (note that $\forall v,w \in V$, $0 \leq d(v,w) \leq 2$) and $\forall v,w \in V$, $d(v,w) = d(w,v)$. It is also clear that $d(v,w) = 0 \iff v = w$. It remains to prove the trianguar inequality for d.
For $x,y \in \mathbb{R}_+$ we have

$$x \geq y \iff x + xy \geq y + yx \iff \frac{x}{1+x} \geq \frac{y}{1+y}.$$

The map $\mathbb{R}_+ \ni x \mapsto \frac{x}{1+x}$ is hence increasing. As a consequence, if $v, w, u \in V$

$$d(v,w) = \sum_{n\geq 0} 2^{-n} \frac{\|v-w\|_n}{1+\|v-w\|_n} \leq \sum_{n\geq 0} 2^{-n} \frac{\|v-u\|_n + \|u-w\|_n}{1+\|v-u\|_n + \|u-w\|_n}$$

$$= \sum_{n\geq 0} 2^{-n} \frac{\|v-u\|_n}{1+\|v-u\|_n + \|u-w\|_n} + \sum_{n\geq 0} 2^{-n} \frac{\|u-w\|_n}{1+\|v-u\|_n + \|u-w\|_n}$$

$$\leq \sum_{n\geq 0} 2^{-n} \frac{\|v-u\|_n}{1+\|v-u\|_n} + \sum_{n\geq 0} 2^{-n} \frac{\|u-w\|_n}{1+\|u-w\|_n} = d(v,u) + d(u,w).$$

Let now $x \in V, r > 0$ and we will show, that there are norms $\| \ \|_k$ for $k = 1, \ldots, N$ and some $\epsilon > 0$, so that $U_{x,\epsilon,1,\ldots,N} \subset B(x,r)$. Since $\sum_{k\geq 0} 2^{-k} = 2$, there is an $N \in \mathbb{N}$, so that $\sum_{k>N} 2^{-k} < \frac{r}{2}$. Set $\epsilon := \frac{r}{2(N+1)}$. For any $y \in U_{x,\epsilon,1,\ldots,N}$ we thus have

$$d(y,x) = \sum_{0\leq k\leq N} 2^{-k} \frac{\|x-y\|_k}{1+\|x-y\|_k} + \sum_{n>N} 2^{-k} \frac{\|x-y\|_n}{1+\|x-y\|_n}$$

$$< \sum_{0\leq k\leq N} \|x-y\|_k + \sum_{n>N} 2^{-k} < \sum_{0\leq k\leq N} \|x-y\|_k + \frac{r}{2} < (N+1)\frac{r}{2(N+1)} + \frac{r}{2} = r.$$

Therefore, $U_{x,\epsilon,1,\ldots,N} \subset B(x,r)$. This being true for any $x \in V$ and any $r > 0$, we may conclude, that all open balls $B(x,r)$ are open for the topology induced by the family of norms $\{\| \ _n\}_{n\in\mathbb{N}}$ which is hence finer than τ_d.
Reciprocally, consider some $U_{x,\epsilon,1,\ldots,N}$ for a given $\epsilon > 0$, $N \in \mathbb{N}$ and $x \in V$. Let $y \in B(x,r)$. Then $r > d(x,y) > 2^{-k}\frac{\|y-x\|_k}{1+\|y-x\|_k}$, so that for $k = 1, \ldots, N$ one has $1 + \|y-x\|_k > 2^{-N} r^{-1} \|y-x\|_k$, which is to say $1 > (2^{-N} r^{-1} - 1)\|y-x\|_k$. If one choses $r < 2^{-N}\frac{\epsilon}{1+\epsilon}$, then $B(x,r) \subset U_{x,\epsilon,1,\ldots,N}$. This shows that τ_d is finer than the topology induced by the norms $\{\| \ _n\}_{n\in\mathbb{N}}$.

3. Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map. Identify any $z \in S^1$ with some $x \in \mathbb{R}/\mathbb{Z}$. This permits one to identify any $f \in C^\infty(S^1)$ with the periodic smooth function $f \circ q \in C^\infty(\mathbb{R})$. For a compact $K \subset S^1$ let $C^\infty(S^1 \setminus K) := \{f \in C^\infty(S^1) \ : \ \forall \alpha \in \mathbb{N}^N, \ \partial^\alpha f\big|_K = 0\}$. On $C^\infty(S^1 \setminus K)$ consider the norms

$$p_n(f) := \max_{\alpha \in \mathbb{N}_{\leq n}} \{\|\partial^\alpha f\|_\infty\}.$$

Show that $(C^\infty(S^1 \setminus K), \{p_n\}_{n\in\mathbb{N}})$ is a Fréchet space.

The fact that p_n are norms on $C^\infty(S^1 \setminus K)$ for all $n \in \mathbb{N}$ is clear. To show completeness of $C^\infty(S^1)$ under the topology induced by the family of norms $\{p_n\}_{n\in\mathbb{N}}$, one may use again the fact that $C(S^1)$, S^1 being compact, is complete for the uniform norm $\| \ \|_\infty$ and then invoke uniform continuity of the Riemann integral.
The fact that $C^\infty(S^1 \setminus K)$ is complete under the topology induced by the family of norms $\{p_n\}_{n\in\mathbb{N}}$ follows now from the fact that a sequence of smooth functions $(f_n)_{n\in\mathbb{N}} \subset C^\infty(S^1 \setminus K)$ can only converge to 0 on K.
$(C^\infty(S^1 \setminus K), \{p_n\}_{n\in\mathbb{N}})$ is hence complete and the topology may also be induced by the distance $d(f,g) := \sum_{n\geq 0} 2^{-n} \frac{p_n(f-g)}{1+p_n(f-g)}$.

4. Let $T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ be a linear map. Show that the following statements are equivalent:

(a) T is continuous

(b) For any norm $\| \ \|_n$, there is a positive constant C and a norm $\|\|_m$, so that for any $f \in \mathcal{S}(\mathbb{R}^N)$, $\|T(f)\|_n \leq C\|f\|_m$.

(c) For any sequence $(f_k)_{k\in\mathbb{N}}$, if $\lim_k f_k = 0$ for $\tau_{\mathcal{S}}$, then $\lim_k T(f_k) = 0$ for $\tau_{\mathcal{S}}$.

(d) For any sequence $(f_k)_{k\in\mathbb{N}}$, if $\lim_k f_k = f \in \mathcal{S}(\mathbb{R}^N)$ for $\tau_{\mathcal{S}}$, then $\lim_k T(f_k) = T(f)$ for $\tau_{\mathcal{S}}$.

$(a) \Rightarrow (b)$: By continuity of T, the set $T^{-1}\{U_{0,1,n}\}$ is open in $\tau_{\mathcal{S}}$. Since $0 \in T^{-1}\{U_{0,1,n}\}$, there is an $\epsilon > 0$ and some $K \in \mathbb{N}$, so that if $f \in U_{0,\epsilon,0,1,\ldots,K}$, then $T(f) \in U_{0,1,n}$. The norms in $\mathcal{S}(\mathbb{R}^N)$ being increasing, one has that $U_{0,\epsilon,K} = U_{0,\epsilon,0,1,\ldots,K}$. Setting therefore $m = K$ and $C = \frac{1}{\epsilon}$, one has that $\|f\|_m < \frac{1}{C}$ implies $\|T(f)\|_n < 1$. Hence, $\|f\|_m \leq \frac{1}{C}$ implies $\|T(f)\|_n \leq 1$, so that for any $f \in \mathcal{S}(\mathbb{R}^N) \setminus \{0\}$, $\|T(f)\|_n = \|T(C\|f\|_m \frac{f}{C\|f\|_m})\|_n = C\|f\|_m\|T(\frac{f}{C\|f\|_m})\|_n \leq C\|f\|_m$.

$(b) \Rightarrow (c)$: We have to show, that for any norm $\|\|_n$, one has $\lim_k \|T(f_k)\|_n = 0$ if $\lim_k f_k = 0$ for $\tau_{\mathcal{S}}$. By assumption, there is a positive constant C and a norm $\|\|_m$, so that $\|T(f_k)\|_m \leq C\|f_k\|_m$. But $\lim_k f_k = 0$ for $\tau_{\mathcal{S}}$ implies in particular, that $\lim_k \|f_k\|_m = 0$. Hence the conclusion.

$(c) \Rightarrow (d)$: Since all norms $\| \ \|_m$ are clearly translation invariant, $\lim_k f_k = f$ is equivalent to $\lim_k(f - f_k) = 0$ in $\tau_{\mathcal{S}}$. By assumption we then have $\lim_k(T(f - f_k)) = 0$ for $\tau_{\mathcal{S}}$, which then is again equivalent to $\lim_k T(f_k) = T(f)$.

$(d) \Rightarrow (a)$: We prove the contraposition. Suppose T is not continuous. There is therefore an open set $U \in \tau_{\mathcal{S}}$ so that $T^{-1}\{U\} \notin \tau_{\mathcal{S}}$. Thus, there is an $f \in T^{-1}\{U\}$, so that for any open set $f \ni V \in \tau_{\mathcal{S}}$, $V \not\subset T^{-1}\{U\}$. In particular, for any $n \in \mathbb{N}^*$, there is an $f_n \in U_{f,n^{-1},n}$ so that $f_n \notin T^{-1}(U)$. Clearly, $\lim_n f_n = f$, but $T(f) \in U$ and $\forall n > 0$, $T(f_n) \notin U$. This contradicts (d).

5. Show that if the sequences $(f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ converge for $\tau_{\mathcal{S}}$ to f and g respectively, then one has $\lim_k f_k g_k = fg$ and $\lim_k(f_k + g_k) = f + g$.
Show that if the sequences $(\varphi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$ converge for $\tau(\mathcal{S}'(\mathbb{R}^N, \mathcal{S}(\mathbb{R}^N))$ to φ and η respectively, then one has $\lim_k(\varphi_k + \eta_k) = \varphi + \eta$.

Since the sequences $(\varphi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$ converge for $\tau(\mathcal{S}'(\mathbb{R}^N, \mathcal{S}(\mathbb{R}^N))$ to φ and η respectively, this implies, that for any fixed $f \in \mathcal{S}(\mathbb{R}^N)$, one has

$$\lim_k \varphi_k(f) = \varphi(f) \quad \text{and} \quad \lim_k \eta_k(f) = \eta(f).$$

By linearity of all the functionals in play and of the limit in complex numbers, one has therefore that

$$\lim_k(\varphi_k + \eta_k)(f) = \lim_k \varphi_k(f) + \lim_k \eta_k(f) = \varphi(f) + \eta(f) = (\varphi + \eta)(f).$$

Note first, that for two Schwartz functions f and g, one has

$$\begin{aligned}
|||fg|||_n &= \max\{\|(1+x\cdot x)^n\partial^\alpha(fg)\|_\infty : \alpha\in\mathbb{N}^N_{\le n}\} \\
&= \max\{\|(1+x\cdot x)^n \sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}(\partial^\beta f)(\partial^\gamma g)\|_\infty : \alpha\in\mathbb{N}^N_{\le n}\} \\
&\le \max\{\sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}\|(1+x\cdot x)^n\partial^\beta f\|_\infty\|\partial^\gamma g)\|_\infty : \alpha\in\mathbb{N}^N_{\le n}\} \\
&\le \binom{N+n}{n} n!|||f|||_n|||g|||_n = \frac{(N+n)!}{N!}|||f|||_n|||g|||_n.
\end{aligned}$$

Let $0\in U\in\tau_{\mathcal{S}}$ and we shall prove, that for sufficiently large $k\in\mathbb{N}$, $fg-f_kg_k, f+g-f_k-g_k\in U$.
Since U is open and contains 0, there is an $n\in\mathbb{N}$, so that $U_{0,\epsilon,|||\,|||_n}\subset U$.
Since $\lim_k f_k = f$ and $\lim_k g_k = g$ for $\tau_{\mathcal{S}}$:

- there is a $k_1\in\mathbb{N}$ so that $k\ge k_1$ implies $|||f-f_k|||_n < \epsilon\frac{N!}{2(|||g|||_n+1)(N+n)!}$,
- there is a $k_2\in\mathbb{N}$ so that $k\ge k_2$ implies $|||g-g_k|||_n < \epsilon\frac{N!}{2(|||f|||_n+1)(N+n)!}$,
- there is a $k_3\in\mathbb{N}$ so that $k\ge k_3$ implies $|||g_k|||_n < |||g|||_n+1$,

For $k\ge\max\{k_1,k_2,k_3\}$ one has

$$\begin{aligned}
|||fg-f_kg_k|||_n = |||fg-fg_k+fg_k-f_kg_k|||_n &\le |||fg-fg_k|||_n + |||fg_k-f_kg_k|||_n \\
&= |||f(g-g_k)|||_n + |||(f-f_k)g_k|||_n \\
&\le \frac{(N+n)!}{N!}(|||f|||_n|||g-g_k|||_n + |||f-f_k|||_n|||g_k|||_n) < \epsilon, \\
|||f+g-f_k-g_k|||_n &\le |||f-f_k|||_n + |||g-g_k|||_n < \epsilon.
\end{aligned}$$

6. Prove that if $T:\mathcal{S}(\mathbb{R}^N)\to\mathcal{S}(\mathbb{R}^N)$ is a continuous linear map, then

$$T^t:\mathcal{S}'(\mathbb{R}^N)\to\mathcal{S}'(\mathbb{R}^N),\quad (T^t\varphi)(f):=\varphi(Tf)$$

is well-defined and continuous for $\tau(\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N))$.
Prove that for a given $\alpha\in\mathbb{N}^N$, the maps

$$\mathcal{S}(\mathbb{R}^N)\ni f\mapsto\partial^\alpha f\in\mathcal{S}(\mathbb{R}^N)\quad\text{and}\quad\mathcal{S}(\mathbb{R}^N)\ni f(x)\mapsto x^\alpha f(x)\in\mathcal{S}(\mathbb{R}^N)$$

are continuous for $\tau_{\mathcal{S}}$.

Since $T:\mathcal{S}(\mathbb{R}^N)\to\mathcal{S}(\mathbb{R}^N)$ and if $\varphi\in\mathcal{S}'(\mathbb{R}^N)$, then $T^t(\varphi)=\varphi\circ T$ is certainly a linear map from $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$.
If U is an open set in $\mathbb{C}$, then $(T^t(\varphi)^{-1}\{U\}=(\varphi\circ T)^{-1}\{U\}=T^{-1}\{\varphi^{-1}\{U\}\}$. Since φ is a tempered distribution, $\varphi^{-1}\{U\}\in\tau_{\mathcal{S}}$ and since T is continuous, $T^{-1}\{\varphi^{-1}\{U\}\}\in\tau_{\mathcal{S}}$ again. Hence, $T^t(\varphi)$ is a continuous linear map on $\mathcal{S}(\mathbb{R}^N)$ for $\tau_{\mathcal{S}}$.
T is then indeed a linear map from $\mathcal{S}'(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$. Let us show that it is also continuous for $\tau(\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N))$.

Let $U \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ and let $\varphi \in (T^t)^{-1}\{U\}$. Since U is open, there are Schwartz functions $f_1, \ldots, f_n$, so that $U_{T^t(\varphi),f_1,\ldots,f_n} = \{\eta \in \mathcal{S}'(\mathbb{R}^N) \,:\, \forall k = 1, \ldots, n,\ |\eta(f_k) - T^t(\varphi)(f_k)| < 1\} \subset U$.
Define then $h_k := T(f_k)$ for all $k = 1, \ldots, n$ and consider the open set $U_{\varphi,h_1,\ldots,h_n} \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$. If $\psi \in U_{\varphi,h_1,\ldots,h_n}$, then $\forall k = 1, \ldots, n$, $|\psi(h_k) - \varphi(h_k)| < 1$, so that $T^t(\psi) \in U_{T^t(\varphi),f_1,\ldots,f_n}$. Hence, $U_{\varphi,h_1,\ldots,h_n} \subset (T^t)^{-1}\{U_{T^t(\varphi),f_1,\ldots,f_n}\} \subset (T^t)^{-1}\{U\}$, which shows, that the latter is open.
For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$, we have that

$$\begin{aligned}
\|\!|\partial^\alpha f|\!\|_n &= \max\{\|(1 + x\cdot x)^n \partial^{\alpha+\beta} f\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \\
&\leq \max\{\|(1 + x\cdot x)^{n+|\alpha|} \partial^{\beta} f\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n+|\alpha|}\} = \|\!|f|\!\|_{n+|\alpha|}, \\
\|\!|x^\alpha f|\!\|_n &= \max\{\|(1 + x\cdot x)^n \partial^{\beta} (x^\alpha f)\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \\
&\leq \max\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} \|(1 + x\cdot x)^n (\partial^\gamma x^\alpha)(\partial^\delta f)\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \\
&= \max\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \|(1 + x\cdot x)^n (x^{\alpha-\gamma})(\partial^\delta f)\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \\
&\leq \max\{\sum_{\gamma+\delta=\beta} \binom{\beta}{\gamma} (\alpha!) \|(1 + x\cdot x)^{n+|\alpha|} (\partial^\delta f)\|_\infty \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \\
&\leq \max\{\sum_{\gamma\leq\beta} \binom{\beta}{\gamma} (\alpha!) \|\!|f|\!\|_{n+|\alpha|} \,:\, \beta \in \mathbb{N}^N_{\leq n}\} \leq \binom{N+n}{n} 2^n (\alpha!)\, \|\!|f|\!\|_{n+|\alpha|}.
\end{aligned}$$

7. The Poincaré group is defined as the set of couples $(\Lambda, d) \in \mathbb{M}_4(\mathbb{R}^4) \times \mathbb{R}^4$, so that $\Lambda^t \eta \Lambda = \eta$, where $\eta = \mathrm{diag}(1, -1, -1, -1)$. The group product is defined as $(\Lambda_1, d_1) \cdot (\Lambda_2, d_2) := (\Lambda_1\Lambda_2, d_1 + \Lambda_1 d_2)$.
Define the action of this group on $\mathcal{S}(\mathbb{R}^4)$ by

$$\mathcal{S}(\mathbb{R}^4) \ni f(x) \mapsto ((\Lambda, d) \circ f)(x) := f((\Lambda, d)^{-1} x)$$

and show that this action is continuous for $\tau_{\mathcal{S}}$.
(Hint: for a linear map $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ and a function $f \in C^\infty(\mathbb{R}^N)$, Faà di Bruno's formula reads

$$\partial^\alpha (f \circ \varphi) = \sum_{\beta \in \mathbb{N}^N_{=|\alpha|}} (\partial^\beta f)(\varphi(x)) \sum_{\substack{\gamma_1,\ldots,\gamma_N \in \mathbb{N}^N, \\ \forall 1\leq j\leq N,\, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^N \gamma_j = \beta}} \alpha! \prod_{j=1}^N \frac{1}{\gamma_j!} \left(\frac{\partial \varphi(x)}{\partial x_j}\right)^{\gamma_j}.$$

)

For a given $x \in \mathbb{R}^4$ and a fixed element (Λ, d) in the Poincaré group, let us denote

$x \cdot x$ by $|x|^2$ and $\sup\{|\Lambda x| : |x| \leq 1\}$ by $\|\Lambda\|$. One then has

$$\begin{aligned}
1 + (\Lambda x + d) \cdot (\Lambda x + d) &= 1 + |\Lambda x + d|^2 \\
&\leq 1 + (\|\Lambda\|\,|x| + |d|)^2 = 1 + \|\Lambda\|^2 x \cdot x + 2\|\Lambda\|\,|x|\,|d| + |d|^2 \\
&\leq 1 + x \cdot x + \|\Lambda\|^2(1 + x \cdot x) + \|\Lambda\|\,|d|(1 + x \cdot x) + |d|^2(1 + x \cdot x) \\
&\leq (1 + |d| + \|\Lambda\|)^2(1 + x \cdot x).
\end{aligned}$$

Also, for a fixed $\alpha \in \mathbb{N}^4$, one has

$$\begin{aligned}
&\partial^\alpha\left((\Lambda, d) \circ f\right) = \partial^\alpha\left(f((\Lambda, d)^{-1}x)\right) = \partial^\alpha\left(f(\Lambda^{-1}x - \Lambda^{-1}d)\right) \\
&= \sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} (\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d) \sum_{\substack{\gamma_1,\ldots,\gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4,\, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \alpha! \prod_{j=1}^4 \frac{1}{\gamma_j!} (\frac{\partial \Lambda x}{\partial x_j})^{\gamma_j}.
\end{aligned}$$

Therefore, if $\alpha \in \mathbb{N}^4_{\leq n}$,

$$\begin{aligned}
&|(1 + x \cdot x)^n \partial^\alpha (\Lambda, d) \circ f| \\
&= |(1 + x \cdot x)^n \sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} (\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d) \sum_{\substack{\gamma_1,\ldots,\gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4,\, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \alpha! \prod_{j=1}^4 \frac{1}{\gamma_j!} (\frac{\partial \Lambda x}{\partial x_j})^{\gamma_j}| \\
&\leq \alpha!(1 + x \cdot x)^n \left(\sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d) \sum_{\substack{\gamma_1,\ldots,\gamma_4 \in \mathbb{N}^4, \\ \forall 1 \leq j \leq 4,\, |\gamma_j| = \alpha_j, \\ \sum_{j=1}^4 \gamma_j = \beta}} \|\Lambda\|^{|\beta|}| \right) \\
&\leq \alpha!(1 + x \cdot x)^n \left(\sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d)| \prod_{j=1}^4 \binom{4 + \alpha_j - 1}{\alpha_j} \|\Lambda\|^{|\beta|} \right) \\
&\leq \alpha! \binom{\overline{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + x \cdot x)^n \sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} |(\partial^\beta f)(\Lambda^{-1}x - \Lambda^{-1}d)| \\
&\underset{y = \Lambda^{-1}x - \Lambda^{-1}d}{=} \alpha! \binom{\overline{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + |\Lambda y + d|^2)^n \sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} |(\partial^\beta f)(y)| \\
&\leq \alpha! \binom{\overline{3} + \alpha}{\alpha} \|\Lambda\|^{|\alpha|} (1 + |\Lambda\| + |d|)^{2n} \sum_{\beta \in \mathbb{N}^4_{=|\alpha|}} |(1 + y \cdot y)^n (\partial^\beta f)(y)| \\
&\leq ((3 + n)!)^5 \|\Lambda\|^n (1 + |\Lambda\| + |d|)^{2n} \|\!|f|\!\|_n,
\end{aligned}$$

and taking the supremum over all $x \in \mathbb{R}^4$ shows continuity.

8. Let $f, g \in \mathcal{S}(\mathbb{R}^N)$ and let $h \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$.Show that $f * h \in C^\infty(\mathbb{R}^N)$ and that $f * g \in \mathcal{S}(\mathbb{R}^N)$ again.
(Hint: for differentiability, you might wanna use dominated convergence and Rolle's theorem.)
Prove then that for $\alpha \in \mathbb{N}^N$, $\partial^\alpha(f * g) = (\partial^\alpha f) * g$ and that $\mathcal{S}(\mathbb{R}^N) \ni g \mapsto f * g$ is continuous for $\tau_{\mathcal{S}}$.

$\int_{\mathbb{R}^N} f(x-y)h(y)\mu_L(dy)$ is certainly well-defined for any fixed $x \in \mathbb{R}^N$ since $f(x-y) \in \mathcal{S}(\mathbb{R}^N)$, implying $f(x-y)h(y) \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$.
Moreover, by Rolle's theorem, for any $x, x' \in \mathbb{R}^N$, there is a number $0 < c < 1$, so that $f(x-y)h(y) - f(x'-y)h(y) = (J_f(x'-y+c(x-x')) \cdot (x-x'))h(y)$, where $J_f(x)$ is the gradient of f.
For $(x'-x)\cdot(x'-x) \leq 1$, this then shows, that both $f(x-y)h(y) - f(x'-y)h(y)$ and $|x'-x|^{-1}(f(x-y) - f(x'-y))h(y)$ are absolutely bounded by $N|\!|\!|f|\!|\!|_1 |h(y)|$.
Applying dominated convergence, one gets

$$\lim_{x' \to x} (f * h)(x') = (f * h)(x) \quad \text{and}$$

$$\forall \delta \in \mathbb{N}^N_{=1}, \quad \lim_{a \to 0} \frac{1}{a}\left((f * h)(x + a\delta)) - (f * h)(x)\right) = \left((\partial^{\bar{1}\delta} f) * h\right)(x).$$

Since $\partial^{\bar{1}\delta} f$ s again a Schwartz function, this shows continuity and differentiability of the function $f * h$. By induction, this shows also that $f * h$ is smooth and that for any multi-index α, $\partial^\alpha(f * h) = (\partial^\alpha f) * h$.
If in addition $g \in \mathcal{S}(\mathbb{R}^N)$, then for any fixed $x \in \mathbb{R}^N$, one has

$$\begin{aligned}
|(1 + x \cdot x)^n (f * g)(x)| &= |\int_{\mathbb{R}^N} (1 + x \cdot x)^n f(x-y)g(y)\mu_L(dy)| \\
&\leq \int_{\mathbb{R}^N} (1 + x \cdot x)^n |f(x-y)g(y)|\mu_L(dy) \\
&\leq \int_{\mathbb{R}^N} (1 + |x|)^{2n} |f(x-y)g(y)|\mu_L(dy) \\
&\leq \int_{\mathbb{R}^N} (1 + |x-y|)^{2n}(1 + |y|)^{2n} |f(x-y)g(y)|\mu_L(dy) \\
&\leq \int_{\mathbb{R}^N} 2^n(1 + (x-y)\cdot(x-y))^n 2^n (1 + y \cdot y)^n |f(x-y)g(y)|\mu_L(dy) \\
&\leq 4^n \|(1 + x \cdot x)^n f(x)\|_{L^2} \|(1 + x \cdot x)^n g(x)\|_{L^2} \leq 4^n \|f\|_{2n} \|g\|_{2n},
\end{aligned}$$

which shows that $(1 + x \cdot x)^n (f * g)(x)$ has to remain bounded for any $n \in \mathbb{N}$. Hence, $f * g \in \mathcal{S}(\mathbb{R}^N)$.
But it also shows continuity, since for all $n \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^N_{\leq n}$,

$$|\!|\!|f * g - f * h|\!|\!|_n = |\!|\!|f * (g - h)|\!|\!|_n \leq \binom{N+n}{n} 4^n \|f\|_{2n} \|g - h\|_{2n}.$$

9. For a given Schwartz function f, show that for any $n \in \mathbb{N}$, there is a constant $C_{n,f}$ and a function $h(y)$ with $\lim_{|y|\to 0} h(y) = 0$, so that

$$\|(1 + x \cdot x)^n (f(x + y) - f(x))\|_\infty \leq C_{n,f} h(y).$$

Let $(d_n(x))_{n\in\mathbb{N}}$ be a Dirac sequence. Use the previous estimation to show that

$$\forall n \in \mathbb{N}, \quad \lim_k |||f - d_k * f|||_n = 0.$$

For $f \in \mathcal{S}(\mathbb{R}^N)$ and $x, y \in \mathbb{R}^N$, one has by Rolle's theorem

$$|(1 + x \cdot x)^n (f(x + y) - f(x))| = |(1 + x \cdot x)^n (\nabla f)(x + ty) \cdot y|$$

for some $t \in]0, 1[$ and where (∇f) is the gradient of f. Now, if $|x| = x \cdot x^{1/2}$, then $|x + y| \leq |x| + |y|$ and $2|x| \leq 1 + x \cdot x$, so that

$$\begin{aligned}(1 + x \cdot x) &= (1 + |x + ty - ty|^2) \leq (1 + |x + ty|^2 + 2|x + ty|\,|ty| + |ty|^2)\\ &\leq (1 + |x + ty|^2 + (1 + |x + ty|^2)|ty| + |ty|^2) \leq 2(1 + |x + ty|^2)(1 + |ty|)^2.\end{aligned}$$

Therefore, for some $t \in]0, 1[$,

$$\begin{aligned}|(1 + x \cdot x)^n (f(x + y) - f(x))| &\leq 2^n |(1 + |x + ty|^2)^n (\nabla f)(x + ty) \cdot y(1 + |y|^2)^n|\\ &\leq 2^n N |||f|||_n |y|(1 + |y|^2)^n.\end{aligned}$$

Hence, $|||f(x + y) - f(x)|||_n \leq 2^n |||f|||_{n+1} |y|(1+|y|^2)$, which shows, that $\lim_{|y|\to 0}(f(x+y) - f(y)) = 0$ for $\tau_{\mathcal{S}}$.
For $f \in \mathcal{S}(\mathbb{R}^N)$ one has

$$\begin{aligned}|f(x) - (d_n * f)(x)| &= |f(x) - \int_{\mathbb{R}^N} d_n(x - y) f(y) \mu_L(dy)|\\ &= |f(x) \int_{\mathbb{R}^N} d_n(y) \mu_L(dy) - \int_{\mathbb{R}^N} d_n(y) f(x - y) \mu_L(dy)|\\ &= |\int_{\mathbb{R}^N} d_n(y) \left(f(x) - f(x - y)\right) \mu_L(dy)|.\end{aligned}$$

Since $(d_k)_{k\in\mathbb{N}}$ is a Dirac sequence, there is an $n_r \in \mathbb{N}$, so that $k \geq n_r$ implies $\text{supp}(d_k) \subset D(0, r)$. For such a $k \in \mathbb{N}$, one therefore has

$$\begin{aligned}|(1 + x \cdot x)^n \left(f(x) - (d_k * f)(x)\right)| &\leq \int_{\mathbb{R}^N} (1 + x \cdot x)^n d_k(y) |f(x) - f(x - y)| \mu_L(dy)\\ &= \int_{D(0,r)} d_k(y)(1 + x \cdot x)^n |f(x) - f(x - y)| \mu_L(dy)\\ &\leq r 4^n |||f|||_{n+1} \int_{D(0,r)} d_k(y) \mu_L(dy) = r 4^n |||f|||_{n+1}.\end{aligned}$$

Consequently, $k \geq n_r$ implies $\|(1 + x \cdot x)^n \left(f - d_k * f\right)\|_\infty \leq r 4^n |||f|||_{n+1}$, which converges to 0.

For a given $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N_{\leq n}$, one has that $\partial^\alpha f$ is again a Schwartz function, so that here also, there is an n_α, so that $k \geq n_\alpha$ implies

$$\|(1 + x \cdot x)^n \left(\partial^\alpha f - \partial^\alpha (d_k * f)\right)\|_\infty = \|(1 + x \cdot x)^n \left(\partial^\alpha f - d_k * \partial^\alpha f\right)\|_\infty$$
$$\leq r 4^n |||\partial^\alpha f|||_{n+1} \leq r 4^n |||f|||_{2n+1}.$$

Since there are finitely many such α's in $\mathbb{N}^N_{\leq n}$, one may take $N_r = \max\{n_\alpha : \alpha \in \mathbb{N}^N_{\leq n}\}$ and for $k \geq N_r$, one has

$$|||f - d_k * f|||_n \leq r 4^n |||f|||_{2n+1}.$$

Therefore, $\lim_k d_k * f = f$ for $\tau_{\mathcal{S}}$.

10. Let $(d_n(x))_{n \in \mathbb{N}}$ be a Dirac sequence. Prove that the functionals

$$\mathbb{S}(\mathbb{R}^N) \ni\mapsto \delta_n(f) := \int_{\mathbb{R}^N} (d_n * f) \mu_L(dx)$$

converge in the weak* topology to φ_1.
Show that for $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, $(d_n * \varphi)_{n \in \mathbb{N}}$ defines a sequence of C^∞-functions converging to φ for the topology $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.

Since we clearly have $d_n \in \mathcal{S}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, the map $f \mapsto d_n^* f$ is continuous for $\tau_{\mathcal{S}}$ and for each fixed $n \in \mathbb{N}$. Hence, $(\delta_n)_{n \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^N)$.
Furthermore, for $f \in \mathcal{S}(\mathbb{R}^N)$,then $d_n(y) f(x - y)$ is clearly in $\mathcal{S}(\mathbb{R}^{2N})$, so that one may apply Fubini's theorem to obtain

$$\begin{aligned}
\delta_n(f) = \int_{\mathbb{R}^N} (d_n * f)(x) \mu_L(dx) &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} d_n(x - y) f(y) \mu_L(dy) \right) \mu_L(dx) \\
&= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} d_n(y) f(x - y) \mu_L(dy) \right) \mu_L(dx) \\
&= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} d_n(y) f(x - y) \mu_L(dx) \right) \mu_L(dy) \\
&= \int_{\mathbb{R}^N} d_n(y) \varphi_1(f) \mu_L(dx) = \varphi_1(f).
\end{aligned}$$

It is then clear that $(\delta_n)_{n \in \mathbb{N}}$ converges *-weakly to φ_1.
By the previous exercice, if $f \in \mathcal{S}(\mathbb{R}^N)$ then $\lim_k d_k * f = f$ for the topology $\tau_{\mathcal{S}}$. Since $(P^t d_k)_{k \in \mathbb{N}}$ is again a Dirac sequence, one has $\lim_k P^t(d_k) * f = f$ as well. If $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, then by continuity one has $\lim_k (d_k * \varphi)(f) = \lim_k \varphi(P^t(d_k) * f) = \varphi(\lim_k P^t(d_k) * f) = \varphi(f)$. Therefore, $\lim_k (d_k * \varphi) = \varphi$ for $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.
It remains to show, that for each $k \in \mathbb{N}$, $(d_k * \varphi)(f)$ is integration of f against a C^∞-function g_k. By Schwartz representation theorem, there is a polynomially bounded function $g \in C(\mathbb{R}^N)$ and a multi-index α, so that for any Schwartz function f,

$$\varphi(f) = \int_{\mathbb{R}^N} g \partial^\alpha f \mu_L(dx).$$

Therefore,

$$(d_k * \varphi)(f) = \varphi((P^t d_k) * f) = \int_{\mathbb{R}^N} g \partial^\alpha \left((P^t d_k) * f\right) \mu_L(dx)$$
$$= \int_{\mathbb{R}^N} g(x) \partial^\alpha \left(\int_{\mathbb{R}^N} d_k(y-x) f(y) \mu_L(dy) \right) \mu_L(dx)$$
$$= \int_{\mathbb{R}^N} g(x) \left(\int_{\mathbb{R}^N} d_k(y-x) \partial^\alpha f(y) \mu_L(dy) \right) \mu_L(dx).$$

The function $g(x)$ being polynomially bounded and continuous, there is an $m \in \mathbb{N}$, so that $g(x)(1 + x \cdot x)^{-m} \in \mathcal{L}^1(\mathbb{R}^N, \mu_L)$. Furthermore,

$$|g(x) d_k(y-x) \partial^\alpha f(y)| = |g(x)(1 + x \cdot x)^{-m} (1 + x \cdot x)^m d_k(x-y) \partial^\alpha f(y)|$$
$$\leq |g(x)(1 + x \cdot x)^{-m} (1 + |x|)^{2m} d_k(x-y) \partial^\alpha f(y)|$$
$$\leq |g(x)(1 + x \cdot x)^{-m} (1 + |x-y| + |y|)^{2m} d_k(x-y) \partial^\alpha f(y)|$$
$$\leq |g(x)(1 + x \cdot x)^{-m} (1 + |x-y|)^{2m} d_k(x-y) (1 + |y|)^{2m} \partial^\alpha f(y)|$$
$$\leq 4^{2m} |g(x)(1 + x \cdot x)^{-m} d_k(x-y) (1 + y \cdot y)^m \partial^\alpha f(y)|,$$

which is in $\mathcal{L}^1(\mathbb{R}^{2N}, \mu_L)$, since $(1 + y \cdot y)^m \partial^\alpha f(y)$ is again a Schwartz function. Therefore, we may apply Fubini's theorem and obtain

$$(d_k * \varphi)(f) = \int_{\mathbb{R}^N} g(x) \left(\int_{\mathbb{R}^N} d_k(y-x) \partial^\alpha f(y) \mu_L(dy) \right) \mu_L(dx)$$
$$= \int_{\mathbb{R}^N} \partial^\alpha f(y) \left(\int_{\mathbb{R}^N} d_k(y-x) g(x) \mu_L(dx) \right) \mu_L(dy).$$

The integral $\int_{\mathbb{R}^N} d_k(y-x) g(x) \mu_L(dx)$ results in a function $h_k(y)$ and by using again a Rolle-type argument can be shown to be C^∞. Partial integration permits one to conclude, that the C^∞-function $g_k(y) = (-1)^{|\alpha|} \partial^\alpha h_k(y)$ satisfies

$$(d_k * \varphi)(f) = \int_{\mathbb{R}^N} g_k f d\mu_L.$$

Remark: The sequence of C^∞-functions $(g_k)_{k \in \mathbb{N}}$ is sometimes called a **regularisation** of φ. This sequence however does only converge in the distributional sense (i.e. in the weak* topology) to φ.